

HOMOLOGY THEORIES

INGRID STARKEY

ABSTRACT. This paper will introduce the notion of homology for topological spaces and discuss its intuitive meaning. It will also describe a general method that is used to construct homology theories and will provide two examples: Poset and Singular Homology. Finally, the paper will connect the theories using the Eilenberg-Steenrod Axioms and will give several consequences of the Axioms.

CONTENTS

1. Introduction	1
2. An Informal Construction of a Homology Theory	2
3. Some Examples of Homology Theories	3
3.1. Poset Homology	3
3.2. Singular Homology	7
4. The Eilenberg-Steenrod Axioms	8
5. Some Consequences of the Eilenberg-Steenrod Axioms	10
5.1. The Uniqueness of Homology Theories	10
5.2. Homology for Homotopy Equivalent Spaces	10
Acknowledgments	12
References	12

1. INTRODUCTION

Given a topological space X , that space can be classified by the number and degree of its “holes”; homology is a method for counting and distinguishing the holes in a space. Although homotopy groups also record this information and more, they are much more difficult to calculate, making homology groups preferable in this regard, despite not carrying as much information. For a given nonnegative $n \in \mathbb{Z}$ and space X , $H_n(X)$ is the n -th homology group of X and describes how many “ n -dimensional holes” are in X . The “0-dimensional hole” can be thought of as a gap or separation between components, with $H_0(X)$ describing how many path components make up X .

There are multiple homology theories for different objects, which differ only in their precise construction and ease of computability. In each of the theories that will be studied, their constructions follow the same steps including the defining of simplices, boundary maps, chain complexes, and homology groups. For each theory, the hope is that by looking at simplices and how they are attached to each other, we can find information about the structure of the space as a whole.

In this paper we will first discuss an informal construction of a homology theory, where multiple shared ingredients will be laid out. We will then introduce the Poset Homology and Singular Homology theories. Next, we will state the Eilenberg-Steenrod Axioms which fully determine homology theories. Finally, we shall discuss several consequences of the axioms, which include the uniqueness of homology theories and the relation between the homology groups of homotopy equivalent spaces.

2. AN INFORMAL CONSTRUCTION OF A HOMOLOGY THEORY

The homology theories we shall consider share the same general ingredients and are constructed in similar ways. Although some ingredients are not formally defined, they can be found later in the homology theories we will define.

Informal Definition 2.1. Given a space X , an n -simplex σ_n is an n -dimensional triangle that lives inside X . The n -simplex is generally constructed from an ordered $(n + 1)$ -tuple of (not necessarily distinct) points. A 2-simplex is the triangle with which we are most familiar.

Remark 2.2. In most cases, two n -simplices σ_n and τ_n can be constructed from the same set of points and still be distinct if the points are ordered differently. For example, a simplex $[a, b, c, d]$ is usually distinct from a simplex $[a, c, b, d]$.

Informal Definition 2.3. An i -simplex σ_i is a *subsimplex* of an n -simplex σ_n if σ_i can be formed by removing points from σ_n . Therefore, $[1, 2, 3]$ is a subsimplex of a simplex $[1, 2, 3, 4]$ since the point 4 has been removed.

Informal Definition 2.4. The group of n -chains C_n is the free abelian group generated by all n -simplices. An element δ_n of C_n is referred to as an n -chain, and is the formal sum of a finite number of n -simplices σ_n with integer weights. Therefore, for $c_i \in \mathbb{Z}$ and $\sigma_{n,i} \neq \sigma_{n,j}$ if $i \neq j$,

$$\delta_n = \sum_{i=0}^m c_i \sigma_{n,i}.$$

Non-Definition 2.5. The *boundary map* $\partial_n : C_n \rightarrow C_{n-1}$ takes an n -simplex σ_n to its boundary, regarded as an $(n - 1)$ -chain, with each $(n - 1)$ -simplex being a subsimplex of σ_n . In the more geometric homology theories, the boundary map may take a simplex to its geometric boundary. A more formal definition of specific boundary maps will appear in Section 3.

Definition 2.6. Given a sequence of abelian groups

$$\cdots \xrightarrow{f_{n+1}} X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0$$

with homomorphisms $f_i : X_i \rightarrow X_{i-1}$, the sequence is a *chain complex* if $\text{im} f_i \subseteq \ker f_{i-1}$ for $i > 1$. This definition is equivalent to the requirement that $f_{i-1} \circ f_i = 0$ for all $i > 1$. The sequence is *exact* if $\text{im} f_i = \ker f_{i-1}$ for $i > 1$.

