

BARYCENTRIC SUBDIVISION AND ISOMORPHISMS OF GROUPOIDS

JASHA SOMMER-SIMPSON

ABSTRACT. Given groupoids \mathcal{G} and \mathcal{H} as well as an isomorphism $\Psi : \text{Sd } \mathcal{G} \cong \text{Sd } \mathcal{H}$ between subdivisions, we construct an isomorphism $P : \mathcal{G} \cong \mathcal{H}$. If Ψ equals $\text{Sd}F$ for some functor F , then the constructed isomorphism P is equal to F . It follows that the restriction of Sd to the category of groupoids is conservative. These results do not hold for arbitrary categories.

CONTENTS

1.	Introduction	1
2.	Notation and overview of proof	2
3.	Construction of $\text{Sd } \mathcal{C}$	4
4.	Sd preserves coproducts and identifies opposite categories	8
5.	$\text{Sd } \mathcal{C}$ encodes objects and arrows of \mathcal{C}	10
6.	$\text{Sd } \mathcal{G}$ encodes composition in \mathcal{G} up to opposites	17
7.	Construction of ψ and proof of the main theorem	27
	References	40
	Appendix A. Proof of Propositions 6.1.19 and 6.1.20	41

1. INTRODUCTION

The categorical subdivision functor $\text{Sd} : \mathbf{Cat} \rightarrow \mathbf{Cat}$ is defined as the composite $\Pi \circ \text{Sd}^s \circ N$ of the nerve functor $N : \mathbf{Cat} \rightarrow \mathbf{sSet}$, the simplicial barycentric subdivision functor $\text{Sd}^s : \mathbf{sSet} \rightarrow \mathbf{sSet}$, and the fundamental category functor $\Pi : \mathbf{sSet} \rightarrow \mathbf{Cat}$ (left adjoint to N). The categorical subdivision functor Sd has deficiencies: for example, it is neither a left nor a right adjoint. Nevertheless, Sd bears similarities to its simplicial analog Sd^s , and has many interesting properties. For example, it is known that the second subdivision $\text{Sd}^2 \mathcal{C}$ of any small category \mathcal{C} is a poset [4, ch. 13].

In general, there exist small categories \mathcal{B} and \mathcal{C} such that $\text{Sd } \mathcal{B}$ is isomorphic to $\text{Sd } \mathcal{C}$ but \mathcal{B} is not isomorphic to \mathcal{C} or to \mathcal{C}^{op} . In this paper we show that such examples do not occur in the category of small groupoids: if $\text{Sd } \mathcal{G}$ is isomorphic to $\text{Sd } \mathcal{H}$ for groupoids \mathcal{G} and \mathcal{H} , then there exists an isomorphism between \mathcal{G} and \mathcal{H} .

In broad strokes, the argument is as follows. There is a canonical identification of objects in $\text{Sd } \mathcal{G}$ with non-degenerate simplices in the nerve $N\mathcal{G}$. Any isomorphism between $\text{Sd } \mathcal{G}$ and $\text{Sd } \mathcal{H}$ induces a bijection between the 0-simplices of $N\mathcal{G}$ and the 0-simplices of $N\mathcal{H}$, and similarly for 1-simplices. The objects and morphisms

of a category correspond (respectively) to the 0-simplices and 1-simplices in the nerve of that category, so for any isomorphism $\Psi : \text{Sd } \mathcal{G} \rightarrow \text{Sd } \mathcal{H}$ there are induced bijections $\text{Ob}(\psi) : \text{Ob}(\mathcal{G}) \rightarrow \text{Ob}(\mathcal{H})$ and $\text{Mor}(\psi) : \text{Mor}(\mathcal{G}) \rightarrow \text{Mor}(\mathcal{H})$. We will show that the restriction of these bijections to any connected component of \mathcal{G} determines a possibly-contravariant functor into \mathcal{H} . Proceeding one component at a time and using the fact that any groupoid is isomorphic to its opposite, we can use these maps ψ to construct a covariant isomorphism between \mathcal{G} and \mathcal{H} .

2. NOTATION AND OVERVIEW OF PROOF

In this section we introduce notation used throughout the paper. We then give a sketch of the proof.

Notation. We write $[n]$ for the totally-ordered poset category having objects the non-negative integers $0, \dots, n$. If i and j are integers satisfying $0 \leq i < j \leq n$, we write $[i < j]$ to indicate the morphism from i to j in the category $[n]$.

Given an object c in a small category \mathcal{C} , we shall write $\langle c \rangle$ for the functor $[0] \rightarrow \mathcal{C}$ that represents c . Given a sequence f_1, \dots, f_n of morphisms in \mathcal{C} satisfying $\text{dom } f_n = \text{cod } f_{n-1}$ for $0 < i \leq n$, we will write $\langle f_n | \dots | f_1 \rangle$ for the functor $[n] \rightarrow \mathcal{C}$ that represents the given sequence $\xleftarrow{f_n} \dots \xleftarrow{f_1}$.

Overview.

- In Section 3, we introduce the simplex category Δ and the nerve functor N . We define the notion of degenerate simplices, and introduce $\text{Sd } \mathcal{C}$ as a category whose objects are the non-degenerate simplices of $N\mathcal{C}$.
- In Section 4, we demonstrate that for any small category \mathcal{C} , there is an isomorphism $\text{Sd } \mathcal{C} \cong \text{Sd}(\mathcal{C}^{op})$ which sends $\langle c \rangle$ to $\langle c \rangle$ and $\langle f_n | \dots | f_1 \rangle$ to $\langle f_1 | \dots | f_n \rangle$. For any small groupoid \mathcal{G} , there is an automorphism $\alpha_{\mathcal{G}}$ of $\text{Sd } \mathcal{G}$ that sends $\langle c \rangle$ to $\langle c \rangle$ and $\langle f_n | \dots | f_1 \rangle$ to $\langle f_1^{-1} | \dots | f_n^{-1} \rangle$. We also show that \mathcal{C} is connected if and only if $\text{Sd } \mathcal{C}$ is connected, and that there is an isomorphism $\text{Sd}(\coprod_i \mathcal{C}_i) \cong \coprod_i (\text{Sd } \mathcal{C}_i)$ for any set $\{\mathcal{C}_i\}$ of small categories. These facts will be useful for reducing the problem to the case of connected groupoids, and for obtaining covariant isomorphisms from contravariant ones.
- In Section 5, we identify categorical properties of $\text{Sd } \mathcal{C}$ that encode some structural aspects of \mathcal{C} . This will be useful later when we consider isomorphisms $\text{Sd } \mathcal{B} \cong \text{Sd } \mathcal{C}$ and show that the identified categorical properties are preserved.
 - Given an object y in $\text{Sd } \mathcal{C}$, we write $\text{mt}_n(y)$ for the set

$$\coprod_{x:[n] \rightarrow \mathcal{C}} \text{Sd } \mathcal{C}(x, y)$$

of morphisms targeting y and having domain equal to some (non-degenerate) n -simplex. We write $\text{mt}(y)$ for the set $\coprod_{n \in \mathbb{N}} \text{mt}_n(y)$ of all morphisms targeting y in $\text{Sd } \mathcal{C}$.

- We show that for any non-degenerate n -simplex $y : [n] \rightarrow \mathcal{C}$, regarded as an object in $\text{Sd } \mathcal{C}$, the morphisms of $\text{Sd } \mathcal{C}$ targeting y are in bijection with the monomorphisms of Δ targeting $[n]$. Thus, an object of $\text{Sd } \mathcal{C}$ is an n -simplex if and only if it is the target of $2^{n+1} - 1$ morphisms in $\text{Sd } \mathcal{C}$.
- Given an object y in $\text{Sd } \mathcal{C}$, we define the *faces* of y to be the objects x such that there exists a morphism $x \rightarrow y$. We show that the proper faces

of a non-degenerate 1-simplex $\langle f \rangle$ are precisely the 0-simplices $\langle \text{dom } f \rangle$ and $\langle \text{cod } f \rangle$.

- Given a 2-simplex $\langle f|g \rangle$ in $\text{Sd } \mathcal{C}$, we show that f is left-inverse to g if and only if there are exactly two morphisms in the set $\text{mt}_1(\langle f|g \rangle)$, the domains of which are the 1-simplices $\langle f \rangle$ and $\langle g \rangle$. If f is not inverse to g , then $\text{mt}_1(\langle f|g \rangle)$ contains three morphisms, the domains of which are $\langle f \rangle$, $\langle g \rangle$, and $\langle f \circ g \rangle$.
- Supposing that \mathcal{G} is a small groupoid, we show in section 6 how properties of $\text{Sd } \mathcal{G}$ encode the structure of \mathcal{G} up to opposites. Supposing that f and g are non-identity morphisms in \mathcal{G} , the local structure of $\text{Sd } \mathcal{G}$ near $\langle f \rangle$ and $\langle g \rangle$ determines whether a given 1-simplex $\langle h \rangle$ satisfies one of the equations $h = f \circ g$ or $h = g \circ f$. The two propositions stated at the end of section 6 (and proved in the Appendix) are this paper's most difficult results.
- Section 7 concerns maps between subdivisions of categories. We suppose that \mathcal{G} and \mathcal{H} are groupoids, and that $\Psi : \text{Sd } \mathcal{G} \rightarrow \text{Sd } \mathcal{H}$ is an isomorphism. The results from Sections 5 and 6 are used to show that the structure of \mathcal{G} encoded by $\text{Sd } \mathcal{G}$ must match the structure of \mathcal{H} encoded by $\text{Sd } \mathcal{H}$.
 - For any objects x and y in $\text{Sd } \mathcal{G}$, the map Ψ induces a bijection between $\text{Sd } \mathcal{G}(x, y)$ and $\text{Sd } \mathcal{H}(\Psi x, \Psi y)$, hence faces of y are sent to faces of Ψy . Because the cardinality of $\text{mt}(y)$ is equal that of $\text{mt}(\Psi y)$, Ψ sends n -simplices to n -simplices. It follows that the elements of $\text{mt}_n(y)$ are sent bijectively to elements of $\text{mt}_n(\Psi y)$.
 - We define the map ψ so that

$$\Psi \langle c \rangle = \langle \psi c \rangle \quad \text{and} \quad \psi \text{id}_a = \text{id}_{\psi a} \quad \text{and} \quad \Psi \langle f \rangle = \langle \psi f \rangle$$

are satisfied for each object c and each non-identity morphism f in \mathcal{G} . We show that this map $\psi : \mathcal{G} \rightarrow \mathcal{H}$ is an isomorphism between the (undirected) graphs that underlie \mathcal{G} and \mathcal{H} .

- The structure of $\text{Sd } \mathcal{G}$ near given 1-simplices $\langle f \rangle$, $\langle g \rangle$, and $\langle h \rangle$ is the same as that of $\text{Sd } \mathcal{H}$ near $\langle \psi f \rangle$, $\langle \psi g \rangle$ and $\langle \psi h \rangle$. Therefore, for any f, g , and h in \mathcal{G} , we have $h = f \circ g$ or $h = g \circ f$ if and only if $\psi h = (\psi f) \circ (\psi g)$ or $\psi h = (\psi g) \circ (\psi f)$.
- With help from a group-theoretic result due to Bourbaki, we show that if \mathcal{G} is a connected single-object groupoid (that is, a group) then the map $\psi : \mathcal{G} \rightarrow \mathcal{H}$ is a possibly-contravariant isomorphism. We give an analogous result concerning connected groupoids that have multiple objects.
- Working with arbitrary (non-connected) groupoids \mathcal{G} and \mathcal{H} , we suppose that \mathcal{G} equals the coproduct $\mathcal{G}_1 \amalg \mathcal{G}_2$, and that ψ is contravariant on \mathcal{G}_1 and covariant on \mathcal{G}_2 . There exist some groupoids \mathcal{H}_1 and \mathcal{H}_2 such that $\mathcal{H} = \mathcal{H}_1 \amalg \mathcal{H}_2$, and such that the composite isomorphism

$$\text{Sd } \mathcal{G}_1 \amalg \text{Sd } \mathcal{G}_2 \xrightarrow{\cong} \text{Sd}(\mathcal{G}_1 \amalg \mathcal{G}_2) \xrightarrow{\Psi} \text{Sd}(\mathcal{H}_1 \amalg \mathcal{H}_2) \xrightarrow{\cong} \text{Sd } \mathcal{H}_1 \amalg \text{Sd } \mathcal{H}_2$$

restricts to isomorphisms $\Psi_1 : \text{Sd } \mathcal{G}_1 \rightarrow \text{Sd } \mathcal{H}_1$ and $\Psi_2 : \text{Sd } \mathcal{G}_2 \rightarrow \text{Sd } \mathcal{H}_2$. We have maps ψ_1 and ψ_2 , each of which is a restriction of ψ ; by using the fact that \mathcal{G}_1 is isomorphic to its opposite category, we can flip the variance of ψ_1 to obtain a covariant isomorphism $\psi'_1 : \mathcal{G}_1 \rightarrow \mathcal{H}_1$, and thus a covariant isomorphism $\psi'_1 \amalg \psi_2$ between \mathcal{G} and \mathcal{H} .

- Appendix A is the combinatorial heart of this paper. It is devoted to the proof of Section 6's two most involved results, which concern the way that $\text{Sd } \mathcal{G}$ encodes the relationship between endomorphisms in a given groupoid \mathcal{G} .

3. CONSTRUCTION OF $\text{Sd } \mathcal{C}$

This section defines the nerve functor, explains what (non)degenerate simplices in the nerve are, and gives a construction of the categorical subdivision functor $\text{Sd} : \mathbf{Cat} \rightarrow \mathbf{Cat}$.

3.1. The Nerve of a Category. For each non-negative integer n , let $[n]$ denote the poset category whose objects are the integers $0, \dots, n$ and whose morphisms are given by the usual ordering on \mathbb{Z} . The simplex category $\mathbf{\Delta}$ is the full subcategory of \mathbf{Cat} whose objects are the posets $[n]$. Note that $\mathbf{\Delta}$ is a concrete category: for any m and n , the morphisms $[m] \rightarrow [n]$ in $\mathbf{\Delta}$ can be identified with the order-preserving functions $\{0, \dots, m\} \rightarrow \{0, \dots, n\}$. Note also that epimorphisms in $\mathbf{\Delta}$ are just order-preserving surjections, and monomorphisms in $\mathbf{\Delta}$ are just order-preserving injections.

The *nerve* $N\mathcal{C}$ of a category \mathcal{C} is the restriction from \mathbf{Cat}^{op} to $\mathbf{\Delta}^{op}$ of the hom-functor $\mathbf{Cat}(-, \mathcal{C})$. The nerve construction can be made into a functor $N : \mathbf{Cat} \rightarrow \mathbf{sSet}$ by sending a functor $F : \mathcal{B} \rightarrow \mathcal{C}$ to the natural transformation $N\mathcal{B} \Rightarrow N\mathcal{C}$ given by $\mathbf{Cat}(-, F)$. The elements of $\mathbf{Cat}([n], \mathcal{C})$ are called the *n-simplices* of $N\mathcal{C}$. Given a morphism $\mu : [m] \rightarrow [n]$ in $\mathbf{\Delta}$, write μ^* for the function

$$\mathbf{Cat}([n], \mathcal{C}) \rightarrow \mathbf{Cat}([m], \mathcal{C})$$

defined by $\mu^*x = x \circ \mu$ for each n -simplex x of $N\mathcal{C}$.

A simplex $y : [m] \rightarrow \mathcal{C}$ of $N\mathcal{C}$ is said to be *degenerate* if there exists some simplex x of $N\mathcal{C}$ and some non-identity epimorphism σ in $\mathbf{\Delta}$ satisfying $\sigma^*x = y$. If this simplex x is not degenerate, then it is called the *non-degenerate root* of y .

Proposition 3.1.1. *Given an m -simplex $y : [m] \rightarrow \mathcal{C}$ of $N\mathcal{C}$, the non-degenerate root of y is unique.*

Proof. This is a corollary of the standard general unique decomposition of a morphism in $\mathbf{\Delta}$ as a composite of an epimorphism and a monomorphism. Suppose that $x_1 : [n_1] \rightarrow \mathcal{C}$ and $x_2 : [n_2] \rightarrow \mathcal{C}$ are distinct simplices satisfying $\sigma_1^*x_1 = y = \sigma_2^*x_2$ for some epimorphisms σ_1 and σ_2 . We will prove that one of x_1 and x_2 must be degenerate.

At least one of σ_1 and σ_2 must be a non-identity morphism, for otherwise $x_1 = y = x_2$. Assume without loss of generality that $\sigma_2 \neq \text{id}$, and let $\nu_1 : [m] \rightarrow [n_1]$ be right-inverse to σ_1 . Then there is equality

$$x_1 = (\sigma_1\nu_1)^*x_1 = \nu_1^*\sigma_1^*x_1 = \nu_1^*y = \nu_1^*\sigma_2^*x_2 = (\sigma_2\nu_1)^*x_2.$$

By unique decomposition we have $\sigma_2\nu_1 = \nu'\sigma'$ for some monomorphism ν' and some epimorphism σ' , resulting in the equality

$$x_1 = (\sigma_2\nu_1)^*x_2 = (\nu'\sigma')^*x_2 = \sigma'^*(\nu'^*x_2).$$

Because $\sigma_2 \neq \text{id}$, we must have $\sigma' \neq \text{id}$, and thus the above demonstrates that x_1 is degenerate. \square

The above proposition holds for simplices in arbitrary simplicial sets, not just for simplices in the nerve of a category. We will use the notation $\lambda(y)$ to denote the non-degenerate root of y .

No 0-simplex of $N\mathcal{C}$ is degenerate, and each 0-simplex $[0] \rightarrow \mathcal{C}$ may be regarded as picking out a single object of \mathcal{C} . For a positive integer m , each m -simplex $x : [m] \rightarrow \mathcal{C}$ is identified with the sequence

$$(3.1.2) \quad x(m) \xleftarrow{f_m} x(m-1) \longleftarrow \cdots \longleftarrow x(1) \xleftarrow{f_1} x(0)$$

of morphisms obtained by setting each f_i equal to the image under x of the morphism $[i-1 < i]$ in $[m]$. Such an m -simplex x is non-degenerate if and only if all of the arrows f_i are non-identity morphisms of \mathcal{C} .

3.2. Construction of $\text{Sd } \mathcal{C}$. Let \mathcal{C} be a small category. The objects of $\text{Sd } \mathcal{C}$ are the non-degenerate simplices of $N\mathcal{C}$. Given objects $x : [m] \rightarrow \mathcal{C}$ and $y : [n] \rightarrow \mathcal{C}$ of $\text{Sd } \mathcal{C}$, the morphisms $x \rightarrow y$ in $\text{Sd } \mathcal{C}$ are the pairs (σ, ν) of morphisms in Δ satisfying $\text{cod } \sigma = [m]$ and $\text{cod } \nu = [n]$ such that

- (1) ν is a monomorphism,
- (2) σ is an epimorphism, and
- (3) there is equality $\sigma^*x = \nu^*y$.

The identity morphism $x \rightarrow x$ is given by the pair $(\text{id}_{[m]}, \text{id}_{[m]})$. Condition (3) above requires that $\text{dom } \sigma = [k] = \text{dom } \nu$ for some k , as in the commutative diagram to the right. Note that for each morphism $(\sigma, \nu) : x \rightarrow y$ in $\text{Sd } \mathcal{C}$, there is a multivalued function $\{0, \dots, m\} \rightarrow \{0, \dots, n\}$ given by $\nu \circ \sigma^{-1}$. Here σ^{-1} denotes the subset of $\{0, \dots, m\} \times \{0, \dots, k\}$ that is inverse to σ (as a binary relation), and $\nu \circ \sigma^{-1}$ is the composite of binary relations ν and σ^{-1} . Composition of morphisms in $\text{Sd } \mathcal{C}$ is induced by composition of such multivalued functions. See Figure 1 on page 6 for visualization of a test morphism in $\text{Sd } \mathcal{C}$. Figure 2 (on page 7) depicts two composable morphisms in $\text{Sd } \mathcal{C}$, and Figure 3 depicts the composite.

$$\begin{array}{ccc} [k] & \xrightarrow{\nu} & [n] \\ \sigma \downarrow & & \downarrow y \\ [m] & \xrightarrow{x} & \mathcal{C} \end{array}$$

Example 3.2.1. Let \mathcal{B} be the category $(\cdot \rightarrow \cdot \leftarrow \cdot)$ with three objects and two non-identity arrows, and let \mathcal{C} be the category $(\cdot \leftarrow \cdot \rightarrow \cdot)$ opposite to \mathcal{B} . We see that \mathcal{B} and \mathcal{C} are not isomorphic, yet the barycentric subdivisions $\text{Sd } \mathcal{B}$ and $\text{Sd } \mathcal{C}$ are both isomorphic to the category $(\cdot \rightarrow \cdot \leftarrow \cdot \rightarrow \cdot \leftarrow \cdot)$ with five objects and four non-identity arrows. We will later prove that $\text{Sd } \mathcal{C} \cong \text{Sd}(\mathcal{C}^{op})$ for any category \mathcal{C} .

The following example is due to Jonathan Rubin:¹

Example 3.2.2. Consider \mathbb{N} and \mathbb{Z} as poset categories. There is no isomorphism between \mathbb{N} and \mathbb{Z} , yet the subdivisions $\text{Sd } \mathbb{N}$ and $\text{Sd } \mathbb{Z}$ are isomorphic.

Generally, if T is a totally-ordered set (regarded as a poset category) then the n -simplices of $\text{Sd } T$ are all the linearly-ordered subsets of T having $n + 1$ elements. If x and y are objects in $\text{Sd } T$ and if \bar{x} and \bar{y} are the corresponding subsets of T , then there exists a morphism $x \rightarrow y$ in $\text{Sd } T$ if and only if \bar{x} is a subset of \bar{y} . Thus $\text{Sd } T$ is the poset

$$\left(\{F \subseteq T \mid 0 < |F| < \infty\}, \subseteq \right)$$

of finite non-empty subsets of T , ordered by inclusion.

¹Private communication.

If T and T' are totally-ordered sets then any set bijection between T and T' induces an isomorphism $\text{Sd } T \cong \text{Sd } T'$ of categories, regardless of whether the given bijection between T and T' is order-preserving.

3.3. Concerning $\text{Sd}F$. Let \mathcal{B} and \mathcal{C} be small categories, and let F be a functor $\mathcal{B} \rightarrow \mathcal{C}$. The functor $\text{Sd}F$ sends each 0-simplex $y : [0] \rightarrow \mathcal{B}$ to the 0-simplex $F \circ y : [0] \rightarrow \mathcal{C}$. Using the notation from Section 2, this means that $\langle b \rangle$ is sent to $\langle F(b) \rangle$ for each object b of \mathcal{B} . For larger m , $\text{Sd}F$ sends each m -simplex (3.1.2) to the subsequence of

$$F(x(m)) \xleftarrow{F(f_m)} F(x(m-1)) \longleftarrow \dots \longleftarrow F(x(1)) \xleftarrow{F(f_1)} F(x(0))$$

consisting of non-identity arrows. Formally, each m -simplex $x : [m] \rightarrow \mathcal{B}$ is sent by $\text{Sd}F$ to the non-degenerate root of $F \circ x$. Multivalued functions can be used to calculate where $\text{Sd}F$ sends the morphisms of $\text{Sd} \mathcal{B}$. For the purposes of this paper, a precise formulation of $\text{Sd}F$'s value on morphisms of $\text{Sd} \mathcal{B}$ is not necessary.

Example 3.3.1. Below is a depiction of a sample morphism $x \rightarrow y$ in $\text{Sd} \mathcal{C}$.

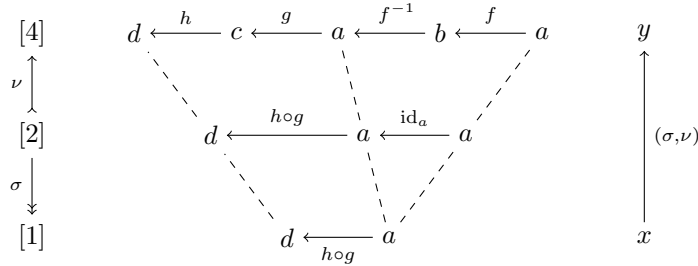


FIGURE 1

Here $x : [1] \rightarrow \mathcal{C}$ picks out the morphism $h \circ g$, and $y : [4] \rightarrow \mathcal{C}$ picks out the sequence

$$\cdot \xleftarrow{h} \cdot \xleftarrow{g} \cdot \xleftarrow{f^{-1}} \cdot \xleftarrow{f} \cdot$$

of arrows in \mathcal{C} . The dashed lines illustrate how h and g compose to $h \circ g$, and how f^{-1} and f compose to id_a . The injection $\nu : [2] \rightarrow [4]$ has image $\{0, 2, 4\}$, and $\sigma : [2] \rightarrow [1]$ is the surjection satisfying $\sigma(0) = 0 = \sigma(1)$. We have assumed that h is not left-inverse to g , and that none of f, f^{-1}, g, h are identity morphisms. The dashed lines determine a multivalued function from $\{0, 1\}$ to $\{0, 1, 2, 3, 4\}$.

