

TRANSPORT ON SMOOTH MANIFOLDS: FIBER BUNDLES, CONNECTIONS, AND COVARIANT DERIVATIVES

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ABSTRACT. In the study of smooth, real manifolds, the most powerful analytical tools available may only be applied locally. It is pertinent, therefore, to consider what global structures may reasonably be imposed on such a manifold, and what analytical tools are gained as a result. In particular, one may ask how local structures at one point of a manifold may be smoothly related to similar structures at a distant point—for example, how the tangent spaces to two distinct points in a manifold may be put in isomorphic correspondence. In this paper, we give an introduction to the study of this problem by developing the basics of the theory of Ehresmann connections on smooth manifolds. To do so, we develop the machinery of fiber bundles, discuss the fiber bundle structure of smooth manifolds, and apply these principles in order to provide a means of globally identifying local structures. Whenever possible, we employ methods that favor geometric intuition and visualization over formalism and computation.

CONTENTS

1. Introduction	1
2. Fiber Bundles – Motivation and Definitions	2
3. The Tangent Bundle of a Smooth Manifold	8
4. Parallel Transport and Covariant Derivatives	10
5. Connections on Fiber Bundles	12
Acknowledgments	18
References	18

1. INTRODUCTION

In \mathbb{R}^n , the principle of *translational invariance* provides a simple means to identify tangent vectors at different points. We may simply draw a line connecting the two points and slide a vector along the line while maintaining the relative angle between the vector and the line.

As far as topological spaces go, however, \mathbb{R}^n is unusually simple. The problem of transporting a tangent vector from one point to another, which in \mathbb{R}^n can be solved by elementary geometric arguments, is much more nuanced in more complicated topological spaces. In particular, we will investigate this problem on *smooth manifolds*, a type of topological structure that is *locally* similar to \mathbb{R}^n but lacks its global structure. In this investigation, we seek to answer the following question.

Question 1.1. *Can the fact that a manifold is locally similar to the real numbers be used to identify tangent vectors at nearby points? If so, can this local identification be extended to yield a broader, global identification?*

Any attempt to answer Question 1.1 will lead almost immediately to the notion of a *fiber bundle*. The fiber bundles encountered in the study of transport on smooth manifolds, however, are difficult to visualize. In order to avoid this difficulty, we begin our investigation in Section 2 with a general, visual introduction to the theory of fiber bundles. In Section 3, we review the basic principles of smooth manifolds and their relation to the theory of fiber bundles. In Section 4, we apply our knowledge of fiber bundles to give a provisional answer to Question 1.1. Finally, in Section 5, we discover that the concepts introduced in the theory of fiber bundles allow us to provide a more general, geometric answer to Question 1.1 that applies to a broader class of topological spaces.

Most of the definitions encountered in this paper are adapted from [1]-[5]. By convention, when we say “manifold” we always mean “smooth, real manifold.” For a topological space X , we denote by $\text{Aut}(X)$ the space of homeomorphisms from X into itself endowed with the compact-open topology.

While it is assumed that the reader has some familiarity with the basic structure of smooth manifolds, most definitions will be introduced as needed. Section 2 will be accessible to readers with a basic understanding of topology. Sections 3 and 4 will be accessible to readers who have at least a hazy notion of what a smooth manifold is, even if they lack precision. The core concepts of Section 5 should be accessible to any reader who was able to understand the previous sections, but some of the proofs involve slightly more advanced structures on smooth manifolds (for example, the differential of a smooth map). Many of the proofs may be safely skipped without losing the core content.

2. FIBER BUNDLES – MOTIVATION AND DEFINITIONS

The definition of a *fiber bundle*, which will be essential to our investigation of transport on smooth manifolds, is a bit obtuse at first glance. We will therefore proceed by giving an example of a fiber bundle, examining its properties, and constructing our definition to match.

Consider an ordinary open cylinder in \mathbb{R}^3 given by the product space $A = S^1 \times (-1, 1)$, pictured below.

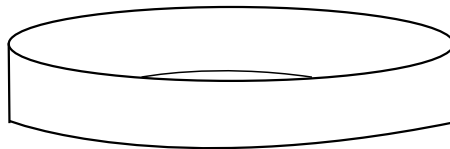
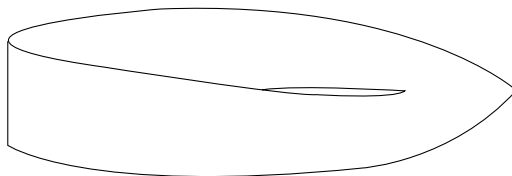


FIGURE 1. The space A , an open cylinder embedded in \mathbb{R}^3 .

The structure of this object is easily described via the product topology. Suppose, however, that we “cut” A crosswise (treating it like a paper model that can be physically deformed), “twist” the surface, and reattach the ends to form the familiar Möbius strip B , pictured below.

FIGURE 2. The space B , a Möbius strip embedded in \mathbb{R}^3 .

Locally, A and B look quite similar. If we cut out and “untwist” any section of B , it can be smoothly identified with a corresponding section of A . However, their global structure is quite different—for example, A is an orientable surface, while the familiar properties of the Möbius strip tell us that B has no well-defined orientation. So while the product topology is valid on any small region of B , it fails to encode the global structure of the space. This failure is precisely what leads to the definition of a *fiber bundle*, which provides a natural description for spaces that “locally look like” the product topology.

Definition 2.1. A *fiber bundle* is a quadruple (E, π, B, F) , where E, B , and F are topological spaces, and π is a map from E to B such that the following conditions hold:

- (1) π is continuous and surjective.
- (2) There exists an open cover of B , denoted $\{U_i\}$, such that for each U_i we may find a corresponding homeomorphism

$$\Phi_{U_i} : \pi^{-1}(U_i) \xrightarrow{\cong} U_i \times F$$

such that the following diagram commutes.

$$(2.2) \quad \begin{array}{ccc} \pi^{-1}(U_i) & \xrightarrow{\Phi_{U_i}} & U_i \times F \\ & \searrow \pi & \downarrow \text{proj}_{U_i} \\ & & U_i \end{array}$$

E is called the *total space*, B is called the *base space*, π is called the *projection map*, and F is called the *fiber*. Each pair (U_i, Φ_{U_i}) is called a *local trivialization*.