Assumption 2.7. Given a space X , it is possible to construct a chain complex $C_*(X)$ of the groups of n -chains on X that is the sequence

$$\cdots \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0.$$

Taking the boundary of any n -simplex or n -chain twice yields 0, by definition.

Definition 2.8. An n -simplex σ_n is a *cycle* if $\sigma_n \in \ker(\partial_n)$. The n -simplex σ_n is a *boundary* if $\sigma_n \in \text{im}(\partial_{n+1})$.

Observation 2.9. Every boundary is a cycle. This follows from the fact that $\text{im} f_i \subseteq \ker f_{i-1}$ for a chain complex.

Definition 2.10. Given a space X , the n -th *homology group* is the quotient group

$$H_n(X) = \ker(\partial_n) / \text{im}(\partial_{n+1}).$$

$H_n(X)$ is defined to be the quotient n -cycles/ n -boundaries.

Remark 2.11. Note that since every boundary is a cycle, the quotient $\ker(\partial_n) / \text{im}(\partial_{n+1})$ makes sense, since $\text{im}(\partial_{n+1}) \subseteq \ker(\partial_n)$ and is a normal subgroup.

3. SOME EXAMPLES OF HOMOLOGY THEORIES

3.1. Poset Homology. Although it might seem unusual to take the homology of an object that is not a topological space, we will show that there is an equivalence between partially ordered sets – posets – and Alexandroff Spaces that are T_0 .

Definition 3.1. Given a set X , a *topology* on X consists of a set \mathcal{U} of subsets of X , called “the open sets of X in the topology \mathcal{U} ” with the following properties:

- Both \emptyset and X are in \mathcal{U} .
- An arbitrary union of sets in \mathcal{U} is in \mathcal{U} .
- A finite intersection of sets in \mathcal{U} is in \mathcal{U} .

An *Alexandroff space* is a set with a collection of open sets that satisfies the aforementioned properties, along with the additional requirement that any arbitrary intersection of open sets is open.

Definition 3.2. If (X, \mathcal{U}) is a topological space, then X is a T_0 *space* if for any two points in X , there is an open set that contains one point, but not the other. Note that one cannot *choose* which point the open set contains.

Non-Example 3.3. Let (X, \mathcal{U}) be a topological space, with $X = \{1, 2, 3, 4, 5\}$ a finite set and $\mathcal{U} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}, X\}$. For some $x, y \in X$ there exists an open set that contains one of x or y but not the other. For example, for $x = 1$ and $y = 5$, there exists the open set $\{1\}$ that contains x but not y . Note that you cannot choose an open set that contains 5 but not 1. However, x and y can be chosen so that there does not exist an open set which contains just one of them. If $x = 4$ and $y = 5$, then there does not exist an open set containing one but not the other, so (X, \mathcal{U}) is not a T_0 space.

Non-Example 3.4. Any space with more than one point and with the indiscrete topology – the open sets are solely the empty set and the entire space – is not a T_0 space since the topology does not distinguish points.

Definition 3.5. A *basis* for a topology on a set X is a set \mathcal{B} of subsets of X such that:

- For each $x \in X$, there exists at least one $B \in \mathcal{B}$ such that $x \in B$.
- If $x \in B' \cap B''$ where $B', B'' \in \mathcal{B}$, then there exists at least one $B \in \mathcal{B}$ such that $x \in B \subseteq B' \cap B''$.

Definition 3.6. Given an Alexandroff space X and an element $x \in X$, let U_x be the intersection of all open sets that contain x . Note U_x is open by the definition of Alexandroff space. Define a relation \leq on X such that $x \leq y$ if and only if $U_x \subseteq U_y$, which would also mean that $x \in U_y$. If the inclusion is proper, write $x < y$.

Lemma 3.7. *Let x and y be two points in X . The set $U_x = U_y$ if and only if $x \in V$ if and only if $y \in V$ for all V open.*

Proof. If $U_x = U_y$, then the intersection of all open sets that contain x and the intersection of all open sets that contain y are the same. Let $U \ni x$ be open. So, $U_x \subseteq U$. Since $U_x = U_y$, then $U_y \subseteq U$ also. This implies that $y \in U$. Thus, x and y are contained in the same open sets. Conversely, if x and y share the same open sets, then the intersection of these open sets will be the same. Therefore, $U_x = U_y$. \square

Remark 3.8. A consequence of this Lemma is that X is T_0 if and only if $U_x = U_y$ implies $x = y$ for all $x, y \in X$.

Definition 3.9. Given a binary relation \diamond , define the following properties:

- A relation \diamond is *reflexive* if $x \diamond x$ for all x .
- A relation \diamond is *transitive* if $x \diamond y$ and $y \diamond z$ implies $x \diamond z$ for all x, y, z .
- A relation \diamond is *antisymmetric* if $x \diamond y$ and $y \diamond x$ implies $x = y$ for all x, y .