Example 3.3.2. Below, two composable morphisms $x \rightarrow y$ and $y \rightarrow z$ in $\text{Sd } \mathcal{C}$ are depicted end-to-end.

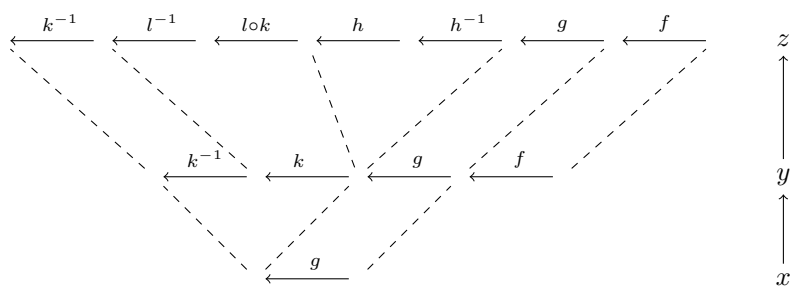


FIGURE 2

The multivalued functions associated to these morphisms $x \rightarrow y$ and $y \rightarrow z$ can be used to calculate the composite arrow in $\text{Sd } \mathcal{C}$. This composite is depicted below.

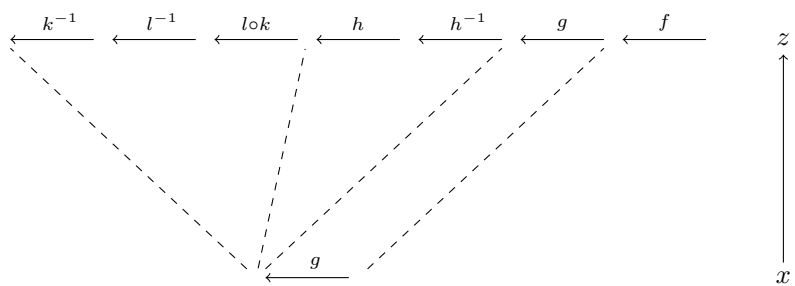


FIGURE 3

Example 3.3.3. The figure below depicts a sample calculation. The functor $\text{Sd}F : \text{Sd} \mathcal{B} \rightarrow \text{Sd} \mathcal{C}$ is applied to a morphism $x \rightarrow y$ of $\text{Sd} \mathcal{B}$.

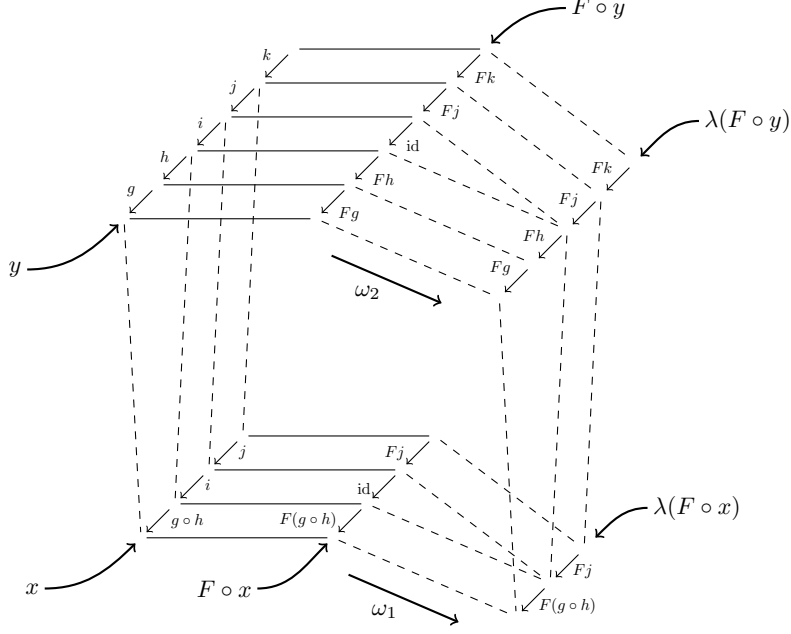


FIGURE 4

The given morphism $x \rightarrow y$ is sent to a morphism $\lambda(F \circ x) \rightarrow \lambda(F \circ y)$ in $\text{Sd} \mathcal{C}$, where $\lambda(F \circ x)$ is the non-degenerate root of x (and similarly for $\lambda(F \circ y)$). The epimorphisms ω_1 and ω_2 are the unique maps in Δ satisfying $\omega_1^* \lambda(F \circ x) = F \circ x$ and $\omega_2^* \lambda(F \circ y) = F \circ y$. We suppose here that F_i is an identity morphism in \mathcal{C} , and that F_g is not left inverse to F_h .

4. Sd PRESERVES COPRODUCTS AND IDENTIFIES OPPOSITE CATEGORIES

This section states two lemmas that will be used later in the paper. In particular, we show in Lemma 4.1 that the functor Sd does not distinguish between a category \mathcal{C} and its opposite category \mathcal{C}^{op} , and in Lemma 4.4 that Sd preserves coproducts. The latter lemma allows for reduction of this paper's main theorem to the case of connected groupoids. The former highlights a fundamental issue: for any small category \mathcal{C} we have an isomorphism $\text{Sd} \mathcal{C} \rightarrow \text{Sd}(\mathcal{C}^{op})$. Therefore, a naive attempt to construct a map $\mathcal{B} \rightarrow \mathcal{C}$ from a given isomorphism $\text{Sd} \mathcal{B} \rightarrow \text{Sd} \mathcal{C}$ could result in contravariance. As mentioned in the introduction, this pitfall can be avoided when working with groupoids, as any groupoid is isomorphic to its opposite via inversion.

Lemma 4.1. *Let \mathcal{C} be a small category. There is a canonical isomorphism*

$$(4.2) \quad \text{Sd} \mathcal{C} \xrightarrow{\cong} \text{Sd}(\mathcal{C}^{op})$$

between the subdivision of \mathcal{C} and the subdivision of the opposite category.

Proof. The claimed isomorphism sends each m -simplex (3.1.2) in $\text{Sd } \mathcal{C}$ to the m -simplex

$$x(m) \xrightarrow{f_m} x(m-1) \longrightarrow \cdots \longrightarrow x(1) \xrightarrow{f_1} x(0)$$

in $\text{Sd}(\mathcal{C}^{op})$. If μ is a arrow $[m] \rightarrow [n]$ in $\mathbf{\Delta}$, write μ' for the arrow $[m] \rightarrow [n]$ defined by $\mu'(m-i) = n - \mu(i)$. The claimed map (4.2) is defined on morphisms of $\text{Sd } \mathcal{C}$ by $(\sigma, \nu) \mapsto (\sigma', \nu')$. \square

Thus, for any statement about the relationship between $\text{Sd } \mathcal{C}$ and \mathcal{C} , there is a dual statement about $\text{Sd } \mathcal{C}$ and \mathcal{C}^{op} .

Example 4.3. For any small groupoid \mathcal{G} we have the inversion isomorphism $\mathcal{G}^{op} \rightarrow \mathcal{G}$ defined on morphisms by $f \mapsto f^{-1}$. The subdivision of this isomorphism is a map $\text{Sd}(\mathcal{G}^{op}) \rightarrow \text{Sd } \mathcal{G}$, which can be composed with the map $\text{Sd } \mathcal{G} \rightarrow \text{Sd}(\mathcal{G}^{op})$ from Lemma 4.1 to obtain an isomorphism $\alpha_{\mathcal{G}} : \text{Sd } \mathcal{G} \rightarrow \text{Sd } \mathcal{G}$. This functor $\alpha_{\mathcal{G}}$ sends each m -simplex (3.1.2) of $\text{Sd } \mathcal{G}$ to the m -simplex

$$x(m) \xrightarrow{f_m^{-1}} x(m-1) \longrightarrow \cdots \longrightarrow x(1) \xrightarrow{f_1^{-1}} x(0).$$

Note that Sd is faithful, as any functor $F : \mathcal{B} \rightarrow \mathcal{C}$ is determined by where $\text{Sd}F$ sends the 0 and 1-simplices of $\text{Sd } \mathcal{B}$. Unless the groupoid \mathcal{G} is a discrete category, the map $\alpha_{\mathcal{G}}$ is not equal to the subdivision of any automorphism $\mathcal{G} \rightarrow \mathcal{G}$. Thus, for any automorphism ξ of a non-discrete groupoid \mathcal{G} we find an automorphism $\alpha_{\mathcal{G}} \circ \text{Sd } \xi$ of the category $\text{Sd } \mathcal{G}$. Therefore, we have the inequality

$$|\text{Aut}(\text{Sd } \mathcal{G})| \geq 2 \cdot |\text{Aut}(\mathcal{G})|.$$

in the case where \mathcal{G} is not discrete.

Lemma 4.4. *The functor Sd preserves coproducts. A category \mathcal{C} is connected if and only if its subdivision $\text{Sd } \mathcal{C}$ is connected.*

Proof. Subdivision Sd is given by the composite $\Pi \circ \text{Sd}^s \circ N$. Here $\Pi : \mathbf{sSet} \rightarrow \mathbf{Cat}$ is left adjoint to N , and $\text{Sd}^s : \mathbf{sSet} \rightarrow \mathbf{sSet}$ denotes barycentric subdivision of simplicial sets. The functors Π and Sd^s are left adjoints, and the nerve functor N preserves coproducts, so Sd does too. Therefore, if $\text{Sd } \mathcal{C}$ is connected then \mathcal{C} must be connected, for otherwise we would have $\mathcal{C} = \coprod_i \mathcal{C}_i$ and hence $\text{Sd } \mathcal{C} = \coprod_i \text{Sd } \mathcal{C}_i$.

Suppose now that \mathcal{C} is a connected category. To prove that $\text{Sd } \mathcal{C}$ is connected we will introduce some new notation. For an object c in \mathcal{C} , let $\langle c \rangle$ denote the 0-simplex $[0] \rightarrow \mathcal{C}$ sending 0 to c . For a morphism f in \mathcal{C} , let $\langle f \rangle$ denote the 1-simplex $[1] \rightarrow \mathcal{C}$ that sends the morphism $[0 < 1]$ to f .

For any non-identity morphism f in \mathcal{C} there are arrows

$$\langle \text{dom } f \rangle \xrightarrow{(\text{id}_{[0]}, \delta)} \langle f \rangle \quad \text{and} \quad \langle \text{cod } f \rangle \xrightarrow{(\text{id}_{[0]}, \delta')} \langle f \rangle$$

in $\text{Sd } \mathcal{C}$, where δ and δ' are the monomorphisms $[0] \rightarrow [1]$ having respective images $\{0\}$ and $\{1\}$. Thus, for any sequence

$$a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} \cdots$$

of objects in \mathcal{C} , there is a sequence

$$\langle a \rangle \rightarrow \langle f \rangle \leftarrow \langle b \rangle \rightarrow \langle g \rangle \leftarrow \langle c \rangle \rightarrow \langle h \rangle \leftarrow \cdots$$

of objects in $\text{Sd } \mathcal{C}$. Therefore, all 0-simplices are in the same connected component. To complete the proof, let $\delta : [0] \rightarrow [m]$ denote the morphism that sends 0 to 0, and

note that for any m -simplex x in $\text{Sd } \mathcal{C}$ there is a morphism $(\text{id}_{[0]}, \delta)$ whose domain is the 0-simplex δ^*x and whose codomain is x . Thus all object of $\text{Sd } \mathcal{C}$ belong to a single connected component. \square

Note that the single-object category $[0]$ is isomorphic to its subdivision $\text{Sd } [0]$. It follows from the previous lemma that any discrete category \mathcal{C} is isomorphic to its own subdivision.

5. $\text{Sd } \mathcal{C}$ ENCODES OBJECTS AND ARROWS OF \mathcal{C}

This section concerns the relationship between a category and its subdivision. We identify categorical properties of $\text{Sd } \mathcal{C}$ that correspond to certain structures in the category \mathcal{C} . The identified properties of $\text{Sd } \mathcal{C}$ are preserved by isomorphism: supposing that $\text{Sd } \mathcal{B} \cong \text{Sd } \mathcal{C}$ for some category \mathcal{B} , the structures of \mathcal{C} will appear the same as in \mathcal{B} . For example, \mathcal{B} is a groupoid if and only if \mathcal{C} is a groupoid.

Notation 5.0.1. As in Section 3, for any objects x and y in $\text{Sd } \mathcal{C}$, we regard the morphisms $x \rightarrow y$ in $\text{Sd } \mathcal{C}$ as pairs (σ, ν) of morphisms in Δ satisfying

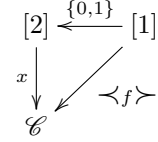
- (1) ν is a monomorphism,
- (2) σ is an epimorphism, and
- (3) there is equality $\sigma^*x = \nu^*y$.

As in Section 4, for any object a of \mathcal{C} we let $\langle a \rangle : [0] \rightarrow \mathcal{C}$ denote the 0-simplex that represents a . Given a morphism f in \mathcal{C} , write $\langle f \rangle : [1] \rightarrow \mathcal{C}$ for the 1-simplex that represents f . We extend this notation as follows: given a sequence f_1, \dots, f_m of morphisms in \mathcal{C} satisfying $\text{dom } f_{n+1} = \text{cod } f_n$ for $1 \leq n < m$, write $\langle f_m | \dots | f_1 \rangle$ for the m -simplex $x : [m] \rightarrow \mathcal{C}$ given by the diagram

$$x(m) \xleftarrow{f_m} x(m-1) \longleftarrow \dots \longleftarrow x(1) \xleftarrow{f_1} x(0)$$

in \mathcal{C} , where f_i is equal to $x([i-1 < i])$ for each i above. This notation is inspired by the ‘‘bar construction’’ on groups.

For a natural number m , identify each nonempty subset S of $\{0, \dots, m\}$ with the monomorphism in Δ having domain $[|S| - 1]$, codomain $[m]$, and image S . For example, if x equals $\langle g | f \rangle$ for some morphisms $f : a \rightarrow b$ and $g : b \rightarrow c$, then we can identify $\{0, 1\}$ and $\{0, 2\}$ with injections $[1] \rightarrow [2]$ to obtain equalities $\{0, 1\}^*x = \langle f \rangle$ and $\{0, 2\}^*x = \langle g \circ f \rangle$.



Recall that the *non-degenerate root* of an m -simplex $y : [m] \rightarrow \mathcal{C}$ is the unique non-degenerate simplex $\lambda(y) : [n] \rightarrow \mathcal{C}$ that satisfies $\omega(y)^*\lambda(y) = y$ for some surjection $\omega(y) : [m] \twoheadrightarrow [n]$. Setting $y = \langle f_m | \dots | f_1 \rangle$, we obtain $\lambda(y)$ by omitting identity arrows from the sequence f_1, \dots, f_m . The map $\omega(y)$ is uniquely determined by the equations

$$(5.0.2) \quad \omega(y)(0) = 0$$

and

$$(5.0.3) \quad \omega(y)(i+1) = \begin{cases} \omega(y)(i) & \text{if } f_{i+1} \text{ is an identity morphism} \\ \omega(y)(i) + 1 & \text{if } f_{i+1} \text{ is a non-identity morphism} \end{cases}$$

where (5.0.3) above holds for all $i < m$. An m -simplex $y : [m] \rightarrow \mathcal{C}$ may be regarded as an object of $\text{Sd } \mathcal{C}$ precisely when it is non-degenerate.

Definition 5.0.4. Given an object y of $\text{Sd } \mathcal{C}$, let $\text{mt}(y)$ denote the set

$$\{f \in \text{Mor}(\text{Sd } \mathcal{C}) \mid \text{cod } f = y\}$$

of morphisms targeting y in $\text{Sd } \mathcal{C}$.

Proposition 5.0.5. *Given an m -simplex y , regarded as an object of $\text{Sd } \mathcal{C}$, there is a bijection between the set of nonempty subsets of $\{0, \dots, m\}$ and the set $\text{mt}(y)$ of morphisms targeting y . This bijection sends a subset S of $\{0, \dots, m\}$ to the morphism $(\omega(S^*y), S) : \lambda(S^*y) \rightarrow y$ in $\text{Sd } \mathcal{C}$.*

Proof. Identifying each subset S with the monomorphism $[|S| - 1] \rightarrow [m]$ having image S , it suffices to show that the map

$$\{\text{monomorphisms of } \Delta \text{ targeting } [m]\} \rightarrow \text{mt}(y)$$

given by $S \mapsto (\omega(S^*y), S)$ is bijective. This pair $(\omega(S^*y), S)$ is to be regarded as a morphism from $\lambda(S^*y)$ to y , witnessed by the equality $\omega(S^*y)^* \lambda(S^*y) = S^*y$. Note that $\omega(S^*y)$ is an identity map if and only if S^*y is non-degenerate.

The given function is injective: if $S_1 \neq S_2$ then $(\omega(S_1^*y), S_1) \neq (\omega(S_2^*y), S_2)$. To check surjectivity, suppose that (σ, ν) is a morphism $z \rightarrow y$ in $\text{Sd } \mathcal{C}$. Writing $S = \text{im}(\nu)$ so that $S^*y = \nu^*y$, we obtain the equality $S^*y = \sigma^*z$ which demonstrates that z is the non-degenerate root of S^*y . Therefore we have $z = \lambda(S^*y)$ and $\sigma = \omega(S^*y)$. \square

Lemma 5.0.6. *An object y of $\text{Sd } \mathcal{C}$ is an m -simplex if and only if the set $\text{mt}(y)$ has cardinality $2^{m+1} - 1$.*

Proof. Every object of $\text{Sd } \mathcal{C}$ is a m -simplex for some m . There are $2^{m+1} - 1$ nonempty subsets of $\{0, \dots, m\}$, and just as many morphisms targeting each m -simplex. \square

Thus the categorical structure of $\text{Sd } \mathcal{C}$ encodes the specific dimension of simplices. Note that if x is an m -simplex and y is an n -simplex such that there exists some non-identity morphism $x \rightarrow y$ in $\text{Sd } \mathcal{C}$, then we must have inequality $m < n$. This is because if (σ, ν) is an epi-mono pair satisfying $\sigma^*x = \nu^*y$, then we have $\text{dom } \sigma = [k] = \text{dom } \nu$ for some k satisfying $m \leq k \leq n$. We obtain a strict inequality $m < n$ if either σ or ν is a non-identity morphism.

Example 5.0.7 (Automorphisms of $\text{Sd}[n]$). The category $[n]$ is a finite poset. Therefore, as discussed in Example 3.2.2, there is a canonical isomorphism between the subdivision $\text{Sd}[n]$ and the poset of non-empty subsets of $\{0, \dots, n\}$, with order given by subset inclusion. Under this isomorphism, each 0-simplex $\langle k \rangle$ of $\text{Sd}[n]$ is sent to the singleton subset $\{k\}$ of $\{0, \dots, n\}$. It follows that each automorphism of $\text{Sd}[n]$ is determined by a permutation of the set $\langle 0 \rangle, \dots, \langle n \rangle$ of 0-simplices. Therefore, we have an isomorphism

$$\text{Aut}(\text{Sd}[n]) \cong S_{n+1}$$

between the group of automorphisms of $\text{Sd}[n]$ and the symmetric group on $n + 1$ elements. For comparison, the group $\text{Aut}([n])$ is trivial for each natural number n .

Definitions 5.0.8. Let x and y be objects in $\text{Sd } \mathcal{C}$ such that the hom-set $\text{Sd } \mathcal{C}(x, y)$ is non-empty. Then we say that x is a *face* of y . If, in addition, x and y are distinct, then x is a *proper face* of y . The *category of faces* of y , written $\lceil y \rceil$, is the full subcategory of $\text{Sd } \mathcal{C}$ whose objects are the faces of y .

The faces of an m -simplex y are precisely the simplices $\lambda(S^*y)$. This is because, by Proposition 5.0.5, the maps into y are precisely of the form $(\omega(S^*y), S)$ for S a non-empty subset of $\{0, \dots, m\}$. The proper faces of y are those simplices $\lambda(S^*y)$ where S is a proper non-empty subset of $\{0, \dots, m\}$.

Here is an example. Let f, g , and h be distinct non-identity morphisms in \mathcal{C} , and assume that $h = g \circ f$. Write $y = \langle g|f \rangle$, and set $a = \text{dom } f$, $b = \text{cod } f = \text{dom } g$, and $c = \text{cod } g$. If the objects a, b, c are all distinct then the category of faces $\ulcorner y \urcorner$ is as in Figure 5. If a equals c and is distinct from b , then $\ulcorner y \urcorner$ is as in Figure 6. If $a = b = c$ then $\ulcorner y \urcorner$ is as in Figure 7.

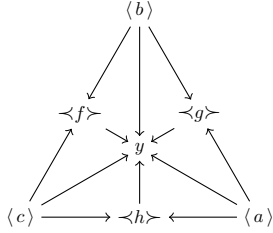


FIGURE 5

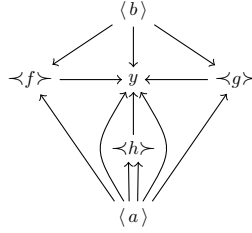


FIGURE 6

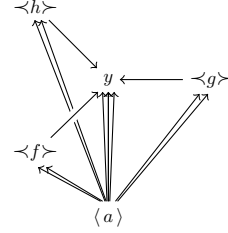


FIGURE 7

Note that identity arrows have been omitted from the above diagrams.

Proposition 5.0.9. *Let $\langle f \rangle : [1] \rightarrow \mathcal{C}$ be a 1-simplex in $\text{Sd } \mathcal{C}$. Then the proper faces of $\langle f \rangle$ are $\langle \text{cod } f \rangle$ and $\langle \text{dom } f \rangle$.*

Proof. By Proposition 5.0.5, the proper faces of $\langle f \rangle$ are the non-degenerate roots $\lambda(\{0\}^* \langle f \rangle)$ and $\lambda(\{1\}^* \langle f \rangle)$. Every 0-simplex is non-degenerate, so we have

$$\lambda(\{0\}^* \langle f \rangle) = \{0\}^* \langle f \rangle = \langle \text{dom } f \rangle$$

and

$$\lambda(\{1\}^* \langle f \rangle) = \{1\}^* \langle f \rangle = \langle \text{cod } f \rangle. \quad \square$$

Corollary 5.0.10. *A non-identity morphism f is an endomorphism if and only if the 1-simplex $\langle f \rangle$ has just one proper face.*

Note that $\text{Sd } \mathcal{C}$ does not distinguish which face of $\langle f \rangle$ corresponds to domain and which to codomain. For example, write f for the arrow $0 \rightarrow 1$ in $[1]$, and consider the category $\text{Sd } [1]$ pictured to the right. Note that the non-identity automorphism of $\text{Sd } [1]$ switches $\langle 0 \rangle$ with $\langle 1 \rangle$. Thus, Sd introduces symmetry, illustrated generally by the canonical isomorphism $\text{Sd } \mathcal{C} \cong \text{Sd}(\mathcal{C}^{op})$. At best, we can hope for $\text{Sd } \mathcal{C}$ to encode \mathcal{C} “up to opposites.”

$$\langle 0 \rangle \longrightarrow \langle f \rangle \longleftarrow \langle 1 \rangle$$

5.1. Encoding triangles. We have seen that the proper faces of 1-simplices correspond to their domain and codomain; we now go up one dimension to see how 2-simplices in $\text{Sd } \mathcal{C}$ codify relationships among morphisms in \mathcal{C} .

Notation 5.1.1. Given an object y of $\text{Sd } \mathcal{C}$, write $\text{mt}_n(y)$ for the set

$$\{f \in \text{Mor}(\text{Sd } \mathcal{C}) \mid \text{cod } f = y \text{ and } \text{dom } f \text{ is an } n\text{-simplex}\}$$

of morphisms targeting y that have source equal to some non-degenerate n -simplex.

Some of the coming proofs will require explicit calculation of the sets $\text{mt}_0(y)$ and $\text{mt}_1(y)$ for 2-simplices y . Below we state some general facts that will make such calculation easier.