The main content of the definition lies in condition (2), which states that every point b in the base space is contained in an open set U for which the preimage under π has the product topology $U \times F$. In particular, the condition that the diagram (2.2) commutes demands that

$$(2.3) \quad \Phi_U^{-1}(b, f) \in \pi^{-1}(b)$$

holds and that Φ_U maps $\pi^{-1}(b)$ into $\{b\} \times F$, which in turn tells us that Φ_U restricts to a homeomorphism on $\pi^{-1}(b)$:

$$\Phi_U : \pi^{-1}(b) \xrightarrow{\cong} \{b\} \times F.$$

This observation constitutes a proof of the following proposition:

Proposition 2.4. *Let (E, π, B, F) be a fiber bundle. For any $b \in B$, we find*

$$\pi^{-1}(b) \simeq F,$$

where \simeq denotes the equivalence relation induced by homeomorphism.

This proposition is crucial to visualizing the structure of a fiber bundle. Since the preimage under π of any point $b \in B$ is topologically equivalent to the fiber F , we may think of a fiber bundle as the union of many copies of F which are “glued” onto each point of the base space B . The way in which the fibers are “glued” together is determined by the structure of the total space E and the projection map $\pi : E \rightarrow B$. A local trivialization gives a way of locally “unwinding” the fibers into the topologically-simpler product space.

Returning to our previous examples, the open cylinder $A = S^1 \times (-1, 1)$ and the Möbius strip B , we see that each can be constructed by “gluing” copies of the segment $(-1, 1)$ onto the points of the unit circle S^1 . While both A and B are fiber bundles with base space S^1 and fiber $(-1, 1)$, they have different topological structures which result from different choices of how the fibers are attached to the base space. In A , the fibers are all attached “parallel” to one another with the same orientation, while in B they are “twisted” around the base space (see Figure 3 below).

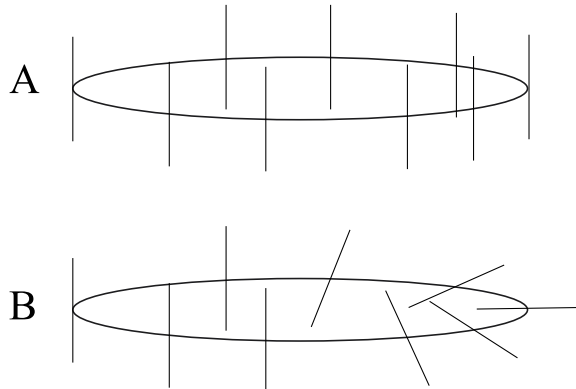


FIGURE 3. The spaces A and B constructed by “gluing” copies of $(-1, 1)$ onto the base S^1 .

Note that we have still not explicitly constructed the Möbius strip as a fiber bundle. This will come shortly once we have developed the machinery of *transition maps*. From geometric intuition, however, it is clear that the Möbius strip as it is seen in Figure 2 satisfies Definition 2.1.

We have one last thing to check about our definition before moving on. If fiber bundles are supposed to describe structures that are locally homeomorphic to product spaces, then all product spaces should certainly be fiber bundles.

Proposition 2.5. *The product space $B \times F$ is a fiber bundle with base space B , fiber F , total space $B \times F$, and projection map $\pi = \text{proj}_B : B \times F \rightarrow B$.*

Proof. B is an open cover of itself, and $\text{proj}_B^{-1}(B) = B \times F$. We may define the map $\Phi_B : \text{proj}_B^{-1}(B) \rightarrow B \times F$ to be the identity map on $B \times F$. It is straightforward

to verify that Φ_B satisfies the definition of a local trivialization. The space $(B \times F, \text{proj}_B, B, F)$ is therefore a fiber bundle with only one local trivialization. \square

The product space $B \times F$ is also some times called the *trivial bundle* over base B with fiber F .

There is one more core idea in the theory of fiber bundles we have yet to explore, which is the issue of compatibility between overlapping local trivializations. The definition of a fiber bundle guarantees the existence of an open cover $\{(U_i, \Phi_{U_i})\}$ of local trivializations, so it is natural to ask what happens in the case that two trivializations intersect—that is, when $U_i \cap U_j \neq \emptyset$.

Proposition 2.6. *Let (E, π, B, F) be a fiber bundle, and let $\{(U_i, \Phi_i)\}$ be a collection of local trivializations that cover B . If two local trivializations have a nontrivial intersection, that is if $U_i \cap U_j \neq \emptyset$, then the map $\Phi_i \circ \Phi_j^{-1}$ is given by*

$$\begin{aligned} \Phi_i \circ \Phi_j^{-1} : (U_i \cap U_j) \times F &\xrightarrow{\cong} (U_i \cap U_j) \times F \\ (p, f) &\mapsto (p, h_{ij}(p)(f)), \end{aligned}$$

where h_{ij} is a map from $U_i \cap U_j$ into $\text{Aut}(F)$ —that is, $h_{ij}(p)$ is a homeomorphism from F into itself for each fixed p .

Each map h_{ij} is called a transition map.