Definition 3.10. A *preorder* is a reflexive and transitive binary relation \leq on a set X .

Definition 3.11. A *partial order* is a preorder (X, \leq) such that \leq is antisymmetric. In this case, (X, \leq) is called a *poset*. We will show that a poset is equivalent to a T_0 Alexandroff Space.

Remark 3.12. Note that for a space X , the set $\{U_x\}_{x \in X}$ satisfies both properties for a basis. For each x , U_x contains x since $x \leq x$. And, if $x \in U_y$ and $x \in U_z$, then $x \leq y$ and $x \leq z$. Since $U_x = \{w \mid w \leq x\}$, then $w \leq y$ and $w \leq z$ for all $w \in U_x$. Thus, $U_x \subseteq U_y \cap U_z$, satisfying the second requirement for a basis.

Theorem 3.13. *Let (X, \leq) be a preorder. Define \mathcal{U} as the collection of arbitrary unions of the sets $U_x = \{y \mid y \leq x\}$. The space (X, \mathcal{U}) is a topology and is an Alexandroff Space. This topology is called the order topology on X .*

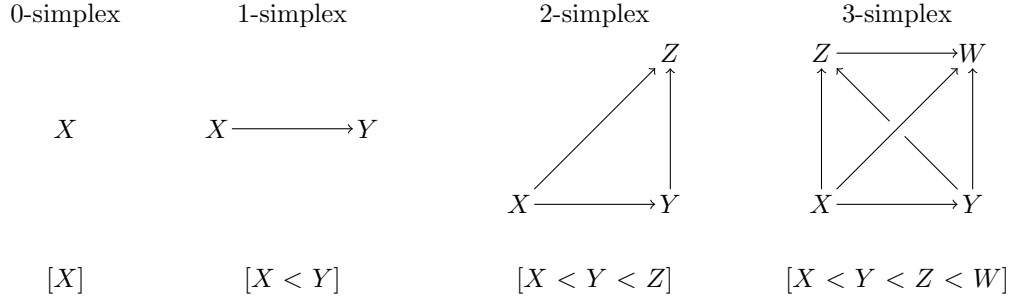
Proof. We first verify that the first two axioms for a topology. So, $X = \bigcup_{x \in X} U_x$. Therefore, $X \in \mathcal{U}$. The union of no set is open, so $\emptyset \in \mathcal{U}$. And, an arbitrary union of arbitrary unions of U_x 's is clearly an arbitrary union of U_x 's, and is thus in \mathcal{U} .

Take an arbitrary intersection $V = \bigcap_{i \in \mathcal{A}} X_i$, where each $X_i \in \mathcal{U}$. If $v \in V$, then $v \in X_i$ for all i . Since each X_i is an arbitrary union of some sets U_x , $v \in U_{x_i}$ for some x_i . Then $U_v \subseteq U_{x_i} \subseteq X_i$. Since a suitable U_{x_i} can be chosen for each X_i , $U_v \subseteq X_i$ for all i . This also holds for all $v \in V$, so $\bigcup_{v \in V} U_v \subseteq \bigcap_{i \in \mathcal{A}} X_i = V$. The other inclusion is clear. Thus, $V = \bigcup_{v \in V} U_v$. Therefore, V is the union of U_x 's, so $V \in \mathcal{U}$. Thus, an arbitrary intersection of open sets is open, making X a topological space, and moreover, an Alexandroff Space. \square

Remark 3.14. Lemma 3.7 shows that (X, \mathcal{U}) is T_0 if and only if (X, \leq) is a poset since $U_x = U_y$ if and only if $x \leq y$ and $y \leq x$, which means that $x = y$.

Definition 3.15. Given a poset C , regard it as a category \mathcal{C} with the objects of \mathcal{C} being elements of C . Let there be a morphism from $X \rightarrow Y$ if $x \leq y$. If $x \leq y$, then use the notation $x \rightarrow y$. Define an n -simplex as $\sigma_n = [x_0 \leq x_1 \leq \dots \leq x_n]$.

Remark 3.16. It can be shown that poset homology can be defined with \leq or with the strict inequality $<$ and the resulting homology theories are naturally isomorphic. For calculations however, the strict inequality $<$ is easier to use since degenerate simplices can be illustrated in multiple different yet valid ways. So, we will use $<$ in the examples and illustrations.