Elements of $\text{mt}_n(y)$ are morphisms $(\omega(S^*y), S) : \lambda(S^*y) \rightarrow y$ such that $\lambda(S^*y)$ is an n -simplex. Note that the sets $\text{mt}_n(y)$ partition $\text{mt}(y)$. Also, note that $\lambda(S^*y)$ has dimension less than or equal to S^*y , which in turn has dimension equal to $|S| - 1$. Therefore, to find $\text{mt}_n(y)$ it is enough to consider only those S with $|S| - 1 \geq n$. Indeed, if $|S| - 1 < n$ then

$$\dim(\lambda(S^*y)) \leq \dim(S^*y) \leq |S| - 1 < n,$$

hence the dimension of $\lambda(S^*y)$ is less than n , and the morphism $(\omega(S^*y), S)$ cannot be in $\text{mt}_n(y)$. Finally, note that if y is an m -simplex and if S is equal to the full set $\{0, \dots, m\}$, then $S^*y = y$ is non-degenerate, in which case we have equality

$$\dim(\lambda(S^*y)) = \dim(y) = m.$$

Therefore, to find $\text{mt}_n(y)$ in the case where $n < m$, it is good enough to consider the proper subsets S of $\{0, \dots, m\}$.

The next order of business will be to show how Sd encodes inversion. This will allow us to determine by looking at $\text{Sd } \mathcal{C}$ whether the category \mathcal{C} is a groupoid. Moreover, if \mathcal{G} is a groupoid and $\langle f \rangle$ is a 1-simplex in $\text{Sd } \mathcal{G}$, being able to find $\langle f^{-1} \rangle$ will help determine whether given 1-simplices $\langle g \rangle$ and $\langle h \rangle$ satisfy $f \circ g = h$.

Proposition 5.1.2. *Let f, g be non-identity morphisms in \mathcal{C} satisfying $\text{dom } f = \text{cod } g$. Set y equal to the 2-simplex $\langle f|g \rangle$. If f is left-inverse to g then $\text{mt}_0(y)$ has four elements and $\text{mt}_1(y)$ has two elements. On the other hand, if the composite $f \circ g$ is not an identity arrow then $\text{mt}_0(y)$ and $\text{mt}_1(y)$ have three elements each.*

Proof. For any object x of $\text{Sd } \mathcal{C}$, the hom-set $\text{Sd } \mathcal{C}(x, x)$ contains only one morphism. We have the identity $\{0, 1, 2\}^*y = y$, therefore the map from Proposition 5.0.5 restricts to a bijection between the set of non-empty proper subsets of $\{0, 1, 2\}$ and the union $\text{mt}_0(y) \sqcup \text{mt}_1(y)$. By counting subsets of $\{0, 1, 2\}$ we find $|\text{mt}_0(y)| + |\text{mt}_1(y)| = 6$.

For each singleton subset $\{i\}$ we have a morphism with domain $\{i\}^*y$, hence $\text{mt}_0(y)$ has at least three elements. The subsets $\{0, 1\}$ and $\{1, 2\}$ correspond to morphisms $\langle g \rangle \rightarrow y$ and $\langle f \rangle \rightarrow y$, thus $\text{mt}_1(y)$ has size at least two.

It remains to consider the subset $\{0, 2\}$. We have $\{0, 2\}^*y = \langle f \circ g \rangle$, and thus the bijection from Proposition 5.0.5 sends $\{0, 2\}$ to an arrow

$$\lambda(\langle f \circ g \rangle) \rightarrow y$$

in $\text{Sd } \mathcal{C}$. If the composite $f \circ g$ is an identity arrow then $\lambda(\langle f \circ g \rangle)$ is a 0-simplex. On the other hand, if f is not left-inverse to g then we have a third element of $\text{mt}_1(y)$. \square

Given the identification of objects in $\text{Sd } \mathcal{C}$ with simplices of $N\mathcal{C}$, we may think of each 2-simplex in $\text{Sd } \mathcal{C}$ as witnessing a triangle-shaped commutative diagram in \mathcal{C} .

Definition 5.1.3. Let f, g , and h be morphisms in \mathcal{C} , and let y be a 2-simplex of $N\mathcal{C}$. Say y is of the form $\begin{array}{c} f \diagdown \\ \\ h \end{array} \diagup g$ if y represents the composite of some pair among

f, g, h yielding the third morphism. Explicitly, y is of the form $\frac{f/\wedge^g}{h}$ if one of the following is satisfied:

- | | |
|--|--|
| 1) $f \circ g = h$ and $y = \langle f g \rangle$, | 4) $h \circ f = g$ and $y = \langle h f \rangle$, |
| 2) $g \circ f = h$ and $y = \langle g f \rangle$, | 5) $h \circ g = f$ and $y = \langle h g \rangle$, |
| 3) $f \circ h = g$ and $y = \langle f h \rangle$, | 6) $g \circ h = f$ and $y = \langle g h \rangle$. |

The expression $\frac{f/\wedge^g}{h}$ does not encode the order of f, g , and h ; the forms $\frac{f/\wedge^g}{h}$ and $\frac{g/\wedge^h}{f}$ are logically equivalent. We will sometimes label vertices of these triangles. For example, if $y : [2] \rightarrow \mathcal{C}$ represents the commutative triangle

$$(5.1.4) \quad \begin{array}{ccc} & b & \\ f \swarrow & & \searrow g \\ c & \xleftarrow{h} & a \end{array} .$$

in \mathcal{C} , then we will say that y is of the form $\frac{f/\wedge^g}{c \xleftarrow{h} a}$. Note that if there is a non-degenerate 2-simplex of the form $\frac{f/\wedge^g}{h}$ then the two morphisms $y[0 < 1]$ and $y[1 < 2]$ must be non-identity, where $[i < j]$ denotes the morphism $i \rightarrow j$ in the category $[2]$. Therefore, if y is 2-simplex in $\text{Sd } \mathcal{C}$ of the form $\frac{f/\wedge^g}{\text{id}}$, then y satisfies either

- 1) $y = \langle f|g \rangle$ and f is left inverse to g , or
- 2) $y = \langle g|f \rangle$ and g is left inverse to f .

The following Lemma shows how this triangle notation summarizes information concerning 2-simplices in $\text{Sd } \mathcal{C}$.

Lemma 5.1.5. *Let $y : [2] \rightarrow \mathcal{C}$ be a non-degenerate 2-simplex. Then y is of the form $\frac{g/\wedge^f}{h}$ for some non-identity arrows f, g , and h if and only if there are three morphisms in the set $\text{mt}_1(y)$, and these morphisms have respective domains $\langle f \rangle$, $\langle g \rangle$, and $\langle h \rangle$. On the other hand, y is of the form $\frac{f/\wedge^g}{\text{id}}$ if and only if there are two morphisms in the set $\text{mt}_1(y)$, and these morphisms have respective domains $\langle f \rangle$ and $\langle g \rangle$.*

Proof. Let y be a non-degenerate 2-simplex $[2] \rightarrow \mathcal{C}$. By definition, y is of the form $\frac{f/\wedge^g}{h}$ for some morphisms f, g , and h in \mathcal{C} . Assume without loss of generality that $y = \langle f|g \rangle$. Then we have $h = g \circ f$, again by definition. Let a, b, c satisfy

$$a = \text{dom } g \quad \text{and} \quad \text{cod } g = b = \text{dom } f \quad \text{and} \quad \text{cod } f = c$$

as in the commutative triangle (5.1.4) above. By Proposition 5.0.5, the non-identity elements of $\text{mt}(y)$ are in bijection with the proper non-empty subsets of $\{0, 1, 2\}$.

Suppose first that all three of f , g , and h are non-identity morphisms. Then the non-identity elements of $\text{mt}(y)$ are the morphisms displayed below:

$$\begin{array}{ll} \langle a \rangle \xrightarrow{(\text{id}_{[0]}, \{0\})} y & \langle g \rangle \xrightarrow{(\text{id}_{[1]}, \{0,1\})} y \\ \langle b \rangle \xrightarrow{(\text{id}_{[0]}, \{1\})} y & \langle f \rangle \xrightarrow{(\text{id}_{[1]}, \{1,2\})} y \\ \langle c \rangle \xrightarrow{(\text{id}_{[0]}, \{2\})} y & \langle h \rangle \xrightarrow{(\text{id}_{[1]}, \{0,2\})} y. \end{array}$$

On the other hand, if f is left-inverse to g then then we must have $a = c$ and $f \circ g = \text{id}_a$. In this case, writing σ for the unique epimorphism $[1] \rightarrow [0]$, the non-identity elements of $\text{mt}(y)$ are the morphisms displayed below:

$$\begin{array}{ll} \langle a \rangle \xrightarrow{(\text{id}_{[0]}, \{0\})} y & \langle g \rangle \xrightarrow{(\text{id}, \{0,1\})} y \\ \langle b \rangle \xrightarrow{(\text{id}_{[0]}, \{1\})} y & \langle f \rangle \xrightarrow{(\text{id}, \{1,2\})} y \\ \langle a \rangle \xrightarrow{(\text{id}_{[0]}, \{2\})} y & \langle a \rangle \xrightarrow{(\sigma, \{0,1\})} y. \end{array}$$

Thus we have $|\text{mt}_0(y)| = 4$ and $|\text{mt}_1(y)| = 2$ if f is left-inverse to g , whereas $|\text{mt}_0(y)| = 3 = |\text{mt}_1(y)|$ if $f \circ g$ is non-identity. \square

Example 5.1.6. By Lemma 5.0.6 we have $|\text{mt}_0(y)| + |\text{mt}_1(y)| = 6$ for any non-degenerate 2-simplex y . If y is of the form

$$\begin{array}{c} b \\ f \swarrow \quad \searrow g \\ c \quad \quad a \\ h \end{array}$$

and satisfies $|\text{mt}_0(y)| = 3$, then the morphisms in $\text{mt}_0(y)$ have respective domains $\langle a \rangle$, $\langle b \rangle$, and $\langle c \rangle$, as in Figures 5 through 7.

Suppose instead that y is of the form

$$\begin{array}{c} b \\ f \swarrow \quad \searrow g \\ a \quad \quad a \\ \text{id}_a \end{array},$$

satisfying $|\text{mt}_0(y)| = 4$. Letting σ denote the unique epimorphism $[1] \rightarrow [0]$, the elements of $\text{mt}_0(y)$ are given by the pairs

$$\begin{array}{l} (\sigma, \{0, 2\}) : \langle a \rangle \rightarrow y, \\ (\text{id}_{[0]}, \{0\}) : \langle a \rangle \rightarrow y, \\ (\text{id}_{[0]}, \{1\}) : \langle b \rangle \rightarrow y, \text{ and} \\ (\text{id}_{[0]}, \{2\}) : \langle a \rangle \rightarrow y. \end{array}$$

If a and b are distinct, then $\lceil y \rceil$ appears as in Figure 8. If $a = b$ but $f \neq g$, then $\lceil y \rceil$ is as in Figure 9. If $f = g$ then $\lceil y \rceil$ is as in Figure 10.

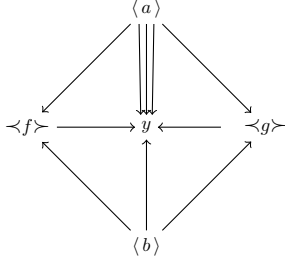


FIGURE 8

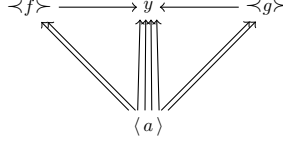


FIGURE 9

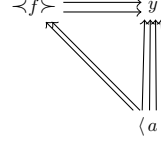


FIGURE 10

Lemma 5.1.7 (Inverse criterion). *A non-identity morphism f in \mathcal{C} is self-inverse if and only if there exists a 2-simplex of the form $\frac{f \wedge f}{\text{id}}$ in $\text{Sd } \mathcal{C}$. For distinct non-identity morphisms f and g in \mathcal{C} , f is the two-sided inverse to g if and only if there exist two distinct 2-simplices of the form $\frac{f \wedge g}{\text{id}}$.*

Proof. If there is a 2-simplex of the form $\frac{f \wedge f}{\text{id}}$ then it must be equal to $\langle f|f \rangle$. In this case f is left inverse to f (by Proposition 5.1.2). Conversely, if f is self-inverse then the 2-simplex $\langle f|f \rangle$ is of the form $\frac{f \wedge f}{\text{id}}$.

Suppose now that f and g are distinct and that $\text{Sd } \mathcal{C}$ contains two 2-simplices of the form $\frac{f \wedge g}{\text{id}}$. These 2-simplices must equal $\langle f|g \rangle$ and $\langle g|f \rangle$, so f is left inverse to g and vice versa. Conversely, if f is the two-sided inverse to g then $\langle f|g \rangle$ and $\langle g|f \rangle$ are of the form $\frac{f \wedge g}{\text{id}}$. \square

Thus, the subdivision $\text{Sd } \mathcal{C}$ encodes whether the morphisms in \mathcal{C} are invertible.

Example 5.1.8 (Automorphisms of $\text{Sd } D_3$). This example illustrates how the categorical structure of a subdivision $\text{Sd } \mathcal{C}$ might fail to distinguish between the composites $f \circ g$ and $g \circ f$ of endomorphisms f and g in \mathcal{C} .

Let r and s be generators for the six-element dihedral group D_3 .

$$D_3 = \langle r, s \mid r^3, s^2, rsrs \rangle$$

Let ϕ denote the automorphism $D_3 \rightarrow D_3$ that sends r to r^2 and s to s . Let α_{D_3} denote the map from Section 4 defined by $\langle f_m | \cdots | f_1 \rangle \mapsto \langle f_1^{-1} | \cdots | f_m^{-1} \rangle$. Below is a comparison of the maps α_{D_3} and $\text{Sd } \phi$ with the composite $\alpha_{D_3} \circ (\text{Sd } \phi)$.

$$\begin{array}{ccc} \alpha_{D_3} : \text{Sd} D_3 \rightarrow \text{Sd} D_3 & \text{Sd } \phi : \text{Sd} D_3 \rightarrow \text{Sd} D_3 & \alpha_{D_3} \circ \text{Sd } \phi : D_3 \rightarrow D_3 \\ \langle r \rangle \mapsto \langle r^{-1} \rangle & \langle r \rangle \mapsto \langle r^{-1} \rangle & \langle r \rangle \mapsto \langle r \rangle \\ \langle s \rangle \mapsto \langle s \rangle & \langle s \rangle \mapsto \langle s \rangle & \langle s \rangle \mapsto \langle s \rangle \\ \langle rs \rangle \mapsto \langle rs \rangle & \langle rs \rangle \mapsto \langle sr \rangle & \langle rs \rangle \mapsto \langle sr \rangle \end{array}$$

Note in particular that the composite $\alpha_{D_3} \circ \text{Sd } \phi$ sends r to r and s to s , but does not send rs to rs .

Given 1-simplices $\langle f \rangle$ and $\langle g \rangle$ corresponding to composable endomorphisms f and g in a groupoid \mathcal{G} , we will show in the following section that the categorical structure of $\text{Sd } \mathcal{G}$ allows to pick out the set $\{\langle f \circ g \rangle, \langle g \circ f \rangle\}$ of 1-simplices corresponding to the composites of $f \circ g$ and $g \circ f$. Of course, these composites may or may not be distinct.

6. $\text{Sd } \mathcal{G}$ ENCODES COMPOSITION IN \mathcal{G} UP TO OPPOSITES

This section builds on the previous one. As Lemma 6.0.2 will demonstrate, the assumption that the morphisms f and g are invertible will make it easier to count 2-simplices of the form $\frac{f \wedge g}{h}$. Therefore, we now restrict our attention from small categories to small groupoids. We will show how the categorical structure of $\text{Sd } \mathcal{G}$ determines which triples $(\langle f \rangle, \langle g \rangle, \langle h \rangle)$ satisfy one of the equations $f \circ g = h$ and $g \circ f = h$.

Definition 6.0.1. Let f, g be non-identity arrows in a groupoid \mathcal{G} , and let y be a 2-simplex in $\text{Sd } \mathcal{G}$. Say that y is a *filler* for the triangle $\frac{f \wedge g}{\text{---}}$ if y is of the form $\frac{f \wedge g}{\text{---}}$ or of the form $\frac{f \wedge g}{h}$ for some morphism h in \mathcal{G} . Such an h is called a *third side* of $\frac{f \wedge g}{\text{---}}$.

Because a fillers of $\frac{f \wedge g}{\text{---}}$ are objects in $\text{Sd } \mathcal{G}$, any such filler y must be non-degenerate. The third sides of a given triangle are classified as follows.

Lemma 6.0.2 (Possible third sides). *Let f and g be non-identity morphisms in a groupoid \mathcal{G} , and suppose that h is a third side of the triangle $\frac{f \wedge g}{\text{---}}$. Then h must equal one of the following six composites:*

$$(6.0.3) \quad f \circ g, \quad g \circ f, \quad f^{-1} \circ g, \quad g \circ f^{-1}, \quad f \circ g^{-1}, \quad g^{-1} \circ f.$$

Proof. Assuming that f and g are invertible, the composites (6.0.3) are the values of h corresponding to each of the cases in Definition 5.1.3. Explicitly, any filler of $\frac{f \wedge g}{\text{---}}$ must be one of the six diagrams [2] $\rightarrow \mathcal{G}$ displayed below:

$$\begin{array}{c|c|c|c|c|c} \begin{array}{c} f \swarrow \quad \nwarrow g \\ \longleftarrow fg \end{array} & \begin{array}{c} f \swarrow \quad \nwarrow g \\ \longrightarrow gf \end{array} & \begin{array}{c} f \swarrow \quad \nwarrow g \\ \longleftarrow f^{-1}g \end{array} & \begin{array}{c} f \swarrow \quad \nwarrow g \\ \longrightarrow gf^{-1} \end{array} & \begin{array}{c} f \swarrow \quad \nwarrow g \\ \longleftarrow fg^{-1} \end{array} & \begin{array}{c} f \swarrow \quad \nwarrow g \\ \longrightarrow g^{-1}f \end{array} \\ \langle f|g \rangle & \langle g|f \rangle & \langle f|f^{-1}g \rangle & \langle gf^{-1}|f \rangle & \langle fg^{-1}|g \rangle & \langle g|g^{-1}f \rangle \end{array}$$

□

Note that some or all of the formal composites (6.0.3) may be undefined, depending on how the domain and codomain of f and g match up.

6.1. Composites in Groupoids. Suppose that f and g are endomorphisms of an object in a groupoid \mathcal{G} , and that the composites $f \circ g$ and $g \circ f$ are distinct. The categorical structure of $\text{Sd } \mathcal{G}$ need not distinguish between the 1-simplices $\langle f \circ g \rangle$ and $\langle g \circ f \rangle$, as in Example 5.1.8 on page 16. It is possible, however, to pick the 1-simplices $\langle f \circ g \rangle$ and $\langle g \circ f \rangle$ out from among the other 1-simplices in $\text{Sd } \mathcal{G}$.

Generally, given non-identity morphisms f and g in \mathcal{G} satisfying $\text{dom } f = \text{cod } g$ or $\text{dom } g = \text{cod } f$, the local structure of $\text{Sd } \mathcal{G}$ near $\langle f \rangle$ and $\langle g \rangle$ determines whether a given 1-simplex $\langle h \rangle$ satisfies $h = f \circ g$ or $h = g \circ f$. This will be key in proving functoriality of the map ψ mentioned in the introduction.

To achieve this result, we introduce some terminology concerning relationships between arrows in \mathcal{G} . Each definition below can be formulated in terms of the proper faces $\{\langle \text{dom } f \rangle, \langle \text{cod } f \rangle\}$ of 1-simplices $\langle f \rangle$.

Definitions 6.1.1.

- (1) *End-to-end morphisms*: $\cdot \longrightarrow \cdot \longrightarrow \cdot$ or $\cdot \longrightarrow \cdot \longleftarrow \cdot$ or $\cdot \longleftarrow \cdot \longrightarrow \cdot$

Morphisms f and g are end-to-end if neither f nor g is an endomorphism and the intersection $\{\text{dom } f, \text{cod } f\} \cap \{\text{dom } g, \text{cod } g\}$ has one element. Note that this implies the three dots are necessarily distinct.

- (2) *Ends-to-ends morphisms*: $\cdot \longleftarrow \cdot \longrightarrow \cdot$ or $\cdot \longrightarrow \cdot \longleftarrow \cdot$

Morphisms f and g are ends-to-ends if neither f nor g is an endomorphism and there is equality $\{\text{dom } f, \text{cod } f\} = \{\text{dom } g, \text{cod } g\}$. This means that the intersection $\{\text{dom } f, \text{cod } f\} \cap \{\text{dom } g, \text{cod } g\}$ has two elements.

- (3) *End-to-endo morphisms*: $\cdot \longrightarrow \cdot \circlearrowright$ or $\cdot \longleftarrow \cdot \circlearrowright$

Morphisms f and g are end-to-endo if f is not an endomorphism, g is an endomorphism, and the intersection $\{\text{dom } f, \text{cod } f\} \cap \{\text{dom } g, \text{cod } g\}$ has one element.

- (4) *Endo-to-endo morphisms*: $\circlearrowleft \cdot \circlearrowright$

Morphisms f and g are endo-to-endo if they are both endomorphisms of a common object.

- (5) *Unrelated morphisms*

Morphisms f and g are unrelated if the sets $\{\text{dom } f, \text{cod } f\}$ and $\{\text{dom } g, \text{cod } g\}$ have no elements in common.

For example, non-endomorphisms f and g in \mathcal{G} are end-to-end if and only if $\langle f \rangle$ and $\langle g \rangle$ have one face in common. A non-identity morphism f is an endomorphism if and only if the 1-simplex $\langle f \rangle$ has exactly one proper face.

Definition 6.1.2. Let f and g be end-to-end morphisms. We say that f and g are *sequential* if $\text{dom } f = \text{cod } g$ or $\text{dom } g = \text{cod } f$. We say that f and g are *cointial* if $\text{dom } f = \text{dom } g$, and that f and g are *coterminal* if $\text{cod } f = \text{cod } g$.

Definition 6.1.3. Let f and g be ends-to-ends morphisms. We say that f and g are *parallel* if $\text{dom } f = \text{dom } g$ and $\text{cod } f = \text{cod } g$. We say that f and g are *opposed* if $\text{dom } f = \text{cod } g$ and $\text{dom } g = \text{cod } f$.

Observe that end-to-end morphisms are sequential if and only if they can be composed in some order. Similarly, ends-to-ends morphisms are opposed if and only if they can be composed in either order. Note that unrelated morphisms are never composable, and that end-to-endo and endo-to-endo morphisms are always composable. Below are criteria for the composability of end-to-end and ends-to-ends morphisms.

Proposition 6.1.4. *Let f and g be end-to-end morphisms in \mathcal{G} . There exists a unique filler for the triangle $\begin{array}{c} f \\ \diagdown \\ \text{---} \\ \diagup \\ g \end{array}$ if and only if f and g are sequential. There is more than one filler for $\begin{array}{c} f \\ \diagdown \\ \text{---} \\ \diagup \\ g \end{array}$ if and only if f and g are coinital or coterminal.*

Proof. It will suffice to show that if f and g are coinital or coterminal then there are exactly two fillers for $\begin{array}{c} f \\ \diagdown \\ \text{---} \\ \diagup \\ g \end{array}$, and that if f and g are sequential, then there is exactly one filler for $\begin{array}{c} f \\ \diagdown \\ \text{---} \\ \diagup \\ g \end{array}$.

Given distinct elements i and j of the set $\{0, 1, 2\}$, we will write $[i < j]$ for the morphism from $i \rightarrow j$ in the category $[2]$. If y is a 2-simplex in $\text{Sd}\mathcal{G}$, i.e. a functor $[2] \rightarrow \mathcal{G}$ that sends $[0 < 1]$ and $[1 < 2]$ to non-identity arrows, then:

- (1) the domain of $y[0 < 1]$ equals the domain of $y[0 < 2]$,
- (2) the codomain of $y[1 < 2]$ equals the codomain of $y[0 < 2]$, and
- (3) $y[1 < 2]$ and $y[0 < 1]$ are composable.

Given a 2-simplex y in $\text{Sd}\mathcal{G}$ that fills $\begin{array}{c} f \\ \diagdown \\ \text{---} \\ \diagup \\ g \end{array}$, we must have $f = y[i < j]$ and $g = y[k < l]$ for some distinct morphisms $[i < j]$ and $[k < l]$ in $[2]$. Functors preserve domain and codomain; by looking at the source and target of f and g , we can rule out combinations of i, j, k, l .

Suppose first that f and g are coinital. Then any filler y of $\begin{array}{c} f \\ \diagdown \\ \text{---} \\ \diagup \\ g \end{array}$ must satisfy either

- $f = y[0 < 1]$ and $g = y[0 < 2]$, or
- $f = y[0 < 2]$ and $g = y[0 < 1]$.