Proof. The only nontrivial statement in this proposition is that each map $\Phi_i \circ \Phi_j^{-1}$ fixes the first factor, which allows us to write the map as

$$\Phi_i \circ \Phi_j^{-1}(p, f) = (p, h_{ij}(p)(f)).$$

To see this, we apply the equality $\text{proj}_U = \pi \circ \Phi_U^{-1}$, which arises from the commuting diagram (2.2). We find

$$\begin{aligned} \text{proj}_{U_i} \circ \Phi_i \circ \Phi_j^{-1}(p, f) &= \pi \circ \Phi_j^{-1}(p, f) \\ &= \text{proj}_{U_j}(p, f) \\ &= p. \end{aligned}$$

This condition states that the composition of two local trivializations leaves the base point unchanged—that is, the orbits of transition maps are contained within individual fibers. \square

Corollary 2.7. *The following identities hold for all $p \in U_i \cap U_j \cap U_k$:*

$$(2.8) \quad h_{ii}(p) = id_F.$$

$$(2.9) \quad h_{ij}(p) \circ h_{ji}(p) = id_F.$$

$$(2.10) \quad h_{ij}(p) \circ h_{jk}(p) \circ h_{ki}(p) = id_F.$$

Remark 2.11. A transition map on a fiber bundle maps each fiber into itself in a manner that is consistent with local trivializations, much in the way that a transition map on an n -dimensional manifold maps \mathbb{R}^n into itself in a manner that is consistent with local coordinate systems. With this analogy in mind it may be instructive to think of each transition map h_{ij} as a “change of coordinates” on individual fibers.

We shall see shortly that transition maps give precise meaning to the hazy notion of “choosing a way to glue fibers onto the base space.” In fact, it is possible to uniquely determine any fiber bundle by specifying its base space, fiber, and transition maps.

Theorem 2.12 ([2, Thm. 3.2]). *Let B and F be topological spaces, and let $\{U_i\}$ be an open cover of B . Let $\{h_{ij}\}$ be a collection of continuous maps such that whenever $U_i \cap U_j \neq \emptyset$, we have $h_{ij} : U_i \cap U_j \rightarrow \text{Aut}(F)$. Additionally, let each map h_{ij} satisfy the properties stated in Corollary 2.7.*

Then there exists a unique fiber bundle (E, π, B, F) with open cover $\{U_i\}$ and transition maps h_{ij} .

Proof (sketch). The general idea of the proof is to construct a fiber bundle with B , F , and $\{h_{ij}\}$ as its base space, fiber, and transition maps, and then to show that any other fiber bundle satisfying the same conditions is equivalent up to isomorphism.

Since our total space should contain homeomorphic copies of each local trivialization $U_i \times F$, we begin by defining a space X which is the disjoint union of all such trivializations:

$$X = \coprod_i U_i \times F.$$

But we expect points in the total space to be identified by the specified transition maps. So we must induce a corresponding equivalence relation on X . We say that two elements $(i, p, f), (j, q, f') \in X$ are equivalent (denoted by the relation \sim) whenever the following relations hold:

$$p = q, \quad h_{ij}(p)(f') = f.$$

Put $E = X / \sim$. We define the projection map $\pi : E \rightarrow B$ by

$$\pi([i, p, f]) = p.$$

With some additional work, it is possible to show that there is a canonical set of local trivializations on E that satisfy the definition of a fiber bundle, and that any fiber bundle with the same base space, fiber, and transition maps must be isomorphic to (E, π, B, F) . The full proof is available in [2].

□

The above theorem gives us a new way of thinking about transition maps. Since a fiber bundle is uniquely specified by its base space, fiber, and transition maps, we may think of transition maps as a set of instructions for *how* the individual fibers should be attached to the base space.

Since all local trivializations of a trivial bundle may be taken as the identity map, all of the transition maps in a trivial bundle should be identity maps. In particular, all of the transition maps defining the bundle $A = S^1 \times (-1, 1)$ should map each point in the base space to the identity map on $(-1, 1)$. Likewise, any bundle with base space S^1 , fiber $(-1, 1)$, and constant transition maps given by the identity on F must be isomorphic to the trivial bundle A by Theorem 2.12. This gives rigorous meaning to the idea the fibers in A are attached to the base space “parallel to one another with the same orientation.”

Both the product topology $A = S^1 \times (-1, 1)$ and the Möbius strip B are bundles with base space S^1 and fiber $(-1, 1)$, so by Theorem 2 we should be able to describe the differences between the two entirely in terms of transition maps.

Example 2.13. In this example, we follow the approach of [1] to formally construct the Möbius strip.

Consider the open cover of S^1 given by

$$U_1 = \left(-\frac{3\pi}{4}, \frac{3\pi}{4} \right).$$

$$U_2 = \left(\frac{\pi}{4}, \frac{7\pi}{4} \right).$$

The intersection of these two sets is

$$U_1 \cap U_2 = \left(\frac{\pi}{4}, \frac{3\pi}{4} \right) \cup \left(\frac{5\pi}{4}, \frac{7\pi}{4} \right).$$

We want to consider fiber bundles with base space S^1 and fiber $(-1, 1)$ that are compatible with this open cover. Since our open cover only contains two sets, there is exactly one transition map we must specify—that is,

$$h_{12} : (U_1 \cap U_2) \rightarrow \text{Aut}((-1, 1)).$$

If we take $h_{12}(p)$ to be the identity for all $p \in S^1$, this uniquely determines the trivial fiber bundle $A = S^1 \times (-1, 1)$. By choosing our transition map differently, we should be able to construct the Möbius strip, as desired.

We think of the Möbius strip as being identical to the product space on one half of the circle, and “twisted” around by 180° on the other half of the circle. We can imitate this effect in our transition map h_{12} . We define $h_{12}(p) : (-1, 1) \rightarrow (-1, 1)$ as follows:

$$h_{12}(p)(f) = \begin{cases} f & \text{if } p \in \left(\frac{\pi}{4}, \frac{3\pi}{4} \right), \\ -f & \text{if } p \in \left(\frac{5\pi}{4}, \frac{7\pi}{4} \right), \end{cases}$$

The overlapping region $U_1 \cap U_2$ of the open cover has two connected components. By letting the transition map be “orientation preserving” on one component (by taking each point in the base space to the identity map on the fiber) and “orientation reversing” on the other component (by taking each point in the base space to the additive inverse map on the fiber), we have defined a transition map that imitates the familiar properties of the Möbius strip. Since specifying the transition map h_{12} uniquely determines a fiber bundle, the above specification of h_{12} constitutes an explicit construction of the Möbius strip.