Definition 3.17. The *boundary map* is $\partial_k : C_k \rightarrow C_{k-1}$ is defined on k -simplices σ_k by the formula

$$\partial_k(\sigma_k) = \sum_{i=0}^k (-1)^i [x_0 \leq \dots \leq \hat{x}_i \leq \dots \leq x_k]$$

and extended to C_k by linearity.

Theorem 3.18. *The boundary maps in Definition 3.17 define a chain complex.*

Proof. Let $\sigma_n = [x_0 \leq \dots \leq x_n]$. From the definition of the boundary map, we know that:

$$\partial_n(\sigma_n) = \sum_{i=1}^n (-1)^i [x_0 \leq \dots \leq \hat{x}_i \leq \dots \leq x_n].$$

The boundary map is constructed to be \mathbb{Z} -linear, so the boundary map ∂_{n-1} can be applied to each of the simplices in $\partial_n(\sigma_n)$. Consider $j, l \in \mathbb{Z}$ such that $0 \leq j, l \leq n$, and $j \neq l$. Without loss of generality, let $j < l$.

We know the following simplices are present in the formal sum $\partial_n(\sigma_n)$:

$$\begin{aligned} \delta_{n-1} &= (-1)^l [x_0 \leq \dots \leq x_j \leq \dots \leq \hat{x}_l \leq \dots \leq x_n], \\ \tau_{n-1} &= (-1)^j [x_0 \leq \dots \leq \hat{x}_j \leq \dots \leq x_l \leq \dots \leq x_n]. \end{aligned}$$

When ∂_{n-1} is applied to δ_{n-1} and τ_{n-1} , the formal sum of $\partial(\delta_{n-1})$ will contain a simplex with x_j removed and $\partial(\tau_{n-1})$ will contain a simplex with x_l removed. These simplices are:

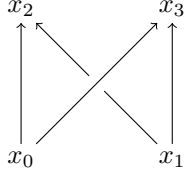
$$\begin{aligned} \alpha_{n-2} &= (-1)^{(l+j)} [x_0 \leq \dots \leq \hat{x}_j \leq \dots \leq \hat{x}_l \leq \dots \leq x_n], \\ \beta_{n-2} &= (-1)^{(l+(j-1))} [x_0 \leq \dots \leq \hat{x}_j \leq \dots \leq \hat{x}_l \leq \dots \leq x_n], \end{aligned}$$

where α_{n-2} is part of the boundary of δ_{n-1} , and β_{n-2} is part of the boundary of τ_{n-1} . Note that the exponent of β_{n-2} is one less than that of α_{n-2} , since in the case of β_{n-2} , the position of x_l had been moved one closer to the beginning of the simplex since x_j had previously been removed and $j < l$. Therefore, α_{n-2} and β_{n-2}

differ by a sign, so their sum is 0. This can be done for any pair of different integers j and l . Thus, the terms in $\partial_{n-1}(\partial_n(\sigma_n))$ cancel in pairs, and the overall sum will equal zero. \square

Remark 3.19. Note that although the previous theorem was only proven for a standard n -simplex, it will also hold for chains of simplices because of \mathbb{Z} -linearity.

Example 3.20. As will be discussed in Section 4, two spaces that are weakly homotopy equivalent will have isomorphic homology groups. Therefore, if there exists a poset model that is weakly homotopy equivalent to the standard unit circle S^1 , then the homology of the poset model can be calculated instead of that of S^1 and the same result will be reached. We will now introduce such a model, which we will call S^* . Consult [3] for a proof that S^1 and S^* are weakly homotopy equivalent.



Define $S^* = [x_0 < x_3] - [x_1 < x_3] + [x_1 < x_2] - [x_0 < x_2]$. Note that the largest simplex making up S^* is a 1-simplex, and that there are no chains of higher order simplices. Therefore, for $n > 1$, $C_n(S^*) \cong 0$ and $H_n(S^*) = 0$. This means that only $H_1(S^*)$ and $H_0(S^*)$ must be calculated.

Consider $H_0(S^*)$ first. We know intuitively that $H_0(S^*)$ indicates the number of path components in a space, so $H_0(S^*)$ should be \mathbb{Z} . The four generators of $C_0(S^*)$ are $[x_0]$, $[x_1]$, $[x_2]$, and $[x_3]$. For $H_0(S^*)$, the pairs of generators making up S^* are identified, so $H_0(S^*) \cong \mathbb{Z}$. This aligns with our knowledge that the circle is path connected.