To see this, note that y sends coinital pairs in $[2]$ to coinital pairs in \mathcal{G} , and similarly for sequential and coterminal pairs. Write $f = y[i < j]$ and $g = y[k < l]$, and note that f and g are neither sequential nor coterminal. By contraposition, $[i < j]$ and $[k < l]$ are neither sequential nor coterminal. Thus the preimages of f and g must be $[0 < 1]$ and $[0 < 2]$.

If $f = y[0 < 1]$ and $g = y[0 < 2]$ then we must have $(y[1 < 2]) \circ f = g$, hence y is the 2-simplex $\langle gf^{-1} | f \rangle$. If $f = y[0 < 2]$ and $g = y[0 < 1]$ then we have $(y[1 < 2]) \circ g = f$, hence y is the 2-simplex $\langle fg^{-1} | g \rangle$. Thus, there are exactly two fillers for $\begin{array}{c} f \\ \diagdown \\ \text{---} \\ \diagup \\ g \end{array}$.

The argument is similar supposing that f and g are coterminal. Of all end-to-end morphism pairs in the category $[2]$, only $[1 < 2]$ and $[0 < 2]$ are neither coinital nor sequential. Therefore, if y fills $\begin{array}{c} f \\ \diagdown \\ \text{---} \\ \diagup \\ g \end{array}$ then we must have $f = y[1 < 2]$ and $g = y[0 < 2]$, or vice versa. These two possibilities correspond to the cases $y = \langle f | f^{-1}g \rangle$ and $y = \langle g | g^{-1}f \rangle$. The coterminal case is dual to the coinital case in a sense made precise by the isomorphism $\text{Sd}\mathcal{G} \rightarrow \text{Sd}(\mathcal{G}^{op})$ from Lemma 4.1, which sends each n -simplex $\langle f_1 | \dots | f_n \rangle$ in $\text{Sd}\mathcal{G}$ to the n -simplex $\langle f_n | \dots | f_1 \rangle$ in $\text{Sd}(\mathcal{G}^{op})$.

Finally, suppose that f and g are sequential. If $\text{dom } f = \text{cod } g$ and if y fills $\begin{array}{c} f \\ \diagdown \\ \text{---} \\ \diagup \\ g \end{array}$ then we have $f = y[1 < 2]$ and $g = y[0 < 1]$, hence y equals $\langle f | g \rangle$. Similarly, if $\text{dom } g = \text{cod } f$ and if y fills $\begin{array}{c} f \\ \diagdown \\ \text{---} \\ \diagup \\ g \end{array}$ then we have $g = y[1 < 2]$ and $f = y[0 < 1]$, hence y equals $\langle g | f \rangle$. In either case, there is only one possible filler y . \square

Note that the above result can fail if f and g are not invertible.

Corollary 6.1.5. *Let f and g be end-to-end morphisms in \mathcal{G} . Then f and g are sequential if and only if there is a unique third side of the triangle $\begin{array}{c} f \\ \diagdown \\ \text{---} \\ \diagup \\ g \end{array}$. This third side is necessarily equal to the composite of f and g .*

Proof. Suppose first that f and g are sequential. The previous proposition shows that there is a unique 2-simplex filler, and thus a unique third side, for the given triangle $\begin{array}{c} f \diagdown \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \diagup \\ g \end{array}$.

For the reverse implication, suppose that f and g are not sequential; we will show that there are two distinct third sides for the given triangle. Note that non-sequential end-to-end morphisms must be either cointial or coterminal.

Supposing first that f and g are cointial, we have two 2-simplex fillers for $\begin{array}{c} f \diagdown \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \diagup \\ g \end{array}$, namely $\langle gf^{-1}|f \rangle$ and $\langle fg^{-1}|g \rangle$. The corresponding third sides are gf^{-1} and fg^{-1} , which must be distinct because $\text{cod } f \neq \text{cod } g$.

Similarly, if f and g are coterminal then we have third sides $g^{-1}f$ and $f^{-1}g$ of $\begin{array}{c} f \diagdown \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \diagup \\ g \end{array}$. These third sides must be distinct because $\text{dom } g \neq \text{dom } f$. \square

The following result is analogous to the previous proposition, concerning ends-to-ends morphisms.

Proposition 6.1.6. *Let f and g be distinct ends-to-ends morphisms in \mathcal{G} . There are four fillers for $\begin{array}{c} f \diagdown \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \diagup \\ g \end{array}$ if and only if f and g are parallel. There are two fillers for the triangle $\begin{array}{c} f \diagdown \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \diagup \\ g \end{array}$ if and only if f and g are opposed.*

Proof. It will suffice to show that if f and g are parallel then there are exactly four fillers for $\begin{array}{c} f \diagdown \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \diagup \\ g \end{array}$, and that if f and g are opposed, then there are exactly two filler for $\begin{array}{c} f \diagdown \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \diagup \\ g \end{array}$. As in the previous proof, we will write $[i < j]$ for the morphism $i \rightarrow j$ in $[2]$.

Suppose first that f and g are parallel, and that y fills $\begin{array}{c} f \diagdown \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \diagup \\ g \end{array}$. Then we must have $f = y[i < j]$ and $g = y[k < l]$ for some distinct morphisms $[i < j]$ and $[k < l]$ in $[2]$. Moreover, the morphisms $[i < j]$ and $[k < l]$ cannot be composable because f and g are not composable. This means that $i \neq l$ and $j \neq k$. Therefore, if y fills $\begin{array}{c} f \diagdown \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \diagup \\ g \end{array}$ then there are four possibilities:

- (1) $f = y[0 < 1]$ and $g = y[0 < 2]$,
- (2) $g = y[0 < 1]$ and $f = y[0 < 2]$,
- (3) $f = y[0 < 2]$ and $g = y[1 < 2]$, or
- (4) $g = y[0 < 2]$ and $f = y[1 < 2]$.

Thus, there are at most four fillers.

The cases (1)-(4) above correspond (respectively) to the 2-simplices

$$y_1 = \langle gf^{-1}|f \rangle, \quad y_2 = \langle fg^{-1}|g \rangle, \quad y_3 = \langle f|f^{-1}g \rangle, \quad y_4 = \langle g|g^{-1}f \rangle.$$

To prove that there are exactly four fillers for $\begin{array}{c} f \diagdown \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \diagup \\ g \end{array}$, we must show that the above fillers y_i are all distinct. By looking at the values $y(0)$, $y(1)$, $y(2)$, we see that the only pairs among the fillers y_i that could be equal are y_1, y_2 and y_3, y_4 . We have $y_1 \neq y_2$ because

$$y_1[0 < 1] = f \neq g = y_2[0 < 1],$$

and similarly we have $y_3 \neq y_4$ because $y_3[1 < 2] \neq y_4[1 < 2]$. Thus, there are exactly four fillers for $\begin{array}{c} f \\ \diagdown \quad \diagup \\ \text{---} \quad \text{---} \\ g \end{array}$.

Suppose now that f and g are opposed. The 2-simplices $\langle f|g \rangle$ and $\langle g|f \rangle$ are of the form $\begin{array}{c} f \\ \diagdown \quad \diagup \\ \text{---} \quad \text{---} \\ g \end{array}$ and $\begin{array}{c} g \\ \diagdown \quad \diagup \\ \text{---} \quad \text{---} \\ f \end{array}$, respectively. We must show that these are the *only* two 2-simplices that fill $\begin{array}{c} f \\ \diagdown \quad \diagup \\ \text{---} \quad \text{---} \\ g \end{array}$. By the proof of Lemma 6.0.2, which lists all potential fillers of $\begin{array}{c} f \\ \diagdown \quad \diagup \\ \text{---} \quad \text{---} \\ g \end{array}$, any filler which is not equal to $\langle f|g \rangle$ or $\langle g|f \rangle$ must be equal to one of the four fillers

$$\langle gf^{-1}|f \rangle, \quad \langle fg^{-1}|g \rangle, \quad \langle f|f^{-1}g \rangle, \quad \langle g|g^{-1}f \rangle.$$

But f and g are opposed, so we have $\text{dom } f \neq \text{dom } g$ and $\text{cod } f \neq \text{cod } g$. Therefore none of the composites gf^{-1} , fg^{-1} , $f^{-1}g$, and $g^{-1}f$ are valid, hence none of the four fillers above are defined. It follows that there are exactly two fillers for $\begin{array}{c} f \\ \diagdown \quad \diagup \\ \text{---} \quad \text{---} \\ g \end{array}$. \square

Corollary 6.1.7. *If f and g are opposed ends-to-ends morphisms in \mathcal{G} , then there are exactly two third sides of $\begin{array}{c} f \\ \diagdown \quad \diagup \\ \text{---} \quad \text{---} \\ g \end{array}$, namely $f \circ g$ and $g \circ f$.*

Proposition 6.1.8. *Let f and g be end-to-endo morphisms in \mathcal{G} . Then h is the composite of f and g if and only if*

- (1) h and f are parallel ends-to-ends morphisms, and
- (2) h is a third side of the triangle $\begin{array}{c} f^{-1} \\ \diagdown \quad \diagup \\ \text{---} \quad \text{---} \\ g \end{array}$.

Proof. There are two cases: either $\text{cod } f = \text{dom } g = \text{cod } g$, or $\text{dom } f = \text{dom } g = \text{cod } g$. Assume first that $\text{cod } f = \text{dom } g = \text{cod } g$, as in the diagram to the right.

$$\begin{array}{c} f \\ \longrightarrow \circlearrowright g \end{array}$$

One implication is clear: the composite $g \circ f$ is parallel to f because g is an endomorphism, and if we set $h = g \circ f$ then the 2-simplex $\langle h|f^{-1} \rangle$ is witness to h being a third side of $\begin{array}{c} f^{-1} \\ \diagdown \quad \diagup \\ \text{---} \quad \text{---} \\ g \end{array}$.

For the reverse implication, suppose that h is a third side of $\begin{array}{c} f^{-1} \\ \diagdown \quad \diagup \\ \text{---} \quad \text{---} \\ g \end{array}$ and that h is parallel to f . Then h and f^{-1} are opposed ends-to-ends morphisms. By the definition of third sides, there must exist some non-degenerate 2-simplex of the form $\begin{array}{c} f^{-1} \\ \diagdown \quad \diagup \\ \text{---} \quad \text{---} \\ h \end{array}$ in $\text{Sd}\mathcal{G}$. Thus, g is a third side of the triangle $\begin{array}{c} f^{-1} \\ \diagdown \quad \diagup \\ \text{---} \quad \text{---} \\ h \end{array}$, and it follows from Corollary 6.1.7 above that g is equal to $f^{-1} \circ h$ or to $h \circ f^{-1}$. But g is an endomorphism of the object $\text{cod } f$, whereas $f^{-1} \circ h$ is an endomorphism of $\text{dom } f$. Because $\text{dom } f \neq \text{cod } f$, we must have $g = h \circ f^{-1}$ and therefore $g \circ f = h$.

If we suppose instead that $\text{dom } f = \text{dom } g = \text{cod } g$, then the proof follows by a similar argument. \square

The remainder of this section establishes a result analogous to Propositions 6.1.4-6.1.8, pertaining to the case where f and g are endo-to-endo. For the time being we will drop the composition symbol, writing (for example) fg for the composite $f \circ g$ and f^2 for the composite $f \circ f$. If f is self-inverse, then we will write $f^2 = \text{id}$. The lemma below concerns composites f^2 .

Lemma 6.1.9 (Square criterion). *Let f and h be non-identity endomorphisms in \mathcal{G} . Then $f^2 = h$ if and only if $\text{Sd}\mathcal{G}$ contains a non-degenerate 2-simplex of the form $\frac{f}{h} \triangleleft$.*

Proof. If $f^2 = h$ then the 2-simplex $\langle f|f \rangle$ is of the form $\frac{f}{h} \triangleleft$. Conversely, suppose that $y : [2] \rightarrow \mathcal{G}$ is of the form $\frac{f}{h} \triangleleft$. By Definition 5.1.3, which lists all possible 2-simplices of the form $\frac{f}{h} \triangleleft$, we must have $f^2 = h$ or $fh = f$ or $hf = f$. If $fh = f$ or if $hf = f$ then we have $h = \text{id}$, which contradicts our assumption that h is non-identity. \square

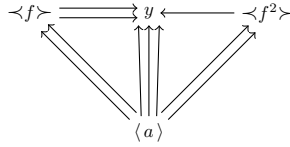


FIGURE 11. Let $f : a \rightarrow a$ be some endomorphism that is not self-inverse, and set $y = \langle f|f \rangle$. Then $\ulcorner y \urcorner$ is given by the diagram above.

Remark 6.1.10. Given a non-degenerate 2-simplex $y : [2] \rightarrow \mathcal{G}$, the category of faces $\ulcorner y \urcorner$ is given by one of the Figures 5 through 11. This is to say, these seven Figures classify the possible categories of faces of 2-simplices. This is true for groupoids, but not for arbitrary categories. For example, in an arbitrary category we might have $f^2 = f$ for some non-identity endomorphism f .

Notation 6.1.11. Given morphisms f , g , and h in \mathcal{G} , we will write $\exists \frac{f}{h} \triangleleft^g$ if there exists a non-degenerate 2-simplex of the form $\frac{f}{h} \triangleleft^g$ in $\text{Sd}\mathcal{G}$. The notation $\exists_n \frac{f}{h} \triangleleft^g$ means that there are exactly n distinct non-degenerate 2-simplices of that form. Similarly, $\nexists \frac{f}{h} \triangleleft^g$ means that no such 2-simplices exist, and $\exists_{\geq n} \frac{f}{h} \triangleleft^g$ means that there are at least n such 2-simplices.

Lemma 6.1.12. *Suppose f , g , h , and h' are endomorphisms in \mathcal{G} , the morphisms f , g , and h are distinct, and a 2-simplex y is simultaneously of the form $\frac{f}{h} \triangleleft^g$ and $\frac{f}{h'} \triangleleft^g$. Then $h = h'$.*

Proof. The morphism h' is one of the $y[i < j]$ for some $0 \leq i < j \leq 1$. Therefore h' equals f or g or h . If h' equals f or g , then for two distinct pairs $i < j$, $k < l$ we have $y[i < j] = y[k < l]$ since y is of the form $\frac{f}{h'} \triangleleft^g$. On the other hand, since y is of the form $\frac{f}{h} \triangleleft^g$ for f, g, h all distinct, this is impossible. Therefore $h = h'$. \square

Lemma 6.1.13. *Let f and h be non-identity endomorphisms in \mathcal{G} , and suppose that $f \neq f^{-1}$ and $f^2 \neq f^{-1}$. If $h = f^{-1}$, then $h = f^3$ if and only if $\exists_4 \frac{f}{h} \triangleleft^{f^2}$. If $h \neq f^{-1}$, then $h = f^3$ if and only if $\exists_2 \frac{f}{h} \triangleleft^{f^2}$.*

Proof. Because f is non-identity and $f \neq f^{-1}$ and $f^2 \neq f^{-1}$, the morphisms

$$\text{id}, \quad f, \quad f^2, \quad \text{and} \quad f^3$$

are all distinct. The cases $h = f^{-1}$ and $h \neq f^{-1}$ above correspond to whether or not f^4 equals id . There are five distinct fillers of $\frac{f}{\triangleleft _ _ _} \searrow^{f^2}$, displayed below:

$$\triangleleft f|f^2 \triangleright \quad \triangleleft f^2|f \triangleright \quad \triangleleft f|f \triangleright \quad \triangleleft f^{-1}|f^2 \triangleright \quad \triangleleft f^2|f^{-1} \triangleright.$$

These fillers are of the form $\frac{f}{\triangleleft _ _ _} \searrow^{f^2}$, $\frac{f}{\triangleleft _ _ _} \searrow^{f^2}$, $\frac{f}{\triangleleft _ _ _} \searrow^{f^2}$, $\frac{f}{\triangleleft _ _ _} \searrow^{f^2}$, and $\frac{f}{\triangleleft _ _ _} \searrow^{f^2}$ (respectively).

If $f^3 = h = f^{-1}$, then there are four distinct 2-simplices of the form $\frac{f}{\triangleleft _ _ _} \searrow^{f^2}$, namely $\triangleleft f|f^2 \triangleright$, $\triangleleft f^2|f \triangleright$, $\triangleleft f^{-1}|f^2 \triangleright$, and $\triangleleft f^2|f^{-1} \triangleright$. Conversely, if $h = f^{-1}$ and there are exactly four distinct 2-simplices of the form $\frac{f}{\triangleleft _ _ _} \searrow^{f^2}$, then at least one of these 2-simplices must be of the form $\frac{f}{\triangleleft _ _ _} \searrow^{f^2}$. This 2-simplex is simultaneously of the form $\frac{f}{\triangleleft _ _ _} \searrow^{f^2}$ and $\frac{f}{\triangleleft _ _ _} \searrow^{f^2}$, so it follows by the previous Lemma that $h = f^3$.

On the other hand, if $h = f^3 \neq f^{-1}$ then there are two distinct 2-simplices of the form $\frac{f}{\triangleleft _ _ _} \searrow^{f^2}$, namely $\triangleleft f|f^2 \triangleright$ and $\triangleleft f^2|f \triangleright$. Conversely, if $h \neq f^{-1}$ and there are exactly two distinct 2-simplices of the form $\frac{f}{\triangleleft _ _ _} \searrow^{f^2}$, then these 2-simplices must be equal to $\triangleleft f|f^2 \triangleright$ and $\triangleleft f^2|f \triangleright$ (for otherwise the previous Lemma would give $f = h = f^3$ or $h = f^{-1}$, contradicting our assumptions). The 2-simplices $\triangleleft f|f^2 \triangleright$ and $\triangleleft f^2|f \triangleright$ are simultaneously of the form $\frac{f}{\triangleleft _ _ _} \searrow^{f^2}$ and $\frac{f}{\triangleleft _ _ _} \searrow^{f^2}$, and it follows from the previous lemma that h equals h^3 . \square

The corollary below follows directly from the lemma above.

Corollary 6.1.14 (Cube criterion). *Let f and h be non-identity endomorphisms in \mathcal{G} , and suppose that $f \neq f^{-1}$ and $f^2 \neq f^{-1}$. Then $h = f^3$ if and only if*

$$\text{either } h = f^{-1} \text{ and } \exists_4 \frac{f}{\triangleleft _ _ _} \searrow^{f^2}, \text{ or } h \neq f^{-1} \text{ and } \exists_2 \frac{f}{\triangleleft _ _ _} \searrow^{f^2}.$$

Given a 1-simplex $\triangleleft f \triangleright$ in $\text{Sd } \mathcal{G}$ such that f is an endomorphism satisfying $f^2 \neq \text{id}$ and $f^3 \neq \text{id}$, the previous results can be used (for example) to find the 1-simplices $\triangleleft f^2 \triangleright$ and $\triangleleft f^3 \triangleright$.

Recall from Lemma 6.0.2 that if a given 2-simplex is of the form $\frac{f}{\triangleleft _ _ _} \searrow^g$, then h must be one of the composites

$$(6.1.15) \quad f \circ g, \quad g \circ f, \quad f^{-1} \circ g, \quad g \circ f^{-1}, \quad f \circ g^{-1}, \quad g^{-1} \circ f.$$

We shall now define notation to set the stage for Lemma 6.1.17, which will state that under certain conditions on f and g , the number of 2-simplices of the form $\frac{f/\backslash^g}{h}$ is equal to the number of composites above that are equal to h .

Notation 6.1.16. Let f and g be non-identity endomorphisms of some object in \mathcal{G} . Suppose that $f \neq g$. Write $C(f, g)$ for the set of quadruples (k, s, l, t) where

- (1) (k, l) is equal to (f, g) or (g, f) , and
- (2) (s, t) is equal to $(1, 1)$ or $(1, -1)$ or $(-1, 1)$.

The six elements of $C(f, g)$ are all distinct, whereas the six composites (6.1.15) might not all be distinct. The evaluation map $\text{ev} : C(f, g) \rightarrow \text{Mor}(\mathcal{G})$, defined by $\text{ev}(k, s, l, t) = k^s l^t$, gives a correspondence between the elements of $C(f, g)$ and the composites (6.1.15). Given this correspondence, we can think of $C(f, g)$ as a set of “formal composites”. We will later make use of the sets $C(f, g)$ to keep track of

The following Lemma shows that, if we assume f and g are endomorphisms satisfying $f^2 \neq g$ and $f \neq g^2$, there is a bijection between $C(f, g)$ and the set of 2-simplex fillers for $\frac{f/\backslash^g}{h}$, sending each formal composite (k, s, l, t) to a filler whose third side equals $k^s l^t$. We will later use the set $C(f, g)$ to simplify a counting argument that involves keeping track of the relationship between 2-simplex fillers and third sides.

Lemma 6.1.17. *Let f and g be non-identity endomorphisms of some object in \mathcal{G} . Suppose that $f \neq g$ and $f^2 \neq g$ and $f \neq g^2$. For any morphism h in \mathcal{G} , the number of quadruples $(k, s, l, t) \in C(f, g)$ satisfying $k^s \circ l^t = h$ is equal to the number of 2-simplices of the form $\frac{f/\backslash^g}{h}$.*

Proof. Note that for any non-identity f and g satisfying $f^2 \neq g$ and $f \neq g^2$ as above, if h is a third side for $\frac{f/\backslash^g}{h}$ then h must be distinct from f and g .

Every 2-simplex filler of $\frac{f/\backslash^g}{h}$ must be one of the six 2-simplices

$$\langle f|g \rangle, \quad \langle g|f \rangle, \quad \langle f|f^{-1}g \rangle, \quad \langle gf^{-1}|f \rangle, \quad \langle fg^{-1}|g \rangle, \quad \langle g|g^{-1}f \rangle$$

displayed in the proof of Lemma 6.0.2. Given the present assumptions on f and g , these 2-simplices are all distinct. For example, $\langle f|g \rangle$ is distinct from $\langle f|f^{-1}g \rangle$ because g^{-1} is a non-identity morphism, and $\langle f|f^{-1}g \rangle$ is distinct from $\langle gf^{-1}|f \rangle$ because $f^2 \neq g$. Thus we have a bijection

$$\begin{array}{ll} (f, 1, g, 1) \mapsto \langle f|g \rangle & (g, 1, f, 1) \mapsto \langle g|f \rangle \\ (f, -1, g, 1) \mapsto \langle f|f^{-1}g \rangle & (g, -1, f, 1) \mapsto \langle g|g^{-1}f \rangle \\ (f, 1, g, -1) \mapsto \langle fg^{-1}|g \rangle & (g, 1, f, -1) \mapsto \langle gf^{-1}|f \rangle \end{array}$$

between $C(f, g)$ and the set of 2-simplex fillers for $\frac{f/\backslash^g}{h}$. We will let

$$\zeta : C(f, g) \rightarrow \{ \text{2-simplex fillers for } \frac{f/\backslash^g}{h} \}$$

denote this bijection. Note that ζ sends each quadruple (k, s, l, t) to a 2-simplex of the form

$$\frac{f \swarrow \searrow g}{k^s \circ l^t}.$$

The map ζ restricts to an injection

$$\zeta|_h : \{\gamma \in C(f, g) \mid \text{ev}(\gamma) = h\} \rightarrow \{2\text{-simplices of the form } \frac{f \swarrow \searrow g}{h}\},$$

so there are at least as many 2-simplices of the form $\frac{f \swarrow \searrow g}{h}$ as there are quadruples (k, s, l, t) satisfying $k^s \circ l^t = h$. Because ζ is a bijection, we can prove that $\zeta|_h$ is a surjection by noting $\text{ev}(\gamma) = h$ whenever $\zeta(\gamma)$ is of the form $\frac{f \swarrow \searrow g}{h}$. Indeed, $\zeta(\gamma)$ is of the form $\frac{f \swarrow \searrow g}{\text{ev}(\gamma)}$, so if $\zeta(\gamma)$ is also of the form $\frac{f \swarrow \searrow g}{h}$ then equality $\text{ev}(\gamma) = h$ follows from Lemma 6.1.12. \square

We now have a way to keep track of 2-simplices of the form $\frac{f \swarrow \searrow g}{h}$ by using formal composites of f , g , f^{-1} , and g^{-1} . We will later define an equivalence relation on formal composites by

$$(k, s, l, t) \sim (k', s', l', t') \iff \text{ev}(k, s, l, t) = \text{ev}(k', s', l', t').$$

This equivalence relation gives a graph structure on the set $C(f, g)$, with edges between equivalent elements. Such graphs will be used to make easier the computations that underlie the proofs of Propositions 6.1.19 and 6.1.20 below. These proofs, found in Appendix A, are the combinatorial heart of this paper.