For the remainder of this paper, we shall be concerned only with *smooth* fiber bundles, where E , B , and F are smooth manifolds, π is a smooth map, and the local trivializations Φ_U are C^∞ diffeomorphisms rather than homeomorphisms. Most properties we have derived may be extended with “continuous” replaced with “smooth” and “homeomorphism” replaced with “ C^∞ diffeomorphism”. We shall also occasionally be concerned with *vector bundles*, where each fiber is a vector space and the transition maps are linear.

3. THE TANGENT BUNDLE OF A SMOOTH MANIFOLD

We may now discuss the applications of the theory of fiber bundles in the study of smooth manifolds. In particular, we shall see that *the set of all tangent spaces on a smooth manifold has a natural fiber bundle structure*. We shall prove this fact shortly, but first we review some basic concepts regarding tangent spaces on manifolds. It is assumed that the reader has some experience working with smooth manifolds, so our discussion will be brief.

Definition 3.1. An n -dimensional manifold is a (second countable, Hausdorff) topological space M together with an open cover $\{U_i\}$ and a collection of maps $\psi_i : U_i \rightarrow \mathbb{R}^n$ such that each ψ_i is a homeomorphism onto its image.

Each pair (U_i, ψ_i) is called a *chart*.

The manifold M is *smooth* if whenever two sets in the open cover have non-trivial intersection, i.e. when $U_i \cap U_j \neq \emptyset$, the *transition map* $\psi_i \circ \psi_j^{-1}$ is a C^∞ diffeomorphism onto its image.

Remark 3.2. The collection of charts $\{(U_i, \psi_i)\}$ is called an *atlas* for the manifold M . The definition of a manifold is often given in terms of equivalence classes of atlases, while our definition depends on a specific choice of atlas. In practice, we will find that Definition 3.1 is sufficient for our investigation.

The existence of a smooth atlas on a manifold allows us to use powerful tools of real analysis in studying the manifold's structure. In particular, it allows us to define what it means for a function on the manifold to be smooth.

Definition 3.3. Let M be a smooth, n -dimensional manifold. For any chart (U_i, ψ_i) , let $W_i \subset \mathbb{R}^n$ be the image of ψ_i .

A map $f : M \rightarrow \mathbb{R}$ is called *smooth* if the map $f \circ \psi_i^{-1} : W_i \rightarrow \mathbb{R}$ is smooth for each chart (U_i, ψ_i) .

We denote the set of all such smooth maps by $C^\infty(M)$.

In order to discuss the tangent bundle of a smooth manifold, of course, we must define the notion of a tangent *vector*. There are many equivalent ways of defining tangent vectors on manifolds. Following the approach of [5], we choose to view tangent vectors as a means of taking directional derivatives of smooth, real functions (in the sense of Definition 3.3). This view of a tangent vector is sometimes called a *derivation*.

Definition 3.4. Let M be a smooth manifold, and fix a point $p \in M$.

A *tangent vector* at the point p is a map $v : C^\infty(M) \rightarrow \mathbb{R}$ that is *linear* and satisfies the *Leibniz rule*. That is, for any $f, g \in C^\infty(M)$ and $\alpha, \beta \in \mathbb{R}$, we have

$$(3.5) \quad v(\alpha f + \beta g) = \alpha v(f) + \beta v(g).$$

$$(3.6) \quad v(fg) = v(f) \cdot g(p) + f(p) \cdot v(g).$$

The following proposition, which is straightforward to prove, justifies the use of the term “tangent vector”.

Proposition 3.7. *Let M be a smooth manifold. The set of all tangent vectors at a point $p \in M$ is a real vector space (denoted $T_p M$).*

Remark 3.8. We have defined tangent spaces only with respect to individual points on a manifold. We will see in sections 5 and 6 that the problem of identifying tangent spaces at distinct points has an elegant solution in the language of fiber bundles.

Even though tangent spaces at different points of a smooth manifold must be treated as different topological objects, they do share some global structure. We find that the tangent space to any point in an n -dimensional smooth manifold *is itself an n -dimensional vector space*. This is shown in the following theorem, which explicitly constructs a basis for the tangent space at any given point.

Theorem 3.9. *Let M be a smooth n -dimensional manifold, and fix $p \in M$ with (U, ψ) a chart covering p (i.e. $p \in U$). Put $W = \text{Im}(\psi) \subset \mathbb{R}^n$.*

For $1 \leq \mu \leq n$, let $X_\mu : C^\infty(M) \rightarrow \mathbb{R}$ be the μ^{th} partial derivative of $f \circ \psi^{-1} : W \rightarrow \mathbb{R}$:

$$X_\mu(f) = \left. \frac{\partial}{\partial x^\mu} (f \circ \psi^{-1}) \right|_{\psi(p)}.$$

Then each differential operator X_μ is a tangent vector in $T_p M$, and the set

$$\{X_1, \dots, X_n\}$$

forms a basis for the tangent space $T_p M$.

Proof. The fact that each X_μ satisfies the linearity and Leibniz conditions at p (and is thus a tangent vector in $T_p M$) is obvious from the corresponding properties of the partial derivative operator.

As for the fact that $\{X_1, \dots, X_n\}$ forms a basis for $T_p M$, we omit the proof, which may be found in [5]. \square

Remark 3.10. It is common to abuse notation by referring to the vector X_μ with the corresponding partial derivative operator $\partial/\partial x^\mu$. We will frequently denote the basis for a tangent space as

$$\left\{ \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n} \right\}$$

Remark 3.11. The existence of this basis allows us to construct a linear diffeomorphism from any tangent space into the real vector space \mathbb{R}^n . That is, if

$$v = v^1 \frac{\partial}{\partial x^1} + \dots + v^n \frac{\partial}{\partial x^n}$$

is a vector in $T_p M$, we may map it to the n -tuple $(v^1, \dots, v^n) \in \mathbb{R}^n$. The tangent space to each point on an n -dimensional manifold is therefore *diffeomorphic* to \mathbb{R}^n . Of course, this diffeomorphism is not canonical, as it depends on a choice of basis and therefore on a choice of coordinate chart.