Finally, consider $H_1(S^*)$. The homology group $H_1(S^*) = \ker(\partial_1)/\text{im}(\partial_2)$. The boundary map is $\partial_2 : C_2(S^*) \rightarrow C_1(S^*)$. However, there are no nondegenerate 2-simplices in S^* . Therefore, $\text{im}(\partial_2) = 0$, which means that $H_1(S^*) = \ker(\partial_1)/\text{im}(\partial_2) = \ker(\partial_1)/0 = \ker(\partial_1)$.

From here, $\ker(\partial_1)$ can be calculated using linear algebra over \mathbb{Z} . The boundary map is $\partial_1 : \langle [x_0 < x_2], [x_1 < x_2], [x_0 < x_3], [x_1 < x_3] \rangle \rightarrow \langle [x_0], [x_1], [x_2], [x_3] \rangle$. The 4×4 matrix B for the linear transformation can be found and is:

$$B = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 \end{bmatrix}.$$

Elementary row operations can be used to find the reduced row echelon form of the matrix, which results in:

$$B' = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since there are three vertical pivots, the rank of the matrix is three, making the dimension of the kernel one. Solving the matrix leads to the equations

$$\begin{aligned} x_0 - x_3 &= 0, \\ x_1 - x_3 &= 0, \\ x_2 - x_3 &= 0. \end{aligned}$$

Therefore, $x_0 = x_1 = x_2 = x_3$. So, $\ker(\partial_1) = \{(t, t, t, t) \mid t \in \mathbb{Z}\}$. Thus, $\ker(\partial_1) \cong \mathbb{Z}$. Thus, $H_1(S^*) = \mathbb{Z}$, and since S^* and S^1 are weakly homotopy equivalent, then $H_1(S^1) = \mathbb{Z}$ also. Logically, this makes sense since a circle has a single “1-dimensional” hole in it, which is counted by the single \mathbb{Z} .

3.2. Singular Homology. Singular Homology is the most general of the homology theories. Its generality makes it useful in proving facts and theorems about homology overall. However, the result of this generality is chain complexes that are too large to use in computations in any reasonable manner.

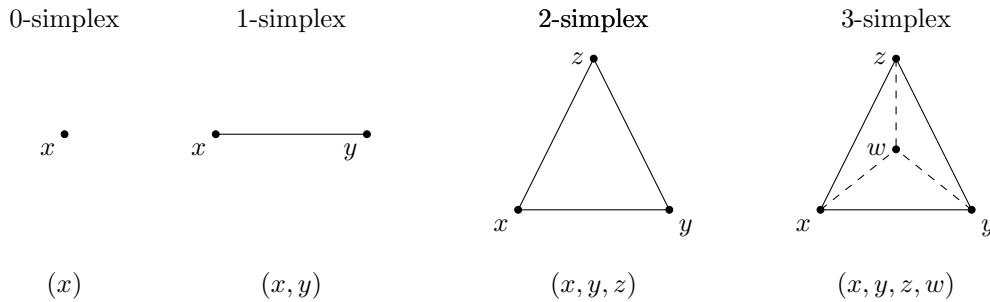
Definition 3.21. The *standard n -simplex* Δ_n is the set:

$$\Delta_n = \{(x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1} \mid 0 \leq x_i \leq 1, \sum_{i=0}^n x_i = 1\}$$

topologized as a subspace of \mathbb{R}^{n+1} .

Some low dimension simplices can be visualized as:

- A 0-simplex is a point.
- A 1-simplex is an edge with two endpoints.
- A 2-simplex is a triangle, with three points and three edge and the face.
- A 3-simplex is the standard tetrahedron, with four points, six edges, four faces, and the interior.



The simplices for singular homology are built off of the standard simplices.

Definition 3.22. Given a standard n -simplex Δ_n and a space X , a *singular n -simplex* is a continuous map $\sigma_n : \Delta_n \rightarrow X$. Note that the map σ_n need not be injective, just continuous - this generality makes it hard (or in many cases impossible) to calculate the singular homology directly.

Definition 3.23. Given a standard n -simplex Δ_{n-1} , consider an element $\alpha_{n-1} \in \Delta_{n-1}$ where $\alpha_{n-1} = (x_0, \dots, x_{n-1})$. Define the i -th *face map* $\delta_n^i : \Delta_{n-1} \rightarrow \Delta_n$ such that $\delta_n^i(\alpha_{n-1}) = (x_0, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_{n-1})$. For a singular n -simplex σ_n , let the i -th *face* be the result of restricting σ_n to the i -th face map.