Under the assumption that f and g satisfy

$$f \neq g, \quad f \neq g^{-1}, \quad f^2 \neq g, \quad \text{and} \quad f \neq g^2,$$

Proposition 6.1.19 below gives conditions that are necessary and sufficient for commutativity $fg = gf$. Assuming that

$$(6.1.18) \quad f \neq g, \quad f \neq g^{-1}, \quad f^2 \neq g, \quad f \neq g^2, \quad f^2 \neq g^{-1}, \quad f^{-1} \neq g^2,$$

Proposition 6.1.20 establishes criteria necessary and sufficient for a given endomorphism h to satisfy one (or both) of the equations $h = fg$ and $h = gf$.

As usual, we aim to encode relationship among f , g , and h in terms of the structure of the category $\text{Sd}\mathcal{G}$ in a neighborhood of its objects $\langle f \rangle$, $\langle g \rangle$, and $\langle h \rangle$.

Proposition 6.1.19 (Commutativity criterion). *Let f and g be non-identity endomorphisms in \mathcal{G} satisfying $f \neq g$ and $f \neq g^{-1}$ and $f^2 \neq g$ and $f \neq g^2$. Then fg equals gf if and only if for every every third side h of the triangle $\frac{f \swarrow \searrow g}{h}$ there are an even number of 2-simplices of the form $\frac{f \swarrow \searrow g}{h}$.*

To prove the above, we define a graph $G(f, g)$ whose vertices are the formal composites $C(f, g)$; the graph is defined to have edge between distinct formal composites γ_1 and γ_2 whenever $\text{ev}(\gamma_1)$ equals $\text{ev}(\gamma_2)$. By Lemma 6.1.17, the number of 2-simplices of the form $\frac{f \swarrow \searrow g}{h}$ is equal to the size of the connected component of $G(f, g)$ whose elements (k, s, l, t) satisfy $h = k^s \circ l^t$. By a combinatorial argument,

we show that fg equals gf if and only if every connected component of $G(f, g)$ has even cardinality. A full proof is in Appendix A.

Proposition 6.1.20. *Let f and g be non-identity endomorphisms of some object in \mathcal{G} satisfying $f \neq g$ and $f \neq g^{-1}$ and $f^2 \neq g$ and $f \neq g^2$ and $f^2 \neq g^{-1}$ and $f^{-1} \neq g^2$. Let h be another non-identity endomorphism of the same object in \mathcal{G} . The cases below give criteria under which h is equal to fg or to gf . Cases 1-2 apply when $f^2 = g^2$, and cases 3-4 apply when $f^2 \neq g^2$. All possibilities are exhausted.*

Case 1: Suppose that $f^2 = \text{id} = g^2$.

- If $fg = gf$, then $h = fg = gf$ if and only if $\exists_6 \frac{f \wedge g}{h}$.
- If $fg \neq gf$, then h equals fg or gf if and only if $\exists_3 \frac{f \wedge g}{h}$.

Case 2: Suppose that $f^2 = g^2$ and $f^2 \neq \text{id}$ and $g^2 \neq \text{id}$.

- If $fg = gf$, then $h = fg = gf$ if and only if $\exists_2 \frac{f \wedge g}{h}$.
- If $fg \neq gf$, then h equals fg or gf if and only if $\exists_1 \frac{f \wedge g}{h}$ or $\exists_3 \frac{f \wedge g}{h}$.

Case 3: Suppose that $f^2 \neq g^2$, and either $f^2 = \text{id}$ or $g^2 = \text{id}$.

- If $fg = gf$, then $h = fg = gf$ if and only if $\exists_4 \frac{f \wedge g}{h}$.
- If $fg \neq gf$, then h equals fg or gf if and only if $\exists_2 \frac{f \wedge g}{h}$ or $\exists_3 \frac{f \wedge g}{h}$.

Case 4: Suppose that $f^2 \neq g^2$ and $f^2 \neq \text{id}$ and $g^2 \neq \text{id}$.

- If $fg = gf$, then $h = fg = gf$ if and only if

$$\exists \frac{f \wedge g}{h} \quad \text{and} \quad \exists \frac{f^{-1} \wedge g}{h} \quad \text{and} \quad \exists \frac{f \wedge g^{-1}}{h}.$$

- If $fg \neq gf$ and $f^2 = g^{-2}$, then h equals fg or gf if and only if $\exists_2 \frac{f^{-1} \wedge g}{h}$ and $\exists_2 \frac{f \wedge g^{-1}}{h}$.
- If $fg \neq gf$ and $f^2 \neq g^{-2}$, then h equals fg or gf if and only if one of the following is satisfied:

$$(1) \exists_1 \frac{f \wedge g}{h} \quad \text{and} \quad \exists_1 \frac{f^{-1} \wedge g}{h} \quad \text{and} \quad \exists_1 \frac{f \wedge g^{-1}}{h}, \text{ or}$$

$$(2) \exists_1 \frac{f^{-1} \wedge g^{-1}}{h} \quad \text{and two of the following three existential statements hold:}$$

$$\exists_2 \frac{f \wedge g}{h}, \quad \exists_2 \frac{f^{-1} \wedge g}{h}, \quad \exists_2 \frac{f \wedge g^{-1}}{h}.$$

See Appendix A for proof.

7. CONSTRUCTION OF ψ AND PROOF OF THE MAIN THEOREM

This section describes how to construct an isomorphism $\mathcal{G} \rightarrow \mathcal{H}$ given an isomorphism $\text{Sd } \mathcal{B} \rightarrow \text{Sd } \mathcal{C}$.

Notation. For convenience, we will write $\text{simp}_m(\text{Sd } \mathcal{C})$ for the set of non-degenerate m -simplices $[m] \rightarrow \mathcal{C}$, regarded as objects in $\text{Sd } \mathcal{C}$.

This section's results are predicated on the fact that any isomorphism $\Psi : \text{Sd } \mathcal{B} \rightarrow \text{Sd } \mathcal{C}$ sends n -simplices to n -simplices (Proposition 7.1.2), so we have induced maps

$$\text{Ob}(\mathcal{B}) \xrightarrow{\cong} \text{simp}_0(\text{Sd } \mathcal{B}) \xrightarrow{\Psi} \text{simp}_0(\text{Sd } \mathcal{C}) \xrightarrow{\cong} \text{Ob}(\mathcal{C})$$

and

$$\mathcal{B}_1 \xrightarrow{\cong} \text{simp}_1(\text{Sd } \mathcal{B}) \xrightarrow{\Psi} \text{simp}_1(\text{Sd } \mathcal{C}) \xrightarrow{\cong} \mathcal{C}_1,$$

where \mathcal{B}_1 denotes the set of non-identity morphisms in \mathcal{B} , and similarly for \mathcal{C}_1 . Together these maps determine an isomorphism ψ between the undirected graphs that underlie \mathcal{B} and \mathcal{C} .

We will show that the map ψ is “well behaved” if \mathcal{B} and \mathcal{C} are groupoids: its restriction to any connected component is a (possibly contravariant) isomorphism of categories. This will be proved in two steps: we first consider groups, that is, groupoids with one object (Subsection 7.3), and then prove an analogous result for connected groupoids that have multiple objects (in Subsection 7.4).

In the general (non-connected) case, we can selectively change the variance of ψ on those connected components where it was originally contravariant, obtaining a map that is covariant everywhere. To do this, we use the fact that any groupoid is isomorphic to its opposite.

7.1. Defining an isomorphism $\mathcal{B} \rightarrow \mathcal{C}$ of undirected graphs.

Lemma 7.1.1. *Let y be an object of $\text{Sd } \mathcal{B}$. Any isomorphism $\Psi : \text{Sd } \mathcal{B} \xrightarrow{\cong} \text{Sd } \mathcal{C}$ sends the proper faces of y bijectively to the proper faces of Ψy .*

Proof. Let x be distinct from y . Then x is a proper face of y if and only if the hom-set $\text{Sd } \mathcal{B}(x, y)$ is non-empty. Given the bijection $\text{Sd } \mathcal{B}(x, y) \cong \text{Sd } \mathcal{C}(\Psi x, \Psi y)$ induced by Ψ , we see that x is a proper face of y if and only if Ψx is a proper face of Ψy . \square

It is a corollary that any isomorphism $\Psi : \text{Sd } \mathcal{B} \rightarrow \text{Sd } \mathcal{C}$ restricts to an isomorphism $\lceil y \rceil \cong \lceil \Psi y \rceil$ for each object y of $\text{Sd } \mathcal{B}$. This follows because a category of faces is a full subcategory.

Proposition 7.1.2. *Any isomorphism $\Psi : \text{Sd } \mathcal{B} \rightarrow \text{Sd } \mathcal{C}$ restricts to a bijection $\text{simp}_m(\text{Sd } \mathcal{B}) \cong \text{simp}_m(\text{Sd } \mathcal{C})$ for each m .*

Proof. Suppose that y is an object of $\text{Sd } \mathcal{B}$. We shall prove that y is an m -simplex only if Ψy is an m -simplex. For each x there is a bijection

$$(7.1.3) \quad \text{Sd } \mathcal{B}(x, y) \cong \text{Sd } \mathcal{C}(\Psi x, \Psi y)$$

induced by Ψ , and therefore we have bijection

$$\text{mt}(y) = \coprod_x \text{Sd } \mathcal{B}(x, y) \cong \coprod_{\Psi x} \text{Sd } \mathcal{C}(\Psi x, \Psi y) = \text{mt}(\Psi y).$$

Recall from Lemma 5.0.6 that y is an m -simplex if and only if the set $\text{mt}(y)$ has $2^{m+1} - 1$ elements. The desired result follows from the equality $|\text{mt}(y)| = |\text{mt}(\Psi y)|$, demonstrating that y has the same dimension as Ψy . \square

Corollary 7.1.4. *If $\Psi : \text{Sd } \mathcal{B} \rightarrow \text{Sd } \mathcal{C}$ is an isomorphism and y is an object of $\text{Sd } \mathcal{B}$, then for each n there is a bijection $\text{mt}_n(y) \cong \text{mt}_n(\Psi y)$ induced by Ψ .*

Construction 7.1.5. Let $\Psi : \text{Sd } \mathcal{B} \rightarrow \text{Sd } \mathcal{C}$ be any isomorphism. Write \mathcal{B}_1 for the set of non-identity morphisms of \mathcal{B} , and define \mathcal{C}_1 similarly. There is a unique map $\psi : \mathcal{B} \rightarrow \mathcal{C}$ whose components make the following diagram commute:

$$\begin{array}{ccccccc} \text{simp}_0(\text{Sd } \mathcal{B}) & \xrightarrow{\langle b \rangle \mapsto b} & \text{Ob } \mathcal{B} & \xrightarrow{\text{id}} & \text{Mor } \mathcal{B} & \xleftarrow{\mathcal{B}_1} & \text{simp}_1(\text{Sd } \mathcal{B}) \\ \downarrow \Psi & & \downarrow \text{Ob}(\psi) & & \downarrow \text{Mor}(\psi) & & \downarrow \Psi \\ \text{simp}_0(\text{Sd } \mathcal{C}) & \xrightarrow{\langle c \rangle \mapsto c} & \text{Ob } \mathcal{C} & \xrightarrow{\text{id}} & \text{Mor } \mathcal{C} & \xleftarrow{\mathcal{C}_1} & \text{simp}_1(\text{Sd } \mathcal{C}) \end{array}$$

The maps $\text{simp}_0(\text{Sd } \mathcal{B}) \rightarrow \text{Ob } \mathcal{B}$ and $\text{simp}_0(\text{Sd } \mathcal{C}) \rightarrow \text{Ob } \mathcal{C}$ above are the bijections sending 0-simplices $\langle a \rangle$ to objects a . Similarly, the maps $\text{simp}_1(\text{Sd } \mathcal{B}) \rightarrow \mathcal{B}_1$ and $\text{simp}_1(\text{Sd } \mathcal{C}) \rightarrow \mathcal{C}_1$ are the bijections sending 1-simplices $\prec f \succ$ to non-identity morphisms f . Note that $\text{Mor } \mathcal{B}$ and $\text{Mor } \mathcal{C}$ are isomorphic to $\text{Ob } \mathcal{B} \amalg \mathcal{B}_1$ and to $\text{Ob } \mathcal{C} \amalg \mathcal{C}_1$, respectively. The components of ψ are uniquely determined by the above commutative diagram; note that ψ might fail to be functorial. This map ψ has the following properties:

- For any object b in \mathcal{B} , we have $\Psi \langle b \rangle = \langle \psi b \rangle$ and $\psi(\text{id}_b) = \text{id}_{\psi b}$.
- For any non-identity morphism f in \mathcal{B} , we have $\Psi \prec f \succ = \prec \psi f \succ$.

The following result shows that this map ψ is an isomorphism between the undirected graphs that underlie \mathcal{B} and \mathcal{C} .

Lemma 7.1.6. *Let $\Psi : \text{Sd } \mathcal{B} \rightarrow \text{Sd } \mathcal{C}$ be an isomorphism. Then ψ is bijective on objects and arrows, satisfying*

$$\{\psi(\text{dom } f), \psi(\text{cod } f)\} = \{\text{dom}(\psi f), \text{cod}(\psi f)\}$$

for each morphism f of \mathcal{B} .

Proof. For any object b in \mathcal{B} we have

$$\begin{aligned} \{\psi(\text{dom } \text{id}_b), \psi(\text{cod } \text{id}_b)\} &= \{\psi b, \psi b\} \\ &= \{\text{dom}(\text{id}_{\psi b}), \text{cod}(\text{id}_{\psi b})\} = \{\text{dom}(\psi \text{id}_b), \text{cod}(\psi \text{id}_b)\}, \end{aligned}$$

so our result is true for identity morphisms.

Suppose that f is a non-identity morphism. By Lemma 5.0.9, the proper faces of $\prec f \succ$ are $\langle \text{dom } f \rangle$ and $\langle \text{cod } f \rangle$, and the proper faces of $\prec \psi f \succ$ are $\langle \text{dom}(\psi f) \rangle$ and $\langle \text{cod}(\psi f) \rangle$. We have $\Psi \prec f \succ = \prec \psi f \succ$. By Lemma 7.1.1, the isomorphism Ψ sends proper faces of $\prec f \succ$ bijectively to proper faces of $\prec \psi f \succ$, therefore

$$\{\langle \psi(\text{dom } f) \rangle, \langle \psi(\text{cod } f) \rangle\} = \{\Psi \langle \text{dom } f \rangle, \Psi \langle \text{cod } f \rangle\} = \{\langle \text{dom}(\psi f) \rangle, \langle \text{cod}(\psi f) \rangle\}. \quad \square$$

If $\Phi : \text{Sd } \mathcal{C} \rightarrow \text{Sd } \mathcal{B}$ is the inverse isomorphism to Ψ , and if we write ϕ for the map $\mathcal{C} \rightarrow \mathcal{B}$ obtained from Φ as per Construction 7.1.5, then ψ is inverse to ϕ (when these maps are regarded as morphisms of graphs).

It is now possible to prove that the restriction of Sd to the category of small groupoids is conservative. Suppose that $F : \mathcal{G} \rightarrow \mathcal{H}$ is any functor between groupoids, write Ψ for the functor $\text{Sd}F : \text{Sd}\mathcal{G} \rightarrow \text{Sd}\mathcal{H}$, and suppose that Ψ is an isomorphism. The induced map $\psi : \mathcal{G} \rightarrow \mathcal{H}$ satisfies $\langle \psi a \rangle = \Psi \langle a \rangle = \langle F a \rangle$ and $\langle \psi g \rangle = \Psi \langle g \rangle = \langle F g \rangle$ for each object a and non-identity morphism g in \mathcal{G} , hence we have $\psi a = F a$ and $\psi g = F g$ by injectivity of $\langle - \rangle$ and $\langle - \rangle$. Thus, we have $\psi a = F a$ and $\psi g = F g$ for all objects a and morphisms g of \mathcal{G} .

By construction, we have $\psi \text{id}_a = \text{id}_{\psi a} = \text{id}_{F a} = F \text{id}_a$ for each object a . Because F is equal to ψ on each object and morphism in \mathcal{G} , and because ψ is bijective on objects and morphisms, F must be an isomorphism.

7.2. Encoding \mathcal{B} with $\text{Sd}\mathcal{B}$, decoding \mathcal{C} from $\text{Sd}\mathcal{C}$. Suppose that some property of \mathcal{B} , e.g. invertibility of morphisms, is encoded by the subdivision $\text{Sd}\mathcal{B}$. Assuming as before that $\Psi : \text{Sd}\mathcal{B} \rightarrow \text{Sd}\mathcal{C}$ is an isomorphism, we can show that $\text{Sd}\mathcal{C}$ encodes the same property in \mathcal{C} . This style of argument will be used again and again: the point of sections 5 and 6 was to encode properties of a given category so that they could be translated to another category by use of an isomorphism between the categories' subdivisions.

The following result shows that ψ assigns morphisms in \mathcal{B} to morphisms in \mathcal{C} in a way that respects faces of 2-simplices.

Proposition 7.2.1. *Let $\Psi : \text{Sd}\mathcal{B} \rightarrow \text{Sd}\mathcal{C}$ be an isomorphism. Then Ψ sends 2-simplices of the form $\frac{f \wedge g}{h}$ bijectively to 2-simplices of the form $\frac{\psi f \wedge \psi g}{\psi h}$.*

Proof. Because Ψ is invertible, it will be good enough to show that each 2-simplex of the form $\frac{f \wedge g}{h}$ is sent by Ψ to a 2-simplex of the form $\frac{\psi f \wedge \psi g}{\psi h}$. The idea of proof is that the form of a 2-simplex y is determined by the set $\text{mt}_1(y)$ of morphisms having target y and source equal to some 1-simplex.

By Lemma 5.1.5, y is of the form $\frac{f \wedge g}{h}$ (for some non-identity morphisms f , g , and h) if and only if there are three morphisms in the set $\text{mt}_1(y)$, and these morphisms have respective domains $\langle f \rangle$, $\langle g \rangle$, and $\langle h \rangle$. The map Ψ gives a bijection

$$\text{Sd}\mathcal{B}(\langle f \rangle, y) \xrightarrow{\cong} \text{Sd}\mathcal{C}(\Psi \langle f \rangle, \Psi y) = \text{Sd}\mathcal{C}(\langle \psi f \rangle, \Psi y),$$

and similar bijections exist for g and h . Therefore, Ψ sends morphisms in $\text{mt}_1(y)$ bijectively to morphisms in $\text{mt}_1(\Psi y)$, which have respective domains $\langle \psi f \rangle$, $\langle \psi g \rangle$, and $\langle \psi h \rangle$. It follows that y is of the form $\frac{\psi f \wedge \psi g}{\psi h}$.

Implicit in this line of reasoning is, for example, the claim that if y is of the form $\frac{f \wedge f}{h}$ for some distinct non-identity arrows f and h , then we have

$$|\text{Sd}\mathcal{B}(\langle f \rangle, y)| = 2 = |\text{Sd}\mathcal{C}(\langle \psi f \rangle, \Psi y)|$$

and

$$|\text{Sd}\mathcal{B}(\langle h \rangle, y)| = 1 = |\text{Sd}\mathcal{C}(\langle \psi h \rangle, \Psi y)|,$$

hence Ψy is of the form $\frac{\psi f / \triangle \psi f}{\psi h}$. Generally, it is important to keep track of how many arrows in $\text{mt}_1(y)$ have domain equal to each given 1-simplex.

By Lemma 5.1.5, a 2-simplex y is of the form $\frac{f / \triangle g}{\text{id}}$ if and only if there are two morphisms in $\text{mt}_1(y)$, having respective domains $\langle f \rangle$ and $\langle g \rangle$. By the same argument as above, the elements of $\text{mt}_1(y)$ are sent bijectively to those of $\text{mt}_1(\Psi y)$, hence Ψy is of the form $\frac{\psi f / \triangle \psi g}{\text{id}}$. \square

Corollary 7.2.2. *Let f and g be morphisms in \mathcal{B} , and let $\Psi : \text{Sd } \mathcal{B} \rightarrow \text{Sd } \mathcal{C}$ be an isomorphism. The fillers of $\frac{f / \triangle g}{\text{---}}$ are sent bijectively by Ψ to the fillers of $\frac{\psi f / \triangle \psi g}{\text{---}}$. Similarly, the third sides of $\frac{f / \triangle g}{\text{---}}$ are sent bijectively by ψ to the third sides of $\frac{\psi f / \triangle \psi g}{\text{---}}$.*

Proof. By invertibility of Ψ and ψ , it suffices to show that each filler or third side of $\frac{f / \triangle g}{\text{---}}$ is sent to a filler or third side of $\frac{\psi f / \triangle \psi g}{\text{---}}$.

If y is a filler for $\frac{f / \triangle g}{\text{---}}$, then y is of the form $\frac{f / \triangle g}{h}$ for some h . The 2-simplex Ψy is of then of the form $\frac{\psi f / \triangle \psi g}{\psi h}$, so Ψy is a filler for $\frac{\psi f / \triangle \psi g}{\text{---}}$.

If h is a third side of the triangle $\frac{f / \triangle g}{\text{---}}$, then there exists some 2-simplex y of the form $\frac{f / \triangle g}{h}$. Because Ψy is of the form $\frac{\psi f / \triangle \psi g}{\psi h}$, the morphism ψh is a third side of the triangle $\frac{\psi f / \triangle \psi g}{\text{---}}$. \square

The result below follows from the fact that a graph isomorphism preserves relationships between edges.

Proposition 7.2.3. *Let $\Psi : \text{Sd } \mathcal{B} \rightarrow \text{Sd } \mathcal{C}$ be an isomorphism. An arrow f of \mathcal{B} is an endomorphism if and only if ψf is an endomorphism. Arrows f and g in \mathcal{B} are end-to-end if and only if ψf and ψg are end-to-end, and similarly for ends-to-ends, end-to-endo, endo-to-endo, and unrelated pairs of morphisms f, g . In the end-to-endo case, ψ preserves which morphism is the endomorphism.*

Proof. Because ψ is an isomorphism of graphs, we have a set bijection

$$\{\text{dom } f, \text{cod } f\} \cong \{\text{dom}(\psi f), \text{cod}(\psi f)\}.$$

Thus, f is an endomorphism if and only if $|\{\text{dom } f, \text{cod } f\}| = 1$, if and only if $|\{\text{dom}(\psi f), \text{cod}(\psi f)\}| = 1$, if and only if ψf is an endomorphism.

Because it is bijective, ψ commutes with \cap . Therefore, we have a set bijection

$$\{\text{dom } f, \text{cod } f\} \cap \{\text{dom } g, \text{cod } g\} \cong \{\text{dom}(\psi f), \text{cod}(\psi f)\} \cap \{\text{dom}(\psi g), \text{cod}(\psi g)\}.$$

The cardinalities of these sets determine the relationship between f and g (in the sense made precise by the Definitions 6.1.1 of end-to-end, ends-to-ends, end-to-endo, endo-to-endo, and unrelated morphisms).

For example, f and g are end-to-end if and only if neither f nor g is an endomorphism and the intersection $\{\text{dom } f, \text{cod } f\} \cap \{\text{dom } g, \text{cod } g\}$ has one element. This

holds if and only if ψf and ψg are non-endomorphisms satisfying

$$|\{\text{dom}(\psi f), \text{cod}(\psi f)\} \cap \{\text{dom}(\psi g), \text{cod}(\psi g)\}| = 1.$$

The other cases are similar. \square

Proposition 7.2.4. *Let $\Psi : \text{Sd } \mathcal{B} \rightarrow \text{Sd } \mathcal{C}$ be an isomorphism, and let f be a morphism of \mathcal{B} . Then f is invertible if and only if ψf is invertible. If f is invertible then $\psi(f^{-1})$ is equal to $(\psi f)^{-1}$.*

Proof. By construction, ψ is bijective on identity arrows. Therefore, f is an identity morphism in \mathcal{B} if and only if ψf is an identity morphism in \mathcal{C} .

Now, suppose that f is a non-identity morphism. The following are equivalent:

- (1) $f^2 = \text{id}$
- (2) There is a 2-simplex in $\text{Sd } \mathcal{B}$ of the form $\frac{f \wedge f}{\text{id}}$
- (3) There is a 2-simplex in $\text{Sd } \mathcal{C}$ of the form $\frac{\psi f \wedge \psi f}{\text{id}}$
- (4) $(\psi f)^2 = \text{id}$.

Equivalence of points (2) and (3) above follows from Proposition 7.2.1. Equivalence of points (1) and (2) and of points (3) and (4) follow from Lemma 5.1.7. Thus, we have equality $\psi(f^{-1}) = \psi f = (\psi f)^{-1}$.