This last remark is quite suggestive. For an n -dimensional manifold M , each point has an associated tangent space which is diffeomorphic to \mathbb{R}^n . We shall see, of course, that the disjoint union of all tangent spaces on an n -dimensional manifold M forms a fiber bundle with M as its base space and \mathbb{R}^n as its fiber.

Example 3.12 (Tangent Bundle of a Smooth Manifold). Note: This construction is taken mostly from [1, Sec 6.6.2]

We construct the tangent bundle using the smooth, vector bundle analogue of Theorem 2.12. If we specify a base space, a fiber, and a collection of linear, smooth transition maps, then we have uniquely specified a smooth vector bundle.

Let M be an n -dimensional manifold. As previously argued, the tangent bundle TM should have base space M and fiber \mathbb{R}^n . In order to uniquely construct a fiber bundle with these properties, we must specify an open cover of the base space and a set of transition maps. M is already naturally endowed with a covering collection of charts (U_i, ϕ_i) , so we take our open cover to be $\{U_i\}$.

For any sets U_i, U_j in the open cover such that $U_i \cap U_j \neq \emptyset$, the composite map $\phi_i \circ \phi_j^{-1}$ is a smooth bijection between two open subsets of \mathbb{R}^n . The *differential* of this map naturally provides the transition map we are looking for. For any fixed $p \in U_i \cap U_j$, the map $d(\phi_i \circ \phi_j^{-1})|_{\phi_j(p)}$ is a smooth, linear map from \mathbb{R}^n into itself. So if we define our transition maps $h_{ij} : U_i \cap U_j \rightarrow \text{Aut}(\mathbb{R}^n)$ by

$$h_{ij}(p)(x) = d(\phi_i \circ \phi_j^{-1})|_{\phi_j(p)}(x),$$

then all the desired properties are satisfied. We call the fiber bundle with base M , fiber \mathbb{R}^n , and transition maps h_{ij} the *tangent bundle* of M , denoted TM .

The fiber bundle structure of TM comes from the transition maps, $d(\phi_i \circ \phi_j^{-1})$, which are chosen to be consistent with the manifold structure given by the charts (U_i, ϕ_i) . In other words, nearby tangent spaces in TM are associated according to the same rules that are used to associate nearby points in M . In this sense, the fiber bundle TM encodes all of the smooth structure of a manifold together with its tangent spaces, and rightfully deserves to be called the “tangent bundle.”

All the intuition we built in our discussion of fiber bundles may now be applied to the study of tangent spaces on a smooth manifold. This is the subject of the next section.

4. PARALLEL TRANSPORT AND COVARIANT DERIVATIVES

We have already mentioned that there is no *canonical* way of identifying tangent spaces at different points of a smooth manifold, but we may still ask what general identification methods are available. Certainly any such identification should be path-dependent, since the analytic properties of manifolds are only defined locally. This process is referred to as *parallel transport*—how may we transport a vector along a curve to move it from one tangent space into another?

One answer to this question comes in the form of an operator called a *covariant derivative*, which provides a means of differentiating a vector field along a curve. It is then possible to determine how a vector should be transported along a curve by requiring that its covariant derivative is zero everywhere. For the remainder of this section, we will explore the properties of this operator.

It is worth noting, before we begin, that all of the definitions and theorems in this section hold for arbitrary vector bundles on smooth manifolds (in particular, it is common to consider covariant derivatives on tensor bundles). For the purposes of this discussion, however, we shall restrict our attention to the tangent bundle.

Since taking the derivative of a function normally requires the function to be defined in a neighborhood of the point at which we want to evaluate the derivative,

it hardly makes sense to talk about taking the derivative of a single vector with respect to another vector. Instead, we will need to define the notion of a *vector field* in the tangent bundle, and decide what it means for such a vector field to be differentiable. This has a simple formulation in the language of fiber bundles, called a *section* or *cross section*.

Definition 4.1. A (smooth) *section* on a smooth fiber bundle (E, π, B, F) is a (smooth) map $\sigma : B \rightarrow E$ such that

$$\sigma(p) \in \pi^{-1}(p)$$

holds for all $p \in B$. That is, σ maps each point in the base space into the fiber connected to that point. The set of all smooth sections on E is denoted $\Gamma(E)$.

A (smooth) section on the tangent bundle TM of a manifold M is called a (*smooth*) *vector field* on M .

A section of the tangent bundle maps each point in the base manifold to a vector in its tangent space, which is precisely what we mean by a *vector field*. In many discussions of tangent fields (e.g. [1]) it is common to call a vector field *smooth* if its action on any arbitrary smooth map $f : M \rightarrow \mathbb{R}$ yields another smooth map. The language of fiber bundles has allowed us to completely bypass this rather obtuse definition—we may simply call a vector field smooth if the section is a smooth map with respect to the topology of the tangent bundle.

We now ask what it means to differentiate a smooth vector field (or really any smooth section of a vector bundle) with respect to another tangent vector field on the manifold—that is, how to take the directional derivative. This gives rise to the definition of a *covariant derivative*.

Definition 4.2. A *covariant derivative* on a smooth vector bundle (E, π, M, F) is an operator $\nabla : \Gamma(TM) \times \Gamma(E) \rightarrow \Gamma(E)$ that takes in a smooth tangent vector field X and a smooth vector bundle section σ and yields another smooth vector bundle section $\nabla_X \sigma$. The covariant derivative is required to satisfy the following properties:

- (1) ∇ must satisfy the *Leibniz rule* (or product rule) with respect to multiplication of the section by smooth functions. If $f \in C^\infty(M)$, then we must have

$$\nabla_X(f\sigma) = X(f)\sigma + f\nabla_X\sigma.$$

- (2) ∇ must be $C^\infty(M)$ -linear in the tangent vector field input. That is, for $X_1, X_2 \in \Gamma(TM)$ and $f, g \in C^\infty(M)$, we have

$$\nabla_{fX_1+gX_2}\sigma = f\nabla_{X_1}\sigma + g\nabla_{X_2}\sigma.$$

- (3) ∇ must be additive in the smooth section input. That is, for $\sigma_1, \sigma_2 \in \Gamma(E)$, we have

$$\nabla_X(\sigma_1 + \sigma_2) = \nabla_X\sigma_1 + \nabla_X\sigma_2.$$

The above definition constitutes the loosest set of assumptions we may place on the operator ∇ while still retaining the core idea of what it means to “take a derivative”. This definition is far from unique—there are infinitely many operators that satisfy Definition 4.2, even when constrained to the particular case of a tangent bundle. The specification of a covariant derivative on a smooth vector bundle is also sometimes called a *Koszul Connection* (see e.g. [1]).