Definition 3.24. Given a singular n -simplex σ_n , the *boundary* of σ_n would be the alternating sum of restricting the map to each of the faces of the standard n -simplex. Thus, the notation for the boundary map that passes the faces is $\sigma_n|_{\delta_n^i} : \Delta_{n-1} \rightarrow X$. Thus, the boundary of an n -simplex σ_n is

$$\sigma_n \circ \partial_n = \sum_{i=0}^n (-1)^i \sigma_n|_{\delta_n^i}.$$

4. THE EILENBERG-STEENROD AXIOMS

The Eilenberg-Steenrod gives an axiomatic description of homology theories. For a theory to be classified as a homology theory, it must satisfy the axioms.

Definition 4.1. A *category* \mathcal{C} consists of a collection of objects, a set $\mathcal{C}(A, B)$ of morphisms (called maps) between any two objects A and B , an identity morphism $id_A \in \mathcal{C}(A, A)$ for each object A , and a unital and associative composition law

$$\circ : \mathcal{C}(B, C) \times \mathcal{C}(A, B) \longrightarrow \mathcal{C}(A, C)$$

for each triple of objects A , B , and C .

Definition 4.2. A *functor* $F : \mathcal{C} \rightarrow \mathcal{D}$ is a map of categories. The functor assigns an object $F(A)$ of \mathcal{D} to an object A of \mathcal{C} and a morphism $F(f) : F(A) \rightarrow F(B)$ of \mathcal{D} to each morphism $f : A \rightarrow B$ of \mathcal{C} so that $F(id_A) = id_{F(A)}$ and $F(g \circ f) = F(g) \circ F(f)$.

Remark 4.3. A definition and diagram describing *natural transformations* can be found in [2] on page 14, along with more information on categories and functors.

Lemma 4.4. *If $A \subseteq X$, the sequence*

$$\cdots \rightarrow C_n(X)/C_n(A) \xrightarrow{\partial} C_{n-1}(X)/C_{n-1}(A) \rightarrow \cdots$$

is a chain complex.

Proof. Given an element $\sigma + C_n(A) \in C_n(X)/C_n(A)$, define the map ∂ such that $\partial(\sigma + C_n(A)) = \partial_n(\sigma) + C_{n-1}(A)$, where ∂_n is the boundary map defined for the chain complex on X and A . Now, we must prove that ∂ is well-defined. If $\sigma + C_n(A) = \tau + C_n(A)$, then $\sigma - \tau \in C_n(A)$. So, $\partial_n(\sigma - \tau) \in C_{n-1}(A)$. Therefore, $\partial_n\sigma - \partial_n\tau \in C_{n-1}(A)$ and so $\partial_n\sigma + C_{n-1}(A) = \partial_n\tau + C_{n-1}(A)$. And finally, taking ∂ twice results in $\partial_{n-1}(\partial_n\sigma) + C_{n-2}(A) = 0 + C_{n-1}(A) = 0$. Thus, the sequence is a chain complex. \square

Definition 4.5. Given a space X with subspace $A \subseteq X$, the homology $H_n(X, A) = H_n(C_*(X)/C_*(A))$ is called the *relative homology*.

Remark 4.6. Intuitively, two spaces are *weakly equivalent* if they have roughly the same basic shape and defining properties. Practically, they are weakly equivalent if there exists a map between them which is a weak homotopy equivalence.

Definition 4.7. A map $f : X \rightarrow Y$ of spaces is a *weak homotopy equivalence* if

$$f_* : \pi_n(X, x) \rightarrow \pi_n(Y, f(x))$$

is an isomorphism for all $x \in X$ and all $n \geq 0$, where $\pi_n(X, x)$ is the n -th homotopy group of X with x as a basepoint. For a complete definition of homotopy groups, consult [1].

Definition 4.8. A map $(X, A) \rightarrow (Y, B)$ of pairs is a *weak equivalence* if its maps $A \rightarrow B$ and $X \rightarrow Y$ are weak equivalences.

Theorem 4.9. For nonnegative integers n , there exist functors $H_n(X, A)$ from the pairs of spaces to the category of abelian groups, with a natural transformation $\partial : H_n(X, A) \rightarrow H_{n-1}(A)$ called the boundary map. Here, $H_n(X)$ is defined to be $H_n(X, \emptyset)$. The following axioms apply to these functors:

- **DIMENSION** If $*$ is a point, then $H_0(*) = \mathbb{Z}$ and $H_n(*) = 0$ for all $n > 0$.
- **EXACTNESS** The following sequence is exact, where the unlabelled arrows are induced by the inclusions $A \rightarrow X$ and $(X, \emptyset) \rightarrow (X, A)$:

$$\cdots \rightarrow H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \rightarrow H_n(X) \rightarrow H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \rightarrow \cdots$$

- **EXCISION** If (X, A) is a pair and U is a subset of X such that the closure of U is contained in the interior of A , then the inclusion map $i : (X \setminus U, A \setminus U) \rightarrow (X, A)$ induces an isomorphism in homology.
- **ADDITIVITY** If $(X, A) = \coprod (X_\alpha, A_\alpha)$ is the disjoint union of pairs (X_α, A_α) , then

$$H_n(X, A) = \bigoplus_{\alpha} H_n(X_\alpha, A_\alpha).$$

- **WEAK EQUIVALENCE** If $f : (X, A) \rightarrow (Y, B)$ is a weak equivalence of pairs, then

$$f_n : H_n(X, A) \rightarrow H_n(Y, B)$$

is an isomorphism for all nonnegative $n \in \mathbb{Z}$.