Finally, supposing that f and g are distinct non-identity morphisms in \mathcal{B} , the following are equivalent:

- (1) $f = g^{-1}$
- (2) There are two 2-simplices in $\text{Sd } \mathcal{B}$ of the form $\frac{f \wedge g}{\text{id}}$
- (3) There are two 2-simplices in $\text{Sd } \mathcal{C}$ of the form $\frac{\psi f \wedge \psi g}{\text{id}}$
- (4) $\psi f = (\psi g)^{-1}$.

As before, equivalence of (2) and (3) follows from Proposition 7.2.1, and Lemma 5.1.7 gives the other equivalences. We conclude that $\psi(f^{-1}) = \psi g = (\psi f)^{-1}$. \square

Corollary 7.2.5. *Let \mathcal{G} be a groupoid and let \mathcal{C} be a category. If $\text{Sd } \mathcal{G}$ is isomorphic to $\text{Sd } \mathcal{C}$ then \mathcal{C} is a groupoid.*

Proof. Take an isomorphism $\Psi : \text{Sd } \mathcal{G} \rightarrow \text{Sd } \mathcal{C}$, and write ψ for the induced isomorphism of undirected graphs. Every morphism g of \mathcal{G} is invertible, so every morphism ψg of \mathcal{C} is invertible, having inverse equal to $\psi(g^{-1})$. \square

The proof of Proposition 7.2.4 uses the fact that 2-simplices of the form $\frac{f \wedge g}{\text{id}}$ encode whether f and g are inverse morphisms. An analogous statement about ψf and ψg is derived by considering 2-simplices of the form $\frac{\psi f \wedge \psi g}{\text{id}}$. Generally, the relationship between morphisms f and g is determined by the 2-simplex fillers for $\frac{f \wedge g}{\text{id}}$, as well as by the fillers for $\frac{f^{-1} \wedge g}{\text{id}}$ and $\frac{f \wedge g^{-1}}{\text{id}}$ and $\frac{f^{-1} \wedge g^{-1}}{\text{id}}$.

We now restrict our attention to groupoids. The construction below concerns a map ψ' that satisfies $\text{dom}(\psi f) = \text{cod}(\psi' f)$ and $\text{cod}(\psi f) = \text{dom}(\psi' f)$ for each morphism f . This will be useful later, as there is no a priori guarantee that the map ψ should be covariant.

Construction 7.2.6. Let \mathcal{G} and \mathcal{H} be groupoids and let $\Psi : \text{Sd } \mathcal{G} \rightarrow \text{Sd } \mathcal{H}$ be an isomorphism. Write $\Psi' : \text{Sd } \mathcal{G} \rightarrow \text{Sd } \mathcal{H}$ for the composite $\Psi \circ \alpha_{\mathcal{G}}$, where $\alpha_{\mathcal{G}} : \text{Sd } \mathcal{G} \rightarrow \text{Sd } \mathcal{G}$ is the map $\langle f_m | \cdots | f_1 \rangle \mapsto \langle f_1^{-1} | \cdots | f_m^{-1} \rangle$ defined in Section 4. Write ψ' for the map obtained from Ψ' per Construction 7.1.5.

Proposition 7.2.7. *Define ψ' as above. We have $\psi' a = \psi a$ for each object a of \mathcal{G} , and $\psi' f = \psi f^{-1}$ for each morphism f of \mathcal{G} .*

Proof. We have

$$\langle \psi' a \rangle = \Psi' \langle a \rangle = \Psi \langle a \rangle = \langle \psi a \rangle$$

and

$$\langle \psi' f \rangle = \Psi' \langle f \rangle = \Psi \langle f^{-1} \rangle = \langle \psi(f^{-1}) \rangle$$

for each object a and non-identity morphism f of \mathcal{G} . By injectivity of the maps sending a to $\langle a \rangle$ and f to $\langle f \rangle$, we have $\psi' a = \psi a$ and $\psi' f = \psi f^{-1}$. \square

Note that the map ψ' defined above is equal to the composite $\psi \circ \gamma$, where γ is the (covariant) functor $\mathcal{G}^{op} \rightarrow \mathcal{G}$ defined on morphisms by $g \mapsto g^{-1}$.

7.3. Single-object groupoids. It is now possible to prove this paper's main result in the special case of groups. We will first state two Lemmas concerning squares and commutativity in groupoids that have one object. Next, we will show that if \mathcal{G} and \mathcal{H} are groups and if $\Psi : \text{Sd } \mathcal{G} \rightarrow \text{Sd } \mathcal{H}$ is an isomorphism, then the identity

$$(7.3.1) \quad \psi(\{fg, gf\}) = \{(\psi f)(\psi g), (\psi g)(\psi f)\}$$

is satisfied for all f and g in \mathcal{G} . Finally, a group-theoretic result of Bourbaki will be used to show that the condition (7.3.1) is sufficient to establish an isomorphism between \mathcal{G} and \mathcal{H} .

We begin by proving that $\psi(f^2) = (\psi f)^2$ and $\psi(f^3) = (\psi f)^3$.

Lemma 7.3.2. *Let \mathcal{G} and \mathcal{H} be groups, let $\Psi : \text{Sd } \mathcal{G} \rightarrow \text{Sd } \mathcal{H}$ be an isomorphism, and let f and h be any morphisms of \mathcal{G} . Then $f^2 = h$ if and only if $(\psi f)^2 = \psi h$.*

Proof. By Lemma 7.2.4, we have $f^2 = \text{id}$ if and only if $(\psi f)^2 = \text{id}$. Thus, if $f^2 = \text{id}$ then

$$f^2 = h \iff h = \text{id} \iff \psi h = \text{id} \iff (\psi f)^2 = \psi h$$

because ψ sends the identity morphism of \mathcal{G} to the identity morphism of \mathcal{H} .

Now, suppose that f^2 is a non-identity morphism. Then the result follows from Lemma 6.1.9, which states that f^2 equals h if and only if there exists a 2-simplex in $\text{Sd } \mathcal{G}$ of the form $\frac{f}{h} \triangleleft f$. The following are equivalent:

$$(1) \quad f^2 = h$$

$$(2) \quad \text{There is a 2-simplex in } \text{Sd } \mathcal{G} \text{ of the form } \frac{f}{h} \triangleleft f$$

(3) There is a 2-simplex in $\text{Sd } \mathcal{H}$ of the form $\frac{\psi f \wedge \psi f}{\psi h}$

(4) $(\psi f)^2 = \psi h$. □

It follows in particular that $\psi(f^2) = \psi(f)^2$.

Lemma 7.3.3. *Let \mathcal{G} and \mathcal{H} be groups, let $\Psi : \text{Sd } \mathcal{G} \rightarrow \text{Sd } \mathcal{H}$ be an isomorphism, and let f and h be any morphisms of \mathcal{G} . Then $f^3 = h$ if and only if $(\psi f)^3 = \psi h$.*

Proof. The following are equivalent:

- (1) f^2 is inverse to f
- (2) $\psi(f^2)$ is inverse to ψf
- (3) $(\psi f)^2$ is inverse to ψf .

Therefore,

$$f^3 = \text{id} \iff f^2 = f^{-1} \iff (\psi f)^2 = (\psi f)^{-1} \iff (\psi f)^3 = \text{id}.$$

Now, suppose that f^3 is not the identity morphism of \mathcal{G} . We cite Corollary 6.1.14, which gives criteria for determining whether $f^3 = h$ by counting 2-simplices of the form $\frac{f \wedge f^2}{h}$, to show that the following are equivalent:

- (1) $f^3 = h$
- (2) either $h = f^{-1}$ and $\exists_4 \frac{f \wedge f^2}{h}$, or $h \neq f^{-1}$ and $\exists_2 \frac{f \wedge f^2}{h}$
- (3) either $\psi h = \psi f^{-1}$ and $\exists_4 \frac{\psi f \wedge (\psi f)^2}{\psi h}$, or $\psi h \neq \psi f^{-1}$ and $\exists_2 \frac{\psi f \wedge (\psi f)^2}{\psi h}$
- (4) $(\psi f)^3 = \psi h$. □

The following two results are the fruit of calculations occurring in the Appendix. The first result shows that commutativity is preserved by ψ ; the second shows that, in the context of groups, $\psi(fg)$ is equal to $(\psi f)(\psi g)$ or to $(\psi g)(\psi f)$.

Lemma 7.3.4. *Let \mathcal{G} and \mathcal{H} be groups, let $\Psi : \text{Sd } \mathcal{G} \rightarrow \text{Sd } \mathcal{H}$ be an isomorphism, and let f and g be morphisms of \mathcal{G} . Then $fg = gf$ if and only if $(\psi f)(\psi g) = (\psi g)(\psi f)$*

Proof. First, suppose that f or g is equal to the identity arrow in \mathcal{G} . Then $fg = gf$ follows trivially, and we have $(\psi f)(\psi g) = (\psi g)(\psi f)$ because either ψf or ψg must be equal to the identity arrow in \mathcal{H} .

Now, suppose that f and g are not identity arrows. Then at least one of the following statements must be true:

- (1) $f = g$,
- (2) $f = g^{-1}$,
- (3) $f^2 = g$,
- (4) $f = g^2$,
- (5) $f \neq g$ and $f \neq g^{-1}$ and $f^2 \neq g$ and $f \neq g^2$.

We will show that the Lemma holds in each of the above cases.

- In case (1), we have $fg = f^2 = gf$ and $(\psi f)(\psi g) = (\psi f)(\psi f) = (\psi g)(\psi f)$.

- In case (2), we have $fg = ff^{-1} = \text{id} = f^{-1}f = gf$ and

$$(\psi f)(\psi g) = (\psi f)(\psi f)^{-1} = \text{id} = (\psi f)^{-1}(\psi f) = (\psi g)(\psi f).$$

- In case (3), we have $fg = f^3 = gf$ and

$$(\psi f)(\psi g) = (\psi f)^3 = (\psi g)(\psi f).$$

- Case (4) is analogous to case (3).
- For case (5), recall from Proposition 6.1.19 that, under the given conditions on f and g , we have $fg = gf$ if and only if there are an even number of 2-simplices of the form $\frac{f/\wedge g}{h}$ for each h . Thus, the following are equivalent:

- (1) $fg = gf$

- (2) There are an even number of $\frac{f/\wedge g}{h}$ for each h

- (3) There are an even number of $\frac{\psi f/\wedge \psi g}{\psi h}$ for each h

- (4) $(\psi f)(\psi g) = (\psi g)(\psi f)$. □

Proposition 7.3.5. *Let \mathcal{G} and \mathcal{H} be groups, and let $\Psi : \text{Sd } \mathcal{G} \rightarrow \text{Sd } \mathcal{H}$ be an isomorphism. For any pair f, g of morphisms in \mathcal{G} there is equality $\psi(fg) = (\psi f)(\psi g)$ or $\psi(fg) = (\psi g)(\psi f)$.*

Proof. Let f and g be some arrows in \mathcal{G} . We will use Proposition 6.1.20, which gives criteria for determining when the a given morphism h is equal to one of the composites fg and gf . When the hypotheses of Proposition 6.1.20 are not satisfied, we resort to more elementary means.

First, if f (resp. g) is equal to the identity arrow of \mathcal{G} , then ψf (resp. ψg) is the identity arrow of \mathcal{H} , in which case the proof follows easily. For example, if $f = \text{id}$ then $\psi(fg) = \psi g = (\psi f)(\psi g)$.

Now, suppose that f and g are not identity arrows. Then at least one of the following statements must be true:

- (1) $f = g$,
- (2) $f = g^{-1}$,
- (3) $f^2 = g$,
- (4) $f = g^2$,
- (5) $f^2 = g^{-1}$,
- (6) $f^{-1} = g^2$,
- (7) $f \neq g$ and $f \neq g^{-1}$ and $f^2 \neq g$ and $f \neq g^2$ and $f^2 \neq g^{-1}$ and $f^{-1} \neq g^2$.

We will show that the result holds in each of the above cases.

- In case (1), we have $\psi(fg) = \psi(f^2) = (\psi f)(\psi f) = (\psi f)(\psi g)$.
- In case (2), we have $\psi(fg) = \psi(\text{id}) = \text{id} = (\psi f)(\psi f)^{-1} = (\psi f)(\psi g)$.
- In case (3), we have $\psi(fg) = \psi(f^3) = (\psi f)(\psi f)^2 = (\psi f)(\psi g)$.
- Case (4) is analogous to case (3).

- In case (5), we have

$$\begin{aligned}\psi(fg) &= \psi(f^{-1}) = (\psi f)^{-1} = (\psi f)(\psi f)^{-2} \\ &= (\psi f)((\psi f)^2)^{-1} = (\psi f)(\psi(g^{-1}))^{-1} = (\psi f)(\psi g).\end{aligned}$$

- Case (6) is analogous to case (5).

Case (7) above is precisely the list of hypotheses for Proposition 6.1.20. To apply that Lemma, we will suppose that h is a morphism in \mathcal{G} , and we will show that

$$h \text{ equals } fg \text{ or } gf \iff \psi h \text{ equals } (\psi f)(\psi g) \text{ or } (\psi g)(\psi f),$$

from which follows the stated result that $\psi(fg)$ is equal to $(\psi f)(\psi g)$ or to $(\psi g)(\psi f)$.

Suppose that conditions (7) above are satisfied by f and g . This section's earlier results guarantee that $\psi(\text{id}) = \text{id}$, and that the following hold for all h in \mathcal{G} :

- $\psi(h^{-1}) = \psi(h)^{-1}$,
- $\psi(h^2) = \psi(h)^2$, and
- $\psi(h^3) = \psi(h)^3$.

It is a corollary of these facts, and of the bijectivity of ψ , that $f^i = g^j$ if and only if $(\psi f)^i = (\psi g)^j$ for any i, j in $\{-2, -1, 0, 1, 2\}$. Therefore, having assumed that the conditions (7) are satisfied by f and g , we deduce that the same conditions are satisfied by ψf and ψg :

$$\begin{aligned}\psi f &\neq \psi g, & \psi f &\neq (\psi g)^{-1}, & (\psi f)^2 &\neq \psi g, \\ \psi f &\neq (\psi g)^2, & (\psi f)^2 &\neq (\psi g)^{-1}, & (\psi f)^{-1} &\neq (\psi g)^2.\end{aligned}$$

The following points demonstrate that each case listed in the statement of Proposition 6.1.20 applies equally as well to f and g as it does to ψf and ψg :

- f and g commute if and only if ψf and ψg commute,
- $f^2 = g^2$ if and only if $(\psi f)^2 = (\psi g)^2$,
- $f^2 = g^{-2}$ if and only if $(\psi f)^2 = (\psi g)^{-2}$, and
- $f^2 = \text{id}$ if and only if $(\psi f)^2 = \text{id}$ (and similarly for g).

We will work through the most complicated case explicitly. Suppose that $fg \neq gf$ and $f^2 \neq g^{-2}$, as in case 4 subcase 3 of Proposition 6.1.20. Then the following are equivalent for any non-identity morphism h in \mathcal{G} :

- (1) h equals fg or gf
- (2) One of the following is satisfied:

$$(a) \exists_1 \frac{f \wedge g}{h} \quad \text{and} \quad \exists_1 \frac{f^{-1} \wedge g}{h} \quad \text{and} \quad \exists_1 \frac{f \wedge g^{-1}}{h}, \text{ or}$$

$$(b) \exists_1 \frac{f^{-1} \wedge g^{-1}}{h} \quad \text{and two of the following three existential statements hold:}$$

$$\exists_2 \frac{f \wedge g}{h}, \quad \exists_2 \frac{f^{-1} \wedge g}{h}, \quad \exists_2 \frac{f \wedge g^{-1}}{h}.$$

- (3) One of the following is satisfied:

$$(a) \exists_1 \frac{\psi f \wedge \psi g}{\psi h} \quad \text{and} \quad \exists_1 \frac{(\psi f)^{-1} \wedge \psi g}{\psi h} \quad \text{and} \quad \exists_1 \frac{\psi f \wedge (\psi g)^{-1}}{\psi h}, \text{ or}$$

(b) $\exists_1 \frac{(\psi f)^{-1} \wedge (\psi g)^{-1}}{\psi h}$ and two of the following three statements hold:

$$\exists_2 \frac{\psi f \wedge \psi g}{\psi h}, \quad \exists_2 \frac{(\psi f)^{-1} \wedge \psi g}{\psi h}, \quad \exists_2 \frac{\psi f \wedge (\psi g)^{-1}}{\psi h}.$$

(4) ψh equals $(\psi f)(\psi g)$ or $(\psi g)(\psi f)$

We have used the fact that 2-simplices of the form $\frac{f \wedge g}{h}$ are in bijection with those of the form $\frac{\psi f \wedge \psi g}{\psi h}$, and similarly when f (resp. g) has been replaced with f^{-1} (resp. g^{-1}). The other cases listed in Proposition 6.1.20 have proofs that are analogous to the above, thus in every case we may conclude that h equals fg or gf if and only if ψh equals $(\psi f)(\psi g)$ or $(\psi g)(\psi f)$. \square

Corollary 7.3.6. *Let \mathcal{G} and \mathcal{H} be groups, and let $\Psi : \text{Sd } \mathcal{G} \rightarrow \text{Sd } \mathcal{H}$ be an isomorphism. For any pair f, g of morphisms in \mathcal{G} there is equality*

$$\psi(\{fg, gf\}) = \{(\psi f)(\psi g), (\psi g)(\psi f)\}.$$

Proof. From the previous Proposition, we know that the right hand side above is a subset of the left hand side:

$$\psi(\{fg, gf\}) = \{\psi(fg), \psi(gf)\} \subseteq \{(\psi f)(\psi g), (\psi g)(\psi f)\}.$$

Let $\phi : \mathcal{H} \rightarrow \mathcal{G}$ denote the inverse map to ψ , induced by the isomorphism $\Phi : \text{Sd } \mathcal{H} \rightarrow \text{Sd } \mathcal{G}$ that is inverse to Ψ . Again by the previous Proposition, we have subset inclusion

$$\begin{aligned} \phi(\{(\psi f)(\psi g), (\psi g)(\psi f)\}) &= \{\phi((\psi f)(\psi g)), \phi((\psi g)(\psi f))\} \\ &\subseteq \{(\phi \psi f)(\phi \psi g), (\phi \psi g)(\phi \psi f)\} = \{fg, gf\}. \end{aligned}$$

It follows that ψ is a bijection between $\{fg, gf\}$ and $\{(\psi f)(\psi g), (\psi g)(\psi f)\}$. \square

We can now make use of the following result from group theory, which is applicable because the category of small groups is equivalent to the category of small single-object groupoids.

Lemma 7.3.7 (Bourbaki).

Let G, G' be two groups, $f : G \rightarrow G'$ a mapping such that, for two arbitrary elements x, y of G , $f(xy) = f(x)f(y)$ or $f(xy) = f(y)f(x)$. It is proposed to prove that f is a homomorphism of G into G' or a homomorphism of G into the opposite group G'^0 (in other words, either $f(xy) = f(x)f(y)$ for every ordered pair (x, y) or $f(xy) = f(y)f(x)$ for every ordered pair (x, y)).

The above is quoted directly from Bourbaki's book [1], in "Exercises for §4", Problem 26, p.139. For an elementary proof, see Lemma 4 in Mansfield's paper [3] on the group determinant. We now state our theorem in the special case of single-object groupoids.

Theorem 7.3.8. *Let \mathcal{G} and \mathcal{H} be groups. and let $\Psi : \text{Sd } \mathcal{G} \rightarrow \text{Sd } \mathcal{H}$ be an isomorphism. Then there exists an isomorphism $P : \mathcal{G} \rightarrow \mathcal{H}$.*

Proof. By the Bourbaki Lemma, ψ is a possibly-contravariant group isomorphism. If ψ is covariant then we set $P = \psi$, and if ψ is contravariant then we set P equal to the map ψ' (from Construction 7.2.6) defined by $\psi' a = \psi a$ and $\psi' f = (\psi f)^{-1}$ for objects a and morphisms f in \mathcal{G} . Thus, for non-identity morphisms f in \mathcal{G} ,

$$P f = \begin{cases} \psi f & \text{if } \psi \text{ is covariant} \\ \psi(f^{-1}) & \text{if } \psi \text{ is contravariant} \end{cases} . \quad \square$$

7.4. Connected multi-object groupoids. This subsection establishes a result analogous to the previous theorem pertaining to connected groupoids that have multiple objects. The combinatorics from the appendix will not be necessary; this subsection's result is significantly easier to prove than that of the previous subsection.

Let \mathcal{G} and \mathcal{H} be connected groupoids that have more than one object each, and let $\Psi : \text{Sd } \mathcal{G} \rightarrow \text{Sd } \mathcal{H}$ be an isomorphism. Given an arrow f of \mathcal{G} that is not an endomorphism, say that ψ is *covariant* at f if

$$(7.4.1) \quad \psi(\text{dom } f) = \text{dom}(\psi f) \quad \text{and} \quad \psi(\text{cod } f) = \text{cod}(\psi f).$$

Similarly, say that ψ is *contravariant* at a given non-endomorphism f if

$$(7.4.2) \quad \psi(\text{dom } f) = \text{cod}(\psi f) \quad \text{and} \quad \psi(\text{cod } f) = \text{dom}(\psi f).$$

Recall from Construction 7.2.6, that there is a map $\Psi' : \text{Sd } \mathcal{G} \rightarrow \text{Sd } \mathcal{H}$ defined as the composite $\Psi \circ \alpha_{\mathcal{G}}$, and that ψ' is the map $\mathcal{G} \rightarrow \mathcal{H}$ induced by Ψ' . This map ψ is characterized by the equalities $\psi'(a) = \psi(a)$ for objects a and $\psi'(f) = \psi(f)^{-1}$ for morphisms f .

We may assume without loss of generality that ψ is covariant at some non-endomorphism f in \mathcal{G} ; if this fails to be the case then ψ is contravariant everywhere, so we may consider ψ' instead of ψ to obtain a map that is covariant. Assuming that ψ is covariant at some non-endomorphism, and that the groupoids \mathcal{G} and \mathcal{H} are connected, we will prove that ψ is covariant everywhere, that is, ψ is a morphism of *directed* graphs.

Note first that ψ is covariant at f if and only if ψ' is contravariant at f . Indeed, by the inversion Lemma 7.2.4 we have equality $\psi' f = (\psi f)^{-1} = \psi(f^{-1})$, hence the domain of ψf equals the codomain of $\psi' f$, and vice versa.

A consequence will be that ψ is functorial: we will show that there is equality $\psi(f \circ g) = (\psi f) \circ (\psi g)$ for every pair f, g of morphisms in \mathcal{G} , thereby establishing an analog to Theorem 7.3.8 pertaining to connected groupoids that have multiple objects.

Proposition 7.4.3. *Let f and g be morphisms of \mathcal{G} satisfying*

- (1) *neither f nor g is an endomorphism, and*
- (2) *the sets $\{\text{dom } f, \text{cod } f\}$ and $\{\text{dom } g, \text{cod } g\}$ are not disjoint, that is, f and g are not unrelated.*

If f and g are end-to-end, f and g are sequential if and only if ψf and ψg are sequential. Similarly, if f and g are ends-to-ends, then f and g are parallel (resp. opposed) if and only if ψf and ψg are parallel (resp. opposed).

Proof. Given the conditions (1) and (2), the morphisms f and g must be end-to-end or ends-to-ends. Recall from Proposition 7.2.3 that f and g are end-to-end if and

only if ψf and ψg are end-to-end, and similarly in the case where f and g are ends-to-ends.

Suppose first that f and g are end-to-end sequential morphisms. By Proposition 6.1.4, this is true if and only if there is a unique filler for the triangle $f \swarrow \searrow^g$. By Corollary 7.2.2, this is true if and only if there is a unique filler for $\psi f \swarrow \searrow^{\psi g}$. Applying Proposition 6.1.4 once more, this is true if and only if ψf and ψg are sequential.

Now, suppose that f and g are non-sequential end-to-end morphisms, that is, if f and g are cointial or coterminal. By Proposition 6.1.4, this is true if and only if there are multiple fillers for $f \swarrow \searrow^g$. By Corollary 7.2.2, $\psi f \swarrow \searrow^{\psi g}$. Applying Proposition 6.1.4 again, this is true if and only if ψf and ψg are coterminal or cointial.