We mentioned before that specifying a covariant derivative on the tangent bundle of a smooth manifold allows us to identify distinct tangent spaces by prescribing how a tangent vector should be smoothly transported along a curve. This is made precise in the following definition.

Definition 4.3. Let M be a smooth, n -dimensional manifold, and let $\sigma \in \Gamma(TM)$ be a smooth vector field on M . Let ∇ be a covariant derivative on TM .

Suppose $\gamma : (0, 1) \rightarrow M$ is a smooth curve in M . At each point t in the domain of γ , we may take the *tangent vector to the curve*, $X(t)$, to be the vector given by the following map for any $f \in C^\infty(M)$:

$$X(t)(f) = \left. \frac{d}{ds}(f \circ \gamma(s)) \right|_{s=t}.$$

The vector field $\sigma \in \Gamma(TM)$ is said to be *parallel transported* by the curve γ if the covariant derivative of σ with respect to the tangent vectors of γ vanishes on the image of γ —that is, if

$$\nabla_{X(t)}\sigma|_{\gamma(t)} = 0$$

holds for all t in the domain of γ .

This definition is extremely powerful. Given a curve γ that connects two points p and q in a manifold M endowed with a covariant derivative, we may identify the tangent spaces T_pM and T_qM by specifying vector fields that are parallel-transported by the curve. Once such a vector field is specified, mapping its value at the point p to its value at the point q provides a natural map from T_pM into T_qM . The vanishing covariant derivative corresponds to the idea that the vector “remains constant” as it is parallel-transported along the curve.

The consequences of specifying a covariant derivative are fascinating and yield completely new subjects of study—choosing a specific type of covariant derivative called the *Levi-Civita connection*, for example, is the first fundamental step in studying Riemannian and Lorentzian geometry.

However, our emphasis in this paper is on building geometric ideas and intuition. Rather than proceeding with the machinery of covariant derivatives, we will take a step back into the territory of smooth fiber bundles in order to develop a broader idea of what it means to identify distant fibers.

5. CONNECTIONS ON FIBER BUNDLES

Covariant derivatives are a powerful computational tool, and see a great deal of use in subjects in physics such as general relativity, but the formulation of transport in terms of covariant derivatives has two main failings. The first failing is that the definition of parallel transport in terms of covariant derivatives lacks a clear geometric picture, and the second is that it is only valid for vector bundles. We

may wish to discuss what happens when we try to transport an element of any fiber in an arbitrary fiber bundle along a curve in its base space. Answering this question will require the notion of an *Ehresmann connection*, which generalizes the idea of a covariant derivative.

Given a smooth fiber bundle (E, π, M, F) , suppose we wish to transport an element in one fiber along a curve in the base space M . That is, if we have a curve $\gamma : [0, 1] \rightarrow M$ satisfying

$$\begin{aligned}\gamma(0) &= p, \\ \gamma(1) &= q,\end{aligned}$$

then we wish to find a way to map elements of $\pi^{-1}(p)$ into elements of $\pi^{-1}(q)$ in a manner that is consistent with the curve γ . This can be accomplished in geometric terms by “lifting” the curve γ , which is a curve in M , to a corresponding curve $\gamma^\#$ in the total space E . When we say that $\gamma^\# : [0, 1] \rightarrow E$ is a “lift” of the curve $\gamma : [0, 1] \rightarrow M$, what we mean is that each point in the image of $\gamma^\#$ lies in the fiber of the corresponding point in the image of γ . In other words, $\gamma^\#$ satisfies

$$\pi \circ \gamma^\#(t) = \gamma(t)$$

for all $t \in [0, 1]$.

The lift $\gamma^\#$ is a smooth curve in the total space E that projects to the guiding curve γ , so $\gamma^\#$ naturally carries the point $\gamma^\#(0)$ in the fiber $\pi^{-1}(\gamma(0))$ to another point $\gamma^\#(1)$ in the fiber $\pi^{-1}(\gamma(1))$. This gives geometric meaning to the idea of identifying points in distant fibers—we may simply take a curve connecting their base points, lift it into the total space, and connect the endpoints of the lifted curve.

Specifying a unique way of lifting base curves into the total space given an initial point will ultimately be called “choosing an *Ehresmann connection*” on the fiber bundle. Since a lifted curve must travel through a set of fibers that is predetermined by the base curve, choosing a connection intuitively amounts to specifying exactly how a curve in the total space should travel through “nearby” fibers.

In order to give this idea precise meaning, we first define the *vertical subspace* to a point in the total space, which corresponds to the set of all vectors in the tangent space that are simultaneously tangent to the fiber passing through the given point.

Definition 5.1. Let (E, π, M, F) be a smooth fiber bundle. Given a point p in the total space E , the differential of the projection map, $d\pi_p$, provides a linear map from $T_p E$ into $T_{\pi(p)} M$.

For any point $p \in E$, the *vertical subspace* of $T_p E$ is given by the kernel of $d\pi_p$. In other terms, the vertical subspace is the set

$$\{v \in T_p E \mid d\pi_p(v) = 0\}.$$

Definition 5.2. Let (E, π, M, F) be a smooth fiber bundle. The subspace of the tangent bundle TE formed by taking the vertical subspace of $T_p E$ for each point $p \in E$ is called the *vertical bundle* of the fiber bundle.

The vertical subspace at a given point consists of vectors that may be thought of as being simultaneously tangent to the total space E and to the fiber F . We shall make this idea precise in Proposition 5.3, but it certainly makes intuitive sense.

The map π carries points in the fiber to points in the base space, so it should carry any vectors that are “tangent to the fiber” to the zero vector in the tangent space of the base point.