Proof. A proof can be found on page 96 in [2] □

Remark 4.10. We have already defined two homology theories, and it can be shown that both satisfy the axioms. In particular, it can easily be seen that the previously defined theories uphold the dimension axiom from their construction. We know that a point $*$ has a single path component, so $H_0(*) = \mathbb{Z}$ for the two theories. Regarding $n > 0$, for poset homology, there are no nondegenerate 1-simplices possible, since a nondegenerate 1-simplex in poset homology requires the presence of at least two points so that one may be less than the other. In a case similar to poset homology, the standard 1-simplex requires more than one point as well. And, since singular homology is constructed by sending standard geometric simplices through a continuous map, then there are no nondegenerate 1-simplices in singular homology either for a single point. Thus, $H_n(*) = 0$ for all theories defined for $n > 0$.

Theorem 4.11. Given a single path component X , $H_0(X) \cong \mathbb{Z}$.

Proof. We know that $H_0(X) = \ker(\partial : C_0(X) \rightarrow 0) / \text{im}(\partial : C_1(X) \rightarrow C_0(X)) = C_0(X) / \sim$, where $x \sim y$ if there exists a path $p : [0, 1] \rightarrow X$ such that $p(0) = x$ and $p(1) = y$. Therefore, all points that are connected by a path are identified, leaving only a single point for each path component. Since we have already shown that $H_0(*) = \mathbb{Z}$, then $H_0(X) = \mathbb{Z}$. □

Remark 4.12. It can also be seen that the additivity axiom is satisfied by the constructed homology theories. We know that for a space X , $H_0(X)$ records the number of disjoint path components. This reflects the additivity axiom since the

space X can be broken up into the disjoint union of topological spaces, each of which is a path component, the direct sum taking the role of counting each part.

5. SOME CONSEQUENCES OF THE EILENBERG-STEENROD AXIOMS

5.1. The Uniqueness of Homology Theories. For a theory to be categorized as a homology theory, however it is constructed, it must satisfy the Eilenberg-Steenrod Axioms. Perhaps the greatest result of the axioms is that the values taken by a homology theory are completely determined by the axioms. Therefore, if one takes the homology of the same shape with multiple theories, the same result will be reached for each theory.

Therefore, once multiple theories are shown to satisfy the axioms, we can select the theory that will be easiest to work with at any given time, since the end result will be the same. So, although singular homology is the most flexible and can be applied to any space, we generally do not use it in calculations. In this case we might instead choose a more simplistic but more calculable theory, like poset homology. Conversely, the flexibility of singular homology theory makes it ideal for proving facts and theorems about homology theory in general, which can then be applied to other stricter and more practical theories.

Remark 5.1. Theorem 4.59 on page 399 of [1] shows that homology theories are isomorphic.

5.2. Homology for Homotopy Equivalent Spaces. Suppose we wish to calculate the homology of a space X . By the Weak Equivalence axiom, it is enough to compute the homology of any weakly equivalent space, and with luck, we may be able to find a simpler, weakly equivalent model to work with. To simplify matters, if two spaces are homotopy equivalent, then they are weakly homotopy equivalent.

Definition 5.2. Two maps $f : X \rightarrow Y$ and $g : X \rightarrow y$ are *homotopic* if there exists a map $h : X \times [0, 1] \rightarrow Y$ such that for all $x \in X$, $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$. This map h is referred to as a *homotopy*.

Remark 5.3. Regarding the map h in the previous definition, the second parameter in the domain acts as a slider between f and g , with one map transformed into the other as the second parameter changes.

Definition 5.4. Two spaces X and Y are *homotopy equivalent* if there are functions $f : X \rightarrow Y$ and $g : Y \rightarrow X$ such that $g \circ f$ is homotopic to the identity map on X and $f \circ g$ is homotopic to the identity map on Y .

Remark 5.5. Both homotopy and homotopy equivalence are equivalence relations.