Suppose now that f and g are parallel ends-to-ends morphisms. From Proposition 6.1.6, we know that this occurs if and only if there are four fillers for $f \swarrow \searrow^g$. By Corollary 7.2.2, this is true if and only if there are four fillers for triangle $\psi f \swarrow \searrow^{\psi g}$ has four fillers. Applying Proposition 6.1.6 once more, this is true if and only if ψf and ψg are parallel. The proof is similar in case f and g are opposed. \square

Corollary 7.4.4. *Let f and g be morphisms of \mathcal{G} satisfying conditions (1) and (2) from Proposition 7.4.3. If ψ is covariant at f , then it is covariant at g . It follows from connectedness that if ψ is covariant at any non-endomorphism f , then ψ is covariant at every non-endomorphism in \mathcal{G} .*

Proof. Suppose first that ψ is covariant at f and that f and g are parallel. Because ψf and ψg are parallel, we have

$$\text{dom}(\psi g) = \text{dom}(\psi f) = \psi(\text{dom } f) = \psi(\text{dom } g)$$

and similarly for codomains, showing that ψ is covariant at g . The proof is similar in case f and g are opposed or sequential.

Suppose now that ψ is covariant at f and that f and g are coterminal or cointial. Then f and g^{-1} are sequential, so ψ is covariant at g^{-1} . It follows that ψ is covariant at g because g^{-1} and g are opposed.

Now, we claim that covariance at any one non-endomorphism f is sufficient to guarantee covariance everywhere. Indeed, suppose that ψ is covariant at f , writing $a = \text{dom } f$ and $b = \text{cod } f$. Letting $g : c \rightarrow d$ be any other non-endomorphism in \mathcal{G} , we will show that ψ is covariant at g . We have proved above that this is true if f and g are related; we still need to consider the case of unrelated morphisms, where the objects a, b, c, d are all distinct.

Because \mathcal{G} is a connected groupoid, there exists some arrow $h : b \rightarrow c$. Then ψ is covariant at h because f and h are sequential, and it follows that ψ is covariant at g because h and g are sequential. \square

Thus we can assume without loss of generality that if \mathcal{G} and \mathcal{H} are connected groupoids and $\Psi : \text{Sd } \mathcal{G} \rightarrow \text{Sd } \mathcal{H}$ is an isomorphism, then $\psi : \mathcal{G} \rightarrow \mathcal{H}$ is an isomorphism of directed graphs; if this fails to be the case, we consider ψ' instead of ψ . It is now possible to demonstrate functoriality.

Proposition 7.4.5. *Assume that \mathcal{G} and \mathcal{H} are connected multi-object groupoids, $\Psi : \text{Sd } \mathcal{G} \rightarrow \text{Sd } \mathcal{H}$ is an isomorphism, and ψ is covariant at every morphism. Then $\psi(f \circ g) = (\psi f) \circ (\psi g)$ for every pair f, g of morphisms in \mathcal{G} .*

Proof. We consider all possible forms a composite could take.

Suppose first that f and g are sequential end-to-end morphisms, and write h for the composite of f and g . Recall Corollary 6.1.5, which says that h is the composite of f and g if and only if there is only one filler of $\begin{array}{c} f \\ \diagdown \\ \text{---} \\ \diagup \\ g \end{array}$, and h is the third side of that filler. We use Corollary 7.2.2 to show that there is only one filler of $\begin{array}{c} \psi f \\ \diagdown \\ \text{---} \\ \diagup \\ \psi g \end{array}$, and ψh is the third side of that filler. It follows that ψh is the composite of ψf and ψg .

If f and g are opposed ends-to-ends morphisms then the equation

$$\{\psi(fg), \psi(gf)\} = \{(\psi f)(\psi g), (\psi g)(\psi f)\}$$

follows from Corollary 6.1.7, which says that there exactly two fillers of $\begin{array}{c} f \\ \diagdown \\ \text{---} \\ \diagup \\ g \end{array}$, and that these fillers have respective third sides equal to fg and to gf . Indeed, by Corollary 7.2.2 there are exactly two fillers of $\begin{array}{c} \psi f \\ \diagdown \\ \text{---} \\ \diagup \\ \psi g \end{array}$, and these fillers have respective third sides equal to $(\psi f)(\psi g)$ and to $(\psi g)(\psi f)$, hence ψ gives a bijection between the respective sets

$$\{fg, gf\} \text{ and } \{(\psi f)(\psi g), (\psi g)(\psi f)\}$$

of third sides. To see that $\psi(fg) = (\psi f)(\psi g)$, note that we cannot have $\psi(fg) = (\psi g)(\psi f)$ because $\psi(fg)$ is an endomorphism of $\text{cod}(\psi f)$ whereas $(\psi g)(\psi f)$ is an endomorphism of $\text{dom}(\psi f)$.

Suppose now that f and g are end-to-endo. Recall Proposition 6.1.8, which states that h is the composite of f and g if and only if h is parallel to f and h is a third side of the triangle $\begin{array}{c} f^{-1} \\ \diagdown \\ \text{---} \\ \diagup \\ g \end{array}$. Thus, ψ sends the composite of f and g to a third side of $\begin{array}{c} (\psi f)^{-1} \\ \diagdown \\ \text{---} \\ \diagup \\ \psi g \end{array}$ that is parallel to ψf . It follows from Proposition 6.1.8 that this third side is the composite of ψf and ψg , proving the desired result. Note that this same proof holds whether $\text{dom } f = \text{dom } g = \text{cod } g$ or $\text{cod } f = \text{dom } g = \text{cod } g$.

Finally, suppose that f and g are both endomorphisms of some object a in \mathcal{G} . Given that \mathcal{G} is connected and has multiple objects, there exists some non-endomorphism k in $\text{Mor}(\mathcal{G})$ satisfying $\text{dom } k = a$. We then have the following:

$$\begin{aligned} \psi(f \circ g) &= \psi(k^{-1} \circ k \circ f \circ g) \\ &= \psi(k^{-1}) \circ \psi(k \circ f \circ g) && (k^{-1} \text{ and } k \circ f \circ g \text{ are ends-to-ends}) \\ &= \psi(k^{-1}) \circ \psi(k \circ f) \circ \psi(g) && (k \circ f \text{ and } g \text{ are end-to-endo}) \\ &= \psi(k^{-1}) \circ \psi(k) \circ \psi(f) \circ \psi(g) && (k \text{ and } f \text{ are end-to-endo}) \\ &= \psi(f) \circ \psi(g) && (\text{because } \psi(k^{-1}) \text{ equals } (\psi k)^{-1}) \quad \square \end{aligned}$$

Theorem 7.4.6. *Let \mathcal{G} and \mathcal{H} be connected multi-object groupoids, and let $\Psi : \text{Sd } \mathcal{G} \rightarrow \text{Sd } \mathcal{H}$ be an isomorphism. Then there exists an isomorphism $P : \mathcal{G} \rightarrow \mathcal{H}$.*

Proof. Pick some non-endomorphism f in $\text{Mor}(\mathcal{G})$. If ψ is covariant at f , then it follows from Corollary 7.4.4 that ψ is covariant everywhere on \mathcal{G} . It then follows from the previous proposition that ψ is functorial, so we can set $P = \psi$.

Suppose instead that ψ is contravariant at f . It follows that ψ' is covariant at f , where ψ' is the map from Construction 7.2.6, defined by $\psi' a = \psi a$ and $\psi' f = (\psi f)^{-1}$. By the same argument as above, it follows that ψ' is covariant everywhere on \mathcal{G} , hence ψ' is functorial. In this case, we set $P = \psi'$. \square

7.5. Statement and proof of the groupoid isomorphism theorem. It is now possible to prove this paper's main result.

Theorem 7.5.1. *Let \mathcal{G} and \mathcal{H} be groupoids. Any isomorphism $\Psi : \text{Sd } \mathcal{G} \rightarrow \text{Sd } \mathcal{H}$ induces an isomorphism $P : \mathcal{G} \rightarrow \mathcal{H}$.*

Proof. Suppose that $\Psi : \text{Sd } \mathcal{G} \rightarrow \text{Sd } \mathcal{H}$ is an isomorphism. Recall from Lemma 4.4 that we have an isomorphism $\text{II}_i(\text{Sd } \mathcal{C}_i) \cong \text{Sd}(\text{II}_i \mathcal{C}_i)$ for any set $\{\mathcal{C}_i\}$ of small categories, and that a category \mathcal{C} is connected if and only if its subdivision $\text{Sd } \mathcal{C}$ is connected.

There exists some index sets I and J , and some collections $\{\mathcal{G}_i\}_{i \in I}$ and $\{\mathcal{H}_j\}_{j \in J}$ of connected groupoids such that $\mathcal{G} = \text{II}_i \mathcal{G}_i$ and $\mathcal{H} = \text{II}_j \mathcal{H}_j$. We obtain a composite isomorphism

$$\Gamma : \prod_i (\text{Sd } \mathcal{G}_i) \xrightarrow{\cong} \text{Sd} \left(\prod_i \mathcal{G}_i \right) \xrightarrow{\Psi} \text{Sd} \left(\prod_j \mathcal{H}_j \right) \xrightarrow{\cong} \prod_j (\text{Sd } \mathcal{H}_j).$$

The image of a connected category is connected. Thus, for each i there is some j such that the image $\Gamma(\text{Sd } \mathcal{G}_i)$ is a subcategory of $\text{Sd } \mathcal{H}_j$. The inverse isomorphism Γ^{-1} must take $\text{Sd } \mathcal{H}_j$ back into $\text{Sd } \mathcal{G}_i$ because $\text{Sd } \mathcal{H}_j$ is connected. The connected components of $\text{Sd } \mathcal{G}$ are in bijection with the connected components of $\text{Sd } \mathcal{H}$, so we can use I to reindex the components $\{\mathcal{H}_j\}$ of \mathcal{H} so as to obtain an isomorphism $\Psi_i : \text{Sd } \mathcal{G}_i \rightarrow \text{Sd } \mathcal{H}_i$ for each i in I . These maps Ψ_i are the evident restrictions of Γ , satisfying $\Gamma = \text{II}_i \Psi_i$.

For each Ψ_i we obtain an isomorphism $P_i : \mathcal{G}_i \xrightarrow{\cong} \mathcal{H}_i$, either from Theorem 7.3.8 or Theorem 7.4.6, depending on whether \mathcal{G}_i has one object or many. Setting $P = \text{II}_i P_i$ we have isomorphism $P : \text{II}_i \mathcal{G}_i \rightarrow \text{II}_i \mathcal{H}_i$ between \mathcal{G} and \mathcal{H} . \square

Acknowledgements. First, I'd like to thank my mentor, Jonathan Rubin, who has been extremely helpful with the organization of this paper, and who has been responsible for much of the content. Henry Chan and Claudio Gonzales offered generous support with proofreading and notation. Many thanks to Peter May for promoting the study of categorical subdivision and for encouraging me to develop these ideas.

REFERENCES

- [1] N. Bourbaki: *Elements of mathematics-Algebra I*. Addison-Wesley, Massachusetts (Trans. 1970).
- [2] P.G. Goerss and J. F. Jardine, *Simplicial Homotopy Theory*. Birkhäuser Verlag (1999).
- [3] R. Mansfield, *A group determinant determines its group*, Proc. Amer. Math. Soc. **116** (1992), 939-941.
- [4] J.P. May, *Finite spaces and larger contexts*, preprint. Retrieved on 9/2/2015 from <http://math.uchicago.edu/~may/FINITE/FINITEBOOK/FINITEBOOKCollatedDraft.pdf>

APPENDIX A. PROOF OF PROPOSITIONS 6.1.19 AND 6.1.20

Here we will prove the final two Lemmas of Section 5. We assume throughout that f and g are distinct non-identity endomorphisms of some common object in a groupoid \mathcal{G} , satisfying the identities $f \neq g$ and $f \neq g^{-1}$ and $f^2 \neq g$ and $f \neq g^2$.

Proposition 6.1.19 states that, under the above assumptions on f and g , we have commutativity $fg = gf$ if and only if for every third side h of the triangle $\begin{array}{c} f \diagdown \diagup g \\ \hline \hline \end{array}$ there are an even number of 2-simplices of the form $\begin{array}{c} f \diagdown \diagup g \\ \hline \hline h \end{array}$. Proposition 6.1.20 gives criteria for a given third side h of the triangle $\begin{array}{c} f \diagdown \diagup g \\ \hline \hline h \end{array}$ to satisfy one of the equations $h = fg$ and $h = gf$.

Proof of Proposition 6.1.19. Write $\text{fill}(f, g)$ for the set of non-degenerate 2-simplex fillers of the triangle $\begin{array}{c} f \diagdown \diagup g \\ \hline \hline \end{array}$. Our assumptions that $f \neq g$ and $f^2 \neq g$ and $f \neq g^2$ guarantee that the set $\text{fill}(f, g)$ contains six distinct 2-simplices (see Lemma 6.1.17). We have assumed $f \neq g^{-1}$ to avoid the trivial case where no non-identity morphisms h satisfy $h = fg$ or $h = gf$.

Recall that the set $C(f, g)$ of formal composites consists of the quadruples (k, s, l, t) satisfying

- (1) (k, l) is equal to (f, g) or (g, f) , and
- (2) (s, t) is equal to $(1, 1)$ or $(1, -1)$ or $(-1, 1)$.

Our proof makes use of the bijection

$$\zeta : C(f, g) \xrightarrow{\cong} \text{fill}(f, g)$$

from Lemma 6.1.17, defined as below:

$$\begin{array}{ll} (f, 1, g, 1) \mapsto \langle f|g \rangle & (g, 1, f, 1) \mapsto \langle g|f \rangle \\ (f, -1, g, 1) \mapsto \langle f|f^{-1}g \rangle & (g, 1, f, -1) \mapsto \langle gf^{-1}|f \rangle \\ (f, 1, g, -1) \mapsto \langle fg^{-1}|g \rangle & (g, -1, f, 1) \mapsto \langle g|g^{-1}f \rangle. \end{array}$$

As before, we write $\text{ev} : C(f, g) \rightarrow \text{Mor}(\mathcal{G})$ for the evaluation map that sends (k, s, l, t) to $k^s l^t$. By Lemma 6.1.17, each element γ of $C(f, g)$ is sent by ζ to a 2-simplex of the form

$$\frac{f \diagdown \diagup g}{\text{ev}(\gamma)}.$$

We define an equivalence relation \sim on $C(f, g)$ by

$$(k, s, l, t) \sim (k', s', l', t') \iff k^s l^t = k'^{s'} l'^{t'}.$$

We can use \sim to define an undirected graph structure on $C(f, g)$ by placing an edge between two formal composites $\gamma, \psi \in C(f, g)$ if and only if $\gamma \sim \psi$ and $\gamma \neq \psi$. Then $\gamma \sim \psi$ exactly when either

- (1) $\gamma = \psi$, or
- (2) there is an edge between γ and ψ .

Consequently, the components of this graph are precisely the fibers of the evaluation map ev .

This graph will be denoted $G(f, g)$, and denoted the *graph of ev* . Working with the connected components of $G(f, g)$, we reduce the problem of counting 2-simplices to a problem of determining fiber sizes.

Note that the graph structure of $G(f, g)$ is encoded by the categorical structure of $Sd \mathcal{G}$: two nodes γ and ψ have an edge between them if and only if the 2-simplices $\zeta(\gamma)$ and $\zeta(\psi)$, each of which fills the triangle $\begin{matrix} f & & g \\ & \searrow & / \\ & d & \end{matrix}$, have third sides that are equal. Explicitly, we have a commutative diagram

$$\begin{array}{ccc}
 C(f, g) & \xrightarrow{ev} & \{\text{non-identity morphisms of } \mathcal{G}\} \\
 \zeta \downarrow \cong & & \downarrow \theta_1^{\mathcal{G}} \\
 \text{fill}(f, g) & \xrightarrow{3rd_{f,g}} & \text{simp}_1(Sd \mathcal{G})
 \end{array}$$

where $\theta_1^{\mathcal{G}}$ is the bijection sending d to $\langle d \rangle$, and where $3rd_{f,g}$ is the function that sends each 2-simplex of the form $\begin{matrix} f & & g \\ & \searrow & / \\ & d & \end{matrix}$ to the third side $\langle d \rangle$.

To prove Proposition 6.1.19, it will suffice to show that $fg = gf$ if and only if every connected component of $G(f, g)$ has even cardinality. This is because the size of each connected component $ev^{-1}\{h\}$ is equal to the number of non-degenerate 2-simplices of the form $\begin{matrix} f & & g \\ & \searrow & / \\ & h & \end{matrix}$.

We will always draw the vertices of $G(f, g)$ in the following configuration

$$\begin{array}{cc}
 (f, 1, g, 1) & (g, 1, f, 1) \\
 (f, -1, g, 1) & (g, 1, f, -1) \\
 (f, 1, g, -1) & (g, -1, f, 1)
 \end{array}$$

and will usually suppress the labels on the vertices to points $*$. Configurations in the graph of ev correspond to equational conditions. Specifically, we have the table below, where the following are equivalent:

- (1) the upper equation holds
- (2) the lower graph is a subgraph of $G(f, g)$
- (3) a single edge of the lower graph is in $G(f, g)$.

Equation	$fg = gf$	$f^2 = id$	$g^2 = id$	$f^2 = g^2$	$fgf = g$	$gfg = f$	$fg^{-1}f = g$
Subgraph							

The equivalence of conditions 1-3 above can be verified by considering if-and-only-if statements such as

$$\begin{aligned} fg = gf &\iff \text{ev}(f, 1, g, 1) = \text{ev}(g, 1, f, 1) \\ &\iff \text{ev}(f, -1, g, 1) = \text{ev}(g, 1, f, -1) \\ &\iff \text{ev}(f, 1, g, -1) = \text{ev}(g, -1, f, 1). \end{aligned}$$

If we suppose that $fg = gf$, then the graph

$$(A.1) \quad \begin{array}{c} * \text{---} * \\ * \text{---} * \\ * \text{---} * \end{array}$$

is a subgraph of $G(f, g)$. It follows that each connected component of $G(f, g)$ must have cardinality equal to 2, 4, or 6. This is because equivalence classes in a coarser equivalence relation are unions of equivalence classes in a finer one – the equivalence relation illustrated by graph A.1 is the finest possible among graphs $G(f, g)$ satisfying $fg = gf$.

We now prove the converse implication: if each connected component of $G(f, g)$ has even cardinality, then $fg = gf$. Let $P(f, g)$ denote the partition of $C(f, g)$ defined by

$$\left\{ \left\{ (f, 1, g, 1), (g, 1, f, 1) \right\}, \left\{ (f, -1, g, 1), (g, 1, f, -1) \right\}, \left\{ (f, 1, g, -1), (g, -1, f, 1) \right\} \right\}.$$

Note that we have commutativity $fg = gf$ if and only if the evaluation map $\text{ev} : C(f, g) \rightarrow \text{Mor}(\mathcal{G})$ factors through $P(f, g)$. Indeed we have $fg = gf$ if and only if there is equality

$$\text{ev}(f, s, g, t) = f^s g^t = g^t f^s = \text{ev}(g, t, f, s)$$

for every pair $s, t \in \{-1, 1\}$, if and only if each fiber $\text{ev}^{-1}\{h\}$ is a union of some classes in $P(f, g)$.

Note that there are three possible partitions of 6 into even integers, namely

$$6 \quad \text{and} \quad 2 + 4 \quad \text{and} \quad 2 + 2 + 2.$$

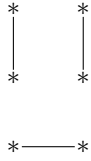
We will prove first that if the partition of $C(f, g)$ into fibers of ev corresponds to either 6 or $2 + 4$, then $fg = gf$. Supposing that $C(f, g)$ is partitioned as 6 or as $2 + 4$, there exists some morphism h in \mathcal{G} such that $\text{ev}^{-1}\{h\}$ has four or more elements. It follows that $\text{ev}^{-1}\{h\}$ must contain an entire class in $P(f, g)$. This implies $f^s g^t = g^t f^s$ for some s and t , and commutativity follows.

To complete our proof, suppose that the partition of $C(f, g)$ given by ev corresponds to $2 + 2 + 2$. Then, the graph $G(f, g)$ consists of exactly three disjoint edges. Our strategy will be to show that, under these conditions, $f^2 \neq \text{id}$ and $g^2 \neq \text{id}$ and $f^2 \neq g^2$. We then use these facts to prove that $fg = gf$.

Suppose for contradiction that $f^2 = \text{id}$. Then $G(f, g)$ must contain

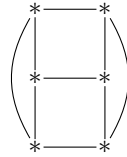


as a subgraph, hence $G(f, g)$ is exactly equal to the graph

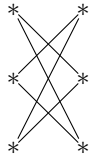


But the bottom edge implies $fg^{-1} = g^{-1}f$, and multiplying on the left and right by g implies $fg = gf$. This, however, implies that all three horizontal edges must be in $G(f, g)$, contradicting our assumption of a $2 + 2 + 2$ partition. Therefore, $f^2 \neq \text{id}$. The arguments for $g^2 \neq \text{id}$ and $f^2 \neq g^2$ are entirely analogous.

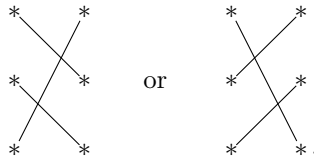
Now, suppose for contradiction that $fg \neq gf$, so that no edge below



is contained in the graph $G(f, g)$. In this case, $G(f, g)$ must be a subgraph of



Since $G(f, g)$ must consist of three disjoint edges, it must be of the form



In either case we have a contradiction, for



is a subgraph of $G(f, g)$ if and only if



is a subgraph of $G(f, g)$. We conclude that if $G(f, g)$ is partitioned into connected components each having size 2, then the assumption $fg \neq gf$ is contradictory. \square

Proof of Proposition 6.1.20. We will prove Proposition 6.1.20 case-by-case, making use of the graph $G(f, g)$ defined above. Recall that the number of non-degenerate 2-simplices of the form $\frac{f/\backslash g}{h}$ is equal to the size of the fiber $ev^{-1}\{h\}$, where $ev : C(f, g) \rightarrow Mor(\mathcal{G})$ is the evaluation map defined by $(k, s, l, t) \mapsto k^s l^t$. In addition to our previous assumptions that f and g are non-identity endomorphisms of some common object in \mathcal{G} satisfying $f \neq g$ and $f \neq g^{-1}$ and $f^2 \neq g$ and $f \neq g^2$, we assume also that $f^2 \neq g^{-1}$ and $f^{-1} \neq g^2$. These last two assumptions guarantee that there are bijections

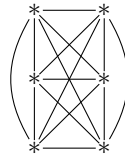
$$C(f^{-1}, g) \cong \text{fill}(f^{-1}, g) \quad \text{and} \quad C(f, g^{-1}) \cong \text{fill}(f, g^{-1})$$

as per Lemma 6.1.17. These two bijections will be useful in the proof of case 4.

As before, configurations in the graph of ev correspond to equational conditions involving f and g . Recall that any distinct points belonging to the same connected component of $G(f, g)$ must have an edge between them. In other words, every full connected subgraph of $G(f, g)$ is a complete graph.

Case 1, subcase 1: Suppose that $f^2 = \text{id} = g^2$ and $fg = gf$.

Under these assumptions, the graph of ev consists of a single connected component. Indeed, $G(f, g)$ is the complete graph on six vertices, as below:



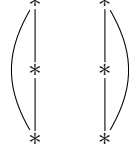
Thus, every third side of $\frac{f/\backslash g}{h}$ is equal to fg and to gf . In other words, all six 2-simplices in $\text{fill}(f, g)$ are of the form $\frac{f/\backslash g}{fg}$.

$$h = fg = gf \iff \exists_6 \frac{f/\backslash g}{h}$$

$$h \neq fg \text{ and } h \neq gf \iff \nexists \frac{f/\backslash g}{h}$$

Case 1, subcase 2: Suppose that $f^2 = \text{id} = g^2$ and $fg \neq gf$.

Consulting the table on page 42, the condition $f^2 = \text{id} = g^2$ guarantees that



is a subgraph of $G(f, g)$. There can be no edges connecting the two halves of the graph above; if there were such a connection, then $G(f, g)$ would have only one connected component, in which case it would be complete and we would have $fg = gf$. Thus, $G(f, g)$ consists of two connected components, and is exactly equal to the graph above.

The fibers $\text{ev}^{-1}\{fg\}$ and $\text{ev}^{-1}\{gf\}$ have three elements each, and every third side of $\frac{f}{\triangleleft} \frac{g}{\triangleright}$ is equal to fg or to gf . In other words, every 2-simplex in $\text{fill}(f, g)$ is of the form $\frac{f}{\triangleleft} \frac{g}{fg}$ or $\frac{f}{gf} \frac{g}{\triangleright}$, and we have the following:

$$h = fg \text{ or } h = gf \iff \exists_3 \frac{f}{\triangleleft} \frac{g}{h}$$

$$h \neq fg \text{ and } h \neq gf \iff \nexists \frac{f}{\triangleleft} \frac{g}{h} .$$

Case 2, subcase 1: Suppose that $f^2 = g^2$ and $fg = gf$ and $f^2 \neq \text{id}$ and $g^2 \neq \text{id}$.

