COMPACT RIEMANN SURFACES: A THREEFOLD CATEGORICAL EQUIVALENCE

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Abstract. We define and prove basic properties of Riemann surfaces, which we follow with a discussion of divisors and an elementary proof of the Riemann-Roch theorem for compact Riemann surfaces. The Riemann-Roch theorem is used to prove the existence of a holomorphic embedding from any compact Riemann surface into n-dimensional complex projective space $\mathbb{P}^n$. Using comparison principles such as Chow's theorem we construct functors from the category of compact Riemann surfaces with nonconstant holomorphic maps to the category of smooth projective algebraic curves with regular algebraic maps and the category of function fields over $\mathbb{C}$ of transcendence degree one with morphisms of complex algebras. We conclude by proving that these functors establish a threefold equivalence of categories.

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1. Riemann Surfaces

Definition 1.1. An n-dimensional *complex manifold* X is a Hausdorff, connected, second-countable topological space with conformal transition maps between charts, that is, there is a covering of X by a family of open sets $\{U_\alpha\}$ and homeomorphisms $\varphi_\alpha : U_\alpha \to V_\alpha$ where $V_\alpha \subset \mathbb{C}^n$ is some open set for which

$$\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$$

is holomorphic. Such a collection of charts is a *complex atlas*. A complex manifold of (complex) dimension 1 is called a *Riemann surface*.

Since the transition maps are holomorphic and hence smooth, complex manifolds have a natural smooth structure. Riemann surfaces are of particular interest since

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they are locally homeomorphic to $\mathbb{C}$. This allows many theorems concerning the complex plane to generalize to Riemann surfaces via