The term *vertical* comes from the idea that if we visualize the fibers F as sticking straight up out of some base space M , the vectors that are tangent to F look “vertical.”

In the following proposition, we show that the vertical subspace at any point has the same dimension as the tangent space to the fiber at that point. This fact, combined with the ideas discussed above, provides justification for treating the vertical bundle as the subset of the tangent bundle TE which is simultaneously tangent to the fibers.

Proposition 5.3. *Let (E, π, M, F) be a smooth fiber bundle such that $\dim(M) = n$ and $\dim(F) = k$.*

For any $p \in E$, the vertical subspace of $T_p E$ has dimension k .

Proof. Note first that since E locally looks like the product space $M \times F$, we must have $\dim(E) = n + k$. The proof follows by applying the rank-nullity theorem.

Fix $p \in E$. We aim to show that the kernel of $d\pi_p : T_p E \rightarrow T_{\pi(p)} M$ has dimension k . Since $\dim(E) = n + k$, this is equivalent to showing that the image of $d\pi_p$ has dimension n .

Since M is n -dimensional, this amounts to showing that the map $d\pi_p : T_p E \rightarrow T_{\pi(p)} M$ is surjective. This will turn out to follow naturally from the fact that $\pi : E \rightarrow M$ is surjective.

Fix $Y \in T_{\pi(p)} M$. Since π is surjective, it has a right inverse. That is, there exists a function $\pi^r : M \rightarrow E$ such that $\pi \circ \pi^r = id_M$. We define a vector $X \in T_p E$ given by the map

$$X(g) = Y(g \circ \pi^r)$$

for any smooth map $g : E \rightarrow \mathbb{R}$.

For any smooth map $f : M \rightarrow \mathbb{R}$, we have

$$\begin{aligned} d\pi_p(X)(f) &= X(f \circ \pi) \\ &= Y(f \circ \pi \circ \pi^r) \\ &= Y(f), \end{aligned}$$

so $d\pi_p(X) = Y$. Since Y was arbitrary, it follows that $d\pi_p$ is surjective, and thus $\dim(\ker(d\pi_p)) = k$, as desired. \square

The natural next step is to ask what might constitute a *horizontal subspace*. The presence of the projection map π gave us a natural way to define the vertical subspace, but there is no such canonical way of defining a horizontal subspace. This ambiguity is what will ultimately give us the notion of an Ehresmann connection. For this reason, we will define a *horizontal subspace* to be any dimension- n subspace which is linearly independent from the vertical subspace.

Definition 5.4. Let (E, π, M, F) be a smooth fiber bundle with vertical bundle V . Fix a point p in the total space E , and denote by V_p the vertical subspace at p .

A *horizontal subspace* at p is a subspace $H_p \subset T_p E$ satisfying the following conditions:

(1)

$$H_p \cap V_p = \{0\}.$$

(2)

$$H_p \oplus V_p = T_p E.$$

That is, the horizontal subspace must intersect the vertical subspace only at the zero vector, and the horizontal and vertical subspaces must together contain a full basis for the tangent space $T_p E$.

A horizontal subspace can essentially be understood as a preferred set of n linearly-independent vectors in the tangent space of a point that are not simultaneously tangent to the fiber containing that point.

At last we may return to the question asked at the beginning of this section with regards to lifting a curve in the base space to a curve in the total space. Since we have found a way to specify a preferred subspace of the tangent space that is “orthogonal” to the fiber at any given point, we may specify a lift into the total space by demanding that the tangent vectors to the lifted curve should lie in that preferred subspace. We shall prove the uniqueness of this lift shortly, but we must first formally define an *Ehresmann connection*.

Definition 5.5. Let (E, π, M, F) be a smooth fiber bundle with vertical bundle V .

If a collection of horizontal subspaces $H_p \subset \pi^{-1}(p)$ at each point $p \in E$ has smooth fiber bundle structure, then their union H is called a *horizontal bundle*.

A specified choice of horizontal bundle H such that $TE = V \oplus H$ is called an *Ehresmann connection* on the fiber bundle.

With this machinery in mind, we may prescribe a means of lifting curves from the base space into the total space, as desired.

Theorem 5.6. Let (E, π, M, F) be a smooth fiber bundle with an Ehresmann connection given by $TE = V \oplus H$.

Let $\gamma : [0, 1] \rightarrow M$ be a smooth curve in the base space with endpoints given by $\gamma(0) = p$ and $\gamma(1) = q$.

Given a point $f \in \pi^{-1}(p)$, there exists a unique smooth curve $\gamma^\# : [0, 1] \rightarrow E$ in the total space satisfying the following properties.

(1)

$$\gamma^\#(0) = f.$$

(2)

The tangent vector to $\gamma^\#$ at any point along its image lies in the horizontal subspace at that point. That is, for $t \in [0, 1]$, we find

$$\dot{\gamma}^\#(t) \in H_{\gamma^\#(t)},$$

where $\dot{\gamma}^\#$ is the tangent vector to $\gamma^\#$.

(3)

$$\pi \circ \gamma^\#(t) = \gamma(t).$$

The curve $\gamma^\#$ is called the *lift* of γ with respect to the given Ehresmann connection.

Proof (sketch). The cleanest expression of this proof requires the definition of a *connection 1-form*, which is a vector-valued 1-form compatible with the specified Ehresmann connection. This methodology is useful for computation, but not particularly instructive. Therefore, we refer the reader to [3, Thm 10.2] for a complete proof, and sketch here an alternate proof that is significantly more instructive, albeit computationally messier.

The proof breaks down into two simple parts. We show that

- (i) There is a unique field of tangent vectors to the total space that is compatible with conditions (2) and (3),
- (ii) There exists a unique curve $\gamma^\#$ in the total space which has tangent vectors given by part (i) and initial value compatible with condition (1).

We shall completely prove part (i) and sketch a proof of part (ii).