Because $f^2 = g^2$ and $fg = gf$, and because each connected full subgraph of $G(f, g)$ is complete, the below



must be a subgraph of $G(f, g)$. There can be no edges connecting the two components of the graph above, for otherwise $G(f, g)$ would be the complete graph, contradicting our assumptions that $f^2 \neq \text{id}$ and $g^2 \neq \text{id}$.

Thus, if $h = fg = gf$ then the fiber $\text{ev}^{-1}\{h\}$ has two elements, and if h is not equal to fg or gf then the fiber $\text{ev}^{-1}\{h\}$ is empty or has four elements.

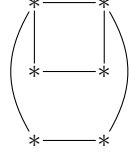
$$h = fg = gf \iff \exists_2 \frac{f}{\triangleleft} \frac{g}{h}$$

$$h \neq fg \text{ and } h \neq gf \iff \nexists \frac{f}{\triangleleft} \frac{g}{h} \text{ or } \exists_4 \frac{f}{\triangleleft} \frac{g}{h}$$

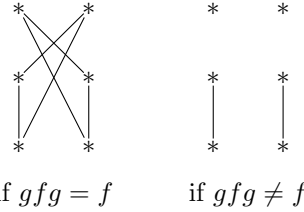
Explicitly, we have $h = fg = gf$ if and only if there are two 2-simplices of the form $\frac{f}{\triangleleft} \frac{g}{h}$, namely $\langle f|g \rangle$ and $\langle g|f \rangle$.

Case 2, subcase 2: Suppose that $f^2 = g^2$ and $fg \neq gf$ and $f^2 \neq \text{id}$ and $g^2 \neq \text{id}$.

Because $fg \neq gf$ and $f^2 \neq \text{id}$ and $g^2 \neq \text{id}$, the graph $G(f, g)$ cannot contain any of the edges below:



We have $f^2 = g^2$, so the only question is what $(f, 1, g, 1)$ and $(g, 1, f, 1)$ are connected to. The graph $G(f, g)$ must be equal to one of the two displayed below.



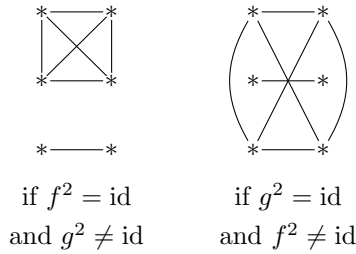
If $gfg = f$ then the fibers of fg and gf both have size 3. If $gfg \neq f$ then the fibers both have size 1. If the fiber $\text{ev}^{-1}\{h\}$ is non-empty and if h is not equal to fg or gf , then we must have $gfg \neq f$ and fiber size 2.

$$h = fg \text{ or } h = gf \iff \exists_1 \frac{f \setminus \setminus g}{h} \text{ or } \exists_3 \frac{f \setminus \setminus g}{h}$$

$$h \neq fg \text{ and } h \neq gf \iff \nexists \frac{f \setminus \setminus g}{h} \text{ or } \exists_2 \frac{f \setminus \setminus g}{h}$$

Case 3, subcase 1: Suppose $f^2 \neq g^2$ and $fg = gf$, and either $f^2 = \text{id}$ or $g^2 = \text{id}$.

The graph $G(f, g)$ contains one of the below two graphs as a subgraph.



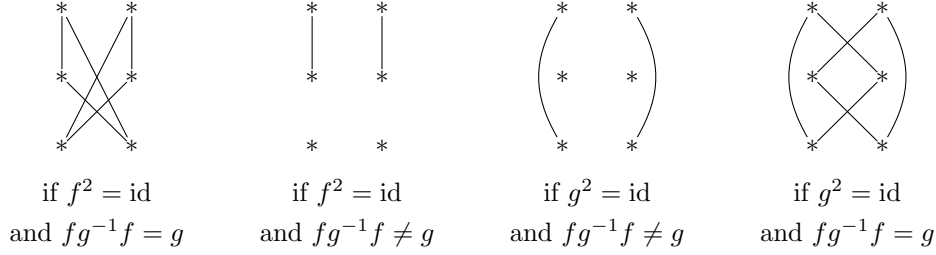
Note that we cannot have both $f^2 = \text{id}$ and $g^2 = \text{id}$, because $f^2 \neq g^2$. We claim that $G(f, g)$ is exactly equal to one of the two graphs above. Indeed, supposing for contradiction that $G(f, g)$ contains one of the above as a *proper* subgraph, we have only one connected component, hence $G(f, g)$ is complete and our assumption $f^2 \neq g^2$ is contradicted.

The above two graphs are isomorphic. In both cases, if $h = fg = gf$ then the fiber of h has four elements. If $h \neq fg$ and $h \neq gf$, then the fiber $\text{ev}^{-1}\{h\}$ is empty or has two elements.

$$h = fg = gf \iff \exists_4 \frac{f \triangleleft g}{h}$$

$$h \neq fg \text{ and } h \neq gf \iff \nexists \frac{f \triangleleft g}{h} \text{ or } \exists_2 \frac{f \triangleleft g}{h}$$

Case 3, subcase 2: Suppose $f^2 \neq g^2$ and $fg \neq gf$, and either $f^2 = \text{id}$ or $g^2 = \text{id}$.
Then $G(f, g)$ is one of the four graphs displayed below.



The two graphs in the middle are guaranteed by the equations $f^2 = \text{id}$ and $g^2 = \text{id}$, respectively. The graph on the far left is the only possible extension of the graph next to it; any other extension would contradict one of the assumptions $f^2 \neq g^2$ and $fg \neq gf$. Similarly, the graph on the far right is the only possible extension of the graph next to it.

Thus, if $h = fg$ or $h = gf$ then the fiber of h has three elements (if $fg^{-1}f = g$) or two elements (if $fg^{-1}f \neq g$). On the other hand, if the fiber $\text{ev}^{-1}\{h\}$ is non-empty and if h is not equal to fg or gf , then we must have $fg^{-1}f = g$ and fiber size 1.

$$h = fg \text{ or } h = gf \iff \exists_2 \frac{f \triangleleft g}{h} \text{ or } \exists_3 \frac{f \triangleleft g}{h}$$

$$h \neq fg \text{ and } h \neq gf \iff \nexists \frac{f \triangleleft g}{h} \text{ or } \exists_1 \frac{f \triangleleft g}{h}$$

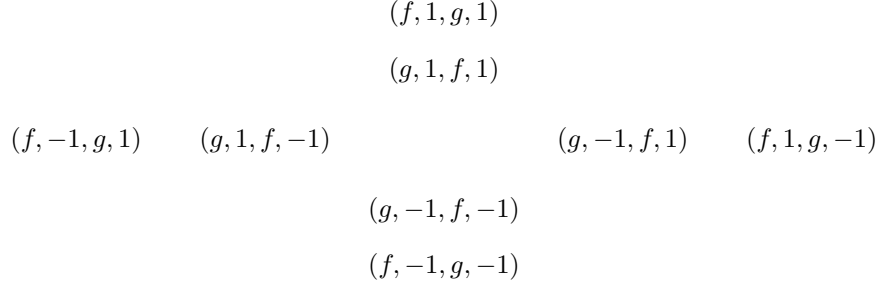
Case 4: Suppose $f^2 \neq g^2$ and $f^2 \neq \text{id}$ and $g^2 \neq \text{id}$.

Under these assumptions, the number of non-degenerate 2-simplices of the form $\frac{f \triangleleft g}{h}$ does not completely determine whether the given morphism h is equal to one of fg or gf . Our strategy here is to consider not only fillers of the triangle $\frac{f \triangleleft g}{h}$, but also to consider the fillers for $\frac{f^{-1} \triangleleft g}{h}$ and $\frac{f \triangleleft g^{-1}}{h}$ and $\frac{f^{-1} \triangleleft g^{-1}}{h}$. Let $C'(f, g)$ denote the superset of $C(f, g)$ consisting of quadruples (k, s, l, t) that satisfy

- (k, l) is equal to (f, g) or (g, f) , and
- s and t are both elements of the set $\{1, -1\}$.

We generalize $G(f, g)$, constructing a graph $W(f, g)$ on the set $C'(f, g)$, placing an edge between distinct elements (k, s, l, t) and (k', s', l', t') whenever the composites $k^s \circ l^t$ and $k'^{s'} \circ l'^{t'}$ are equal in \mathcal{G} . Thus, the connected components of $W(f, g)$ are

the fibers of the evaluation map $\text{ev} : C'(f, g) \rightarrow \text{Mor}(\mathcal{G})$ sending (k, s, l, t) to $k^s l^t$. We will always draw the vertices of $W(f, g)$ in the following configuration



and will usually suppress the labels on the vertices to points $*$. Note that $G(f, g)$ is the full subgraph of $W(f, g)$ defined on the upper six vertices. Similarly, $G(f^{-1}, g)$, $G(f, g^{-1})$, and $G(f^{-1}, g^{-1})$ are (respectively) isomorphic to the left, right, and lower T-shaped subgraphs of $W(f, g)$. For example, the canonical inclusion

$$G(f^{-1}, g) \hookrightarrow W(f, g)$$

is defined by sending (f^{-1}, s, g, t) to $(f, -s, g, t)$ and (g, s, f^{-1}, t) to $(g, s, f, -t)$.

The graph $W(f, g)$ is determined by its subgraphs $G(f^{\pm 1}, g^{\pm 1})$, which are in turn determined by $\text{Sd}\mathcal{G}$. By considering these four graphs simultaneously, one may obtain a good deal of information concerning $W(f, g)$.

As with $G(f, g)$, any two points of $C'(f, g)$ belonging to the same connected component of $W(f, g)$ must have an edge between them. Configurations in $W(f, g)$ correspond to equational conditions. We have the tables below, where the following are equivalent:

- (1) the upper equation holds
- (2) the lower graph is a subgraph of $W(f, g)$
- (3) a single edge of the lower graph is in $W(f, g)$.

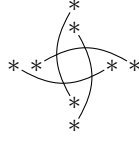
Equation	$fg = gf$	$f^2 = \text{id}$	$g^2 = \text{id}$	$f^2 = g^2$	$f^2 = g^{-2}$
Subgraph					
Equation		$fgf = g$	$gfg = f$	$fg^{-1}f = g$	$fgf = g^{-1}$
Subgraph					

The above graphs have the following symmetries:

- There is an edge in $W(f, g)$ between (k, s, l, t) and (m, u, n, v) if and only if there is an edge between $(l, -t, k, -s)$ and $(n, -v, m, -u)$. This comes from inversion

$$k^s l^t = m^u n^v \iff l^{-t} k^{-s} = n^{-v} m^{-u}.$$

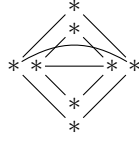
The pairing between vertices (k, s, l, t) and $(l, -t, k, -s)$ is illustrated below:



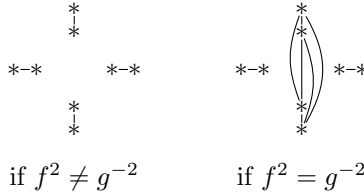
- For each graph above, there is another graph obtained by interchanging the roles of f and g . For example, the graphs corresponding to the equations $f^2 = \text{id}$ and $g^2 = \text{id}$ are one such pair.

Case 4, subcase 1: Suppose $f^2 \neq g^2$ and $fg = gf$ and $f^2 \neq \text{id}$ and $g^2 \neq \text{id}$.

Because $f^2 \neq g^2$ and $f^2 \neq \text{id}$ and $g^2 \neq \text{id}$, the graph $W(f, g)$ cannot contain any of the edges below:



The graph $W(f, g)$ must be equal to one of the two graphs below.



The graph on the left is guaranteed by the condition $fg = gf$. The graph on the right is the only possible extension that is consistent with the assumptions $f^2 \neq g^2$ and $f^2 \neq \text{id}$ and $g^2 \neq \text{id}$.

Suppose that $h = fg = gf$. Then in each of the two graphs above, the subgraph $G(f, g)$ contains two elements of the fiber of h , namely $(f, 1, g, 1)$ and $(g, 1, f, 1)$. Thus we have $\exists_2 \frac{f \wedge g}{h}$. Similarly, the left and right T-shaped subgraphs, which are respectively isomorphic to $G(f^{-1}, g)$ and $G(f, g^{-1})$, each contain either two or four elements of the fiber of h (depending on whether $f^2 = g^{-2}$). Therefore we have

$$h = fg = gf \implies \exists_2 \frac{f \wedge g}{h} \quad \text{and} \quad \exists_{\geq 2} \frac{f^{-1} \wedge g}{h} \quad \text{and} \quad \exists_{\geq 2} \frac{f \wedge g^{-1}}{h}.$$

On the other hand, suppose that h is not equal to fg or gf , and that the fiber of h is non-empty. If $f^2 = g^{-2}$ then $\text{ev}^{-1}\{h\}$ must be one of the connected components

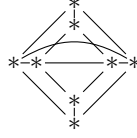
$$\left\{ (f, -1, g, 1), (g, 1, f, -1) \right\} \quad \text{or} \quad \left\{ (f, 1, g, -1), (g, -1, f, 1) \right\},$$

in which case we have (respectively) $\sharp \frac{f \wedge^{g^{-1}}}{h}$ or $\sharp \frac{f^{-1} \wedge^g}{h}$. If $f^2 \neq g^{-2}$ then the fiber of h must be one of the connected components not equal to $\{(f, 1, g, 1), (g, 1, f, 1)\}$, so we have $\sharp \frac{f \wedge^{g^{-1}}}{h}$ or $\sharp \frac{f^{-1} \wedge^g}{h}$ or $\sharp \frac{f \wedge^g}{h}$.

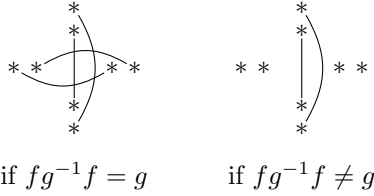
We conclude that $h = fg = gf \iff \exists \frac{f \wedge^g}{h}$ and $\exists \frac{f^{-1} \wedge^g}{h}$ and $\exists \frac{f \wedge^{g^{-1}}}{h}$.

Case 4, subcase 2: $f^2 \neq g^2$ and $fg \neq gf$ and $f^2 \neq \text{id}$ and $g^2 \neq \text{id}$ and $f^2 = g^{-2}$.

Because $f^2 \neq g^2$ and $fg \neq gf$ and $f^2 \neq \text{id}$ and $g^2 \neq \text{id}$, the graph $W(f, g)$ cannot contain any of the edges below:



Therefore, because $f^2 = g^{-2}$, the graph $W(f, g)$ must be equal to one of the two graphs displayed below.



Suppose first that $h = fg$ or $h = gf$. In each case above, the fiber of h must be equal to one of the connected components

$$\left\{ (f, 1, g, 1), (f, -1, g, -1) \right\} \quad \text{or} \quad \left\{ (g, 1, f, 1), (g, -1, f, -1) \right\}.$$

These connected components are both contained in the intersection of the left and right T-shaped subgraphs. Those two subgraphs correspond (respectively) to $G(f^{-1}, g)$ and $G(f, g^{-1})$, so we have $\exists_2 \frac{f^{-1} \wedge^g}{h}$ and $\exists_2 \frac{f \wedge^{g^{-1}}}{h}$.

Suppose now that h is not equal to fg or gf , and that the fiber $\text{ev}^{-1}\{h\}$ is non-empty. If $fg^{-1}f = g$ then $\text{ev}^{-1}\{h\}$ must be equal to one of the connected components

$$\left\{ (f, -1, g, 1), (g, -1, f, 1) \right\} \quad \text{or} \quad \left\{ (g, 1, f, -1), (f, 1, g, -1) \right\}.$$

If $fg^{-1}f \neq g$ then the fiber of h is equal to one of the singleton sets

$$\{(f, -1, g, 1)\}, \{(g, -1, f, 1)\}, \{(g, 1, f, -1)\}, \{(f, 1, g, -1)\}.$$

In either case, the intersection of $\text{ev}^{-1}\{h\}$ with the left T-shaped subgraph contains at most one point, and similarly for the intersection with the right T subgraph. Therefore we have the following:

$$h = fg \text{ or } h = gf \iff \exists_2 \begin{array}{c} f^{-1} \\ \diagdown \\ h \end{array} \diagup^g \quad \text{and} \quad \exists_2 \begin{array}{c} f \\ \diagdown \\ h \end{array} \diagup^{g^{-1}}.$$

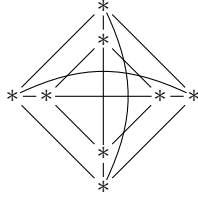
Case 4, subcase 3: $f^2 \neq g^2$ and $fg \neq gf$ and $f^2 \neq \text{id}$ and $g^2 \neq \text{id}$ and $f^2 \neq g^{-2}$.

This is the most complicated case. As before, our strategy is to calculate all possibilities for the graph $W(f, g)$ that are consistent with the above assumptions on f and g .

First note that the nine graphs described in the table on page 49 have pairwise disjoint edges sets, and that there are a total of 28 edges among those graphs. Therefore, the complete graph on 8 vertices has edges given by the disjoint union of the edge sets corresponding to the nine graphs displayed in the table (because the complete graph on 8 vertices has 28 edges).

Next, recall that if $W(f, g)$ contains a single edge from one of the nine graphs in the table, then $W(f, g)$ contains that entire graph as a subgraph. Therefore, $W(f, g)$ is the union of some choice of subgraphs from among the nine displayed in the table (where by “union” we mean that the edges of $W(f, g)$ are obtained as a disjoint union of the edge sets of some subgraphs displayed in the table).

A given graph from the table is contained in $W(f, g)$ if and only if the corresponding equation is satisfied. Thus, the graph $W(f, g)$ determines a set of “generating relations” among the nine equations listed in the table. Explicitly, $W(f, g)$ assigns a value of T/F to each of the nine equations. Given our assumptions that $f^2 \neq g^2$ and $fg \neq gf$ and $f^2 \neq \text{id}$ and $g^2 \neq \text{id}$ and $f^2 \neq g^{-2}$, the graph $W(f, g)$ cannot contain any of the edges below:



We need only to consider the remaining four equations from the table on page 49:

$$(A.2) \quad f g f = g, \quad g f g = f, \quad f g^{-1} f = g, \quad f g f = g^{-1}.$$

Below is the union of the graphs corresponding to the above four equations:



The graph $W(f, g)$ must be a subgraph of the union above, for any edge in $W(f, g)$ belongs to a graph corresponding to one of the generating relations (A.2).

Each assignment of T/F to the four generating relations (A.2) corresponds to a subgraph of the above (A.3). Given that each connected, full subgraph of $W(f, g)$ must be a complete graph, only some assignments of T/F correspond to actual possibilities for the graph $W(f, g)$. Thus, only some subgraphs of the union above could possibly be equal to $W(f, g)$.

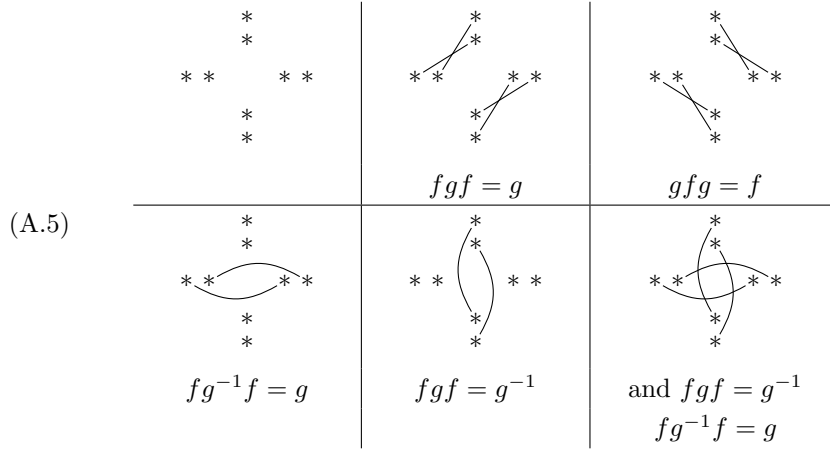
We consider the space $S = \{T, F\}^4$ of quadruples of T's and F's, where the four coordinates correspond respectively to each of the equations (A.2). Given an element γ of S , the corresponding subgraph of (A.3) is given by the union of graphs corresponding to those equations (A.2) for which γ 's value is T. For example, the assignment (F, F, F, F) corresponds to the graph with eight vertices and no edges, the assignment (T, T, T, T) corresponds to the union (A.3), and the assignment (T, F, F, F) gives the graph associated to the equation $fgf = g$ (see the table on page 49).

We will say that a graph is *valid* if each of its connected components is a complete graph. The graph $W(f, g)$ must be a subgraph of (A.3) that is valid, and it and must correspond to one of the elements of S . We claim that the following six elements of S are the only ones whose corresponding graphs are valid:

(A.4)

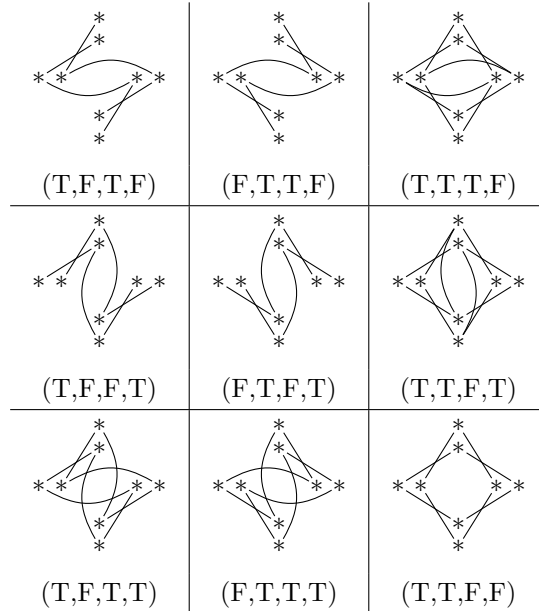
(F, F, F, F)	(T, F, F, F)	(F, T, F, F)
(F, F, T, F)	(F, F, F, T)	(F, F, T, T)

The elements of S above correspond (respectively) to the following six graphs.



It is clear that any connected component in any of the above graphs is complete: each connected component has size at most two.

It remains to show that the elements of S that are *not* displayed in Table (A.4) all correspond to graphs that are non-valid. We have seen that the graph (A.3), which corresponds to the assignment (T, T, T, T), is not valid. The remaining elements of S are displayed below, together with their corresponding subgraphs.



By inspection, each of the above graphs has a connected component that is not complete. Thus, none of the graphs displayed above are valid, so $W(f, g)$ must be equal to one of the six graphs (A.5).

We can now claim that h equals fg or gf if and only if one of the following is satisfied:

- (1) $\exists_1 \frac{f \wedge g}{h}$ and $\exists_1 \frac{f^{-1} \wedge g}{h}$ and $\exists_1 \frac{f \wedge g^{-1}}{h}$, or
 (2) $\exists_1 \frac{f^{-1} \wedge g^{-1}}{h}$ and two of the following three hold:

$$\exists_2 \frac{f \wedge g}{h} \quad \text{or} \quad \exists_2 \frac{f^{-1} \wedge g}{h} \quad \text{or} \quad \exists_2 \frac{f \wedge g^{-1}}{h}.$$

The proof of this claim is given below in two steps: “only if” and “if.”

Only if. Suppose that h is equal to fg or to gf . If $W(f, g)$ is equal to one of the graphs on the left hand side of the table (A.5), corresponding to assignment (F, F, F, F) or (F, F, T, F), then condition (1) is satisfied: each of the subgraphs $G(f, g)$, $G(f^{-1}, g)$ and $G(f, g^{-1})$ of $W(f, g)$ contains exactly one element of the fiber $\text{ev}^{-1}\{h\}$.

If $W(f, g)$ is equal to one of the graphs in the center or on the right hand side of the table, corresponding to (T, F, F, F) or (F, T, F, F) or (F, F, F, T) or (F, F, T, T), then condition (2) is satisfied: the subgraph $G(f^{-1}, g^{-1})$ of $W(f, g)$ contains one element of the fiber of h , and exactly two of the subgraphs $G(f, g)$, $G(f^{-1}, g)$ and $G(f, g^{-1})$ contain two elements of the fiber of h .

If. If $W(f, g)$ is equal to one of the two graphs on the left of the table (A.5), and if condition (1) is satisfied, then $h = fg$ or $h = gf$. If $W(f, g)$ is equal to one of the other four graphs, then (1) is false for every h .

If $W(f, g)$ is equal to one of the four graphs on the right or in the middle of the table, and if condition (2) is satisfied, then we must have $h = fg$ or $h = gf$. If $W(f, g)$ is equal to one of the two graphs on the left of the table, then condition (2) is false for every h . \square