Example 5.6. Consider the space $X = \mathbb{R}^2 \setminus (0,0)$. We will show that this space and the unit circle $Y = S^1$ are homotopy equivalent.

Define $f : X \rightarrow Y$ and $g : Y \rightarrow X$ by:

$$f(x) = \frac{x}{\|x\|},$$

$$g(x) = x.$$

Clearly, $g \circ f = g \left(\frac{x}{\|x\|} \right) = \frac{x}{\|x\|}$. Now, since the identity map on X is $id_X(x) = x$, a homotopy must be formed between $\frac{x}{\|x\|}$ and x . Define $h : X \times [0, 1] \rightarrow X$ such that $h(x, t) = (t \cdot x) + \left((1-t) \cdot \frac{x}{\|x\|} \right)$. It can be seen that h maps into X since any point $x \in X$ and its normalization are in a line leading away from the origin. Since h will map x to a point in the closure of the line segment between the two points, and the two points are on the same side of the origin, h will never map a point to the origin. So, $h(x, 0) = g \circ f$ and $h(x, 1) = id_X$. Similarly, $f \circ g = \frac{x}{\|x\|}$. Since the domain of g is the unit circle, the normalization defined by f will have no effect because the unit circle is already normalized. Therefore, $f \circ g = id_X$.

Therefore, S^1 and $\mathbb{R}^2 \setminus (0, 0)$ are homotopy equivalent. We have already calculated the homology groups of a poset model that is Weakly Equivalent to S^1 : $H_0(S^1) = \mathbb{Z}$, $H_1(S^1) = \mathbb{Z}$, and $H_n(S^1) = 0$ for $n > 1$. And, we just showed that S^1 is homotopy equivalent to $\mathbb{R}^2 \setminus (0, 0)$. Thus, by simply knowing the homology groups of the poset model of the circle, we are able to determine the homology groups of $\mathbb{R}^2 \setminus (0, 0)$ indirectly by showing that there exists a homotopy equivalence between the three spaces.

Remark 5.7. The previous example also makes logical sense with the construction of homology groups. Consider $H_1(S^1) = \ker \partial_1 / \text{im} \partial_2$. The quotient group keeps track of cycles that are not boundaries. Intuitively in $\mathbb{R}^2 \setminus (0, 0)$, a cycle that is not a boundary can be “filled in” without any issues. The hole in the center of the circle keeps it from being “filled in” in the same way as the hole punched through the origin does for $\mathbb{R}^2 \setminus (0, 0)$. Since any closed shape that does not include the origin can be filled in, and is thus part of the quotient, the only part left are the loops around the origin. This similarity between the circle and the rings around the origin that remain after the quotient shows intuitively why the homology is isomorphic.

Example 5.8. Consider \mathbb{R}^n . We will show that \mathbb{R}^n and the space $*$ consisting of a single point $p \in \mathbb{R}^n$ are homotopy equivalent. To do this, we must construct $f : \mathbb{R}^n \rightarrow *$ and $g : * \rightarrow \mathbb{R}^n$ such that $g \circ f$ is homotopic to the $id_{\mathbb{R}^n}$ and $f \circ g$ is homotopic to id_* .

Define functions $f : \mathbb{R}^n \rightarrow *$ and $g : * \rightarrow \mathbb{R}^n$ by $f(x) = p$ and $g(p) = p$ for all x . We can see that $(g \circ f)(x) = g(p) = p$. Let $h : X \times [0, 1] \rightarrow Y$ be the homotopy $h(x, t) = (t \cdot x) + ((1-t) \cdot p)$. Therefore, $h(x, 0) = g \circ f$ and $h(x, 1) = id_{\mathbb{R}^n}$. And, $f \circ g = p = id_*$.

Therefore, \mathbb{R}^n is homotopy equivalent to the space consisting of a single point. This is significant because we now know the homology of the plane \mathbb{R}^n without needing to compute anything – even the homology of a homotopy equivalent space – since the homology of a space consisting of a point is defined in the axioms.

Definition 5.9. A space X is *contractible* if it is homotopy equivalent to a point.

Remark 5.10. Note that any space that is contractible has the same homology as a single point, which is defined in the axioms.

Acknowledgments. I would like to thank my mentors Jonathan Rubin and Henry Chan for constantly clarifying and introducing me to so much new material throughout the program. I would also like to thank Peter May for organizing the REU and giving me this great experience and opportunity.

REFERENCES

- [1] A. Hatcher. *Algebraic Topology*. Cambridge University Press. 2001.
- [2] J. P. May. *A Concise Course in Algebraic Topology*. University of Chicago Press. 1999.
- [3] J. P. May. *Finite spaces and larger contexts*. (in revision/not yet published)