- (i) Condition (3) tells us that every point in the lifted curve is taken to the corresponding point in the base by the projection map π . Moreover, we shall show that if $\gamma^\#$ exists, then its *tangent vector* at each point of the total space is taken to the corresponding tangent vector of the base curve in the base space by the differential of the projection map $d\pi$.

To see this, fix an element t in the domain of γ and a point p in the corresponding fiber $\pi^{-1}(\gamma(t))$ —that is, fix a point p in the total space such that it is possible for the lifted curve $\gamma^\#$ to pass through p .

Suppose $\gamma^\#$ exists and $\gamma^\#(t) = p$. Let $\dot{\gamma}^\#(t)$ be the tangent vector to $\gamma^\#$ at p . Formally, if $f \in C^\infty(E)$, then

$$\dot{\gamma}^\#(t)(f) = \left. \frac{d}{ds}(f \circ \gamma^\#(s)) \right|_{s=t}.$$

Now we consider the image of $\dot{\gamma}^\#(t)$ under the map $d\pi_p$. For $g \in C^\infty(M)$, we have

$$\begin{aligned} d\pi_p(\dot{\gamma}^\#(t))(g) &= \dot{\gamma}^\#(t)(g \circ \pi) \\ &= \left. \frac{d}{ds}(g \circ \pi \circ \gamma^\#(s)) \right|_{s=t}. \end{aligned}$$

Since $\pi \circ \gamma^\# = \gamma$ by condition (3), the above equation tells us that

$$\begin{aligned} d\pi_p(\dot{\gamma}^\#(t))(g) &= \left. \frac{d}{ds}(g \circ \gamma(s)) \right|_{s=t} \\ &= \dot{\gamma}(t)(g). \end{aligned}$$

In other words, the map $d\pi$ maps the tangent vector to $\gamma^\#$ into the corresponding tangent vector of γ at the same value of t , as desired.

This fact, combined with condition (2), is enough to uniquely specify the tangent vector to $\gamma^\#$ at each point along its trajectory. To see this, fix a point $p = \gamma^\#(t)$ along the trajectory of $\gamma^\#$ (again, assuming it exists).

From our Ehresmann connection, we have $T_p E = V_p \oplus H_p$. Since $d\pi_p$ is a linear map from $T_p E$ into $T_{\pi(p)} M$, it restricts to a linear map from H_p into $T_{\pi(p)} M$. Moreover, since V_p is the kernel of $d\pi_p$, it follows that the restriction of $d\pi_p$ to the domain H_p is still a surjective map.

Since H_p and $T_{\pi(p)}M$ have the same dimension as vector spaces, it follows that $d\pi_p$ is a bijection from H_p onto $T_{\pi(p)}M$. That is, for any tangent vector to the point $\pi(p)$ in M , there exists a unique vector in H_p which is mapped to the given tangent vector under $d\pi_p$.

In particular, since the curve γ is specified, and since $p = \gamma^\#(t)$, there exists a unique vector X in H_p such that $d\pi_p(X) = \dot{\gamma}(t)$. Since we showed previously that this condition must be satisfied by the tangent vector to $\gamma^\#$ if it exists, we conclude that if $\gamma^\#$ exists, there is only one possible choice for its tangent vector at each point. That is, if $\gamma^\#$ exists, its tangent field is uniquely specified by the curve γ .

- (ii) Due to the coordinate properties of manifolds, the condition imposed in part (i)—that the tangent vectors to the curve $\gamma^\#$ match elements of a given tangent field—reduces to an ordinary differential equation in any given chart. Let us take a coordinate chart containing the point f , which was specified to be the initial value of our curve $\gamma^\#$. It is possible to show, using the existence and uniqueness theorems of ordinary differential equations, that there exists a unique solution for the curve $\gamma^\#$ in this chart with initial value $\gamma^\#$. It is then possible to take another coordinate chart intersecting the first one and use the initial value given by the first solution to “continue” the lifted curve $\gamma^\#$.

Using the fact that the image of the curve γ is compact in M , it is possible to show that one may reach the fiber of the second endpoint $\gamma(1)$ by repeating the above procedure finitely many times. This gives a global, unique solution for the lifted curve $\gamma^\#$ satisfying conditions (1), (2), and (3), constructed from local solutions found using the fundamental theorem of ODEs.

□

Finally, we have found a way to identify points of distant fibers in a fiber bundle—we specify a curve connecting the two base points, specify an Ehresmann connection, use the connection to lift the base-space curve to a curve in the total space, and identify the two endpoints of the lifted curve. In particular, this may be used as in Section 4 to explain how a tangent vector (or differential form, or tensor) should be transported along a given curve.

By proving this theorem, we have concluded the main points of this paper. In Section 2, we gave precision to the notion of smoothly attaching copies of a given topological space to points on another topological space by introducing the machinery of *fiber bundles*. In Section 3, we expanded this machinery to the discussion of smooth manifolds, showing that the tangent bundle of a manifold is a smooth vector bundle. In Section 4 we used the tangent bundle as a concrete example to discuss the general theory of connections on vector bundles, introducing a tool called a *covariant derivative* to globally identify distant fibers in a smooth vector bundle. Finally, in Section 5, we generalized this notion to arbitrary fiber bundles with less structure than a vector bundle, introducing the geometric idea of an *Ehresmann connection*, which allows the identification of distant fibers in a smooth bundle by providing a unique way to lift arbitrary curves in a base manifold to curves in the total space that pass through the fibers.

Remark 5.7. The theory of Ehresmann connections has a host of fascinating applications. In particular, the reading that resulted in this paper was motivated by the study of gauge theories in theoretical physics.

Yang-Mills theory, for example, which provides a means for unifying the electromagnetic, weak nuclear, and strong nuclear forces, may be expressed in terms of connections on *principal G -bundles*, a kind of fiber bundle where the fibers are diffeomorphic to a Lie group that acts on the total space. Amazingly, many of the familiar properties of fundamental physical forces may be derived by demanding coordinate invariance (and *gauge invariance* between overlapping local trivializations) on a principal G -bundle where the Lie group G encodes the symmetries of the fundamental particles being studied.

For more reading on this topic, see [1] and [3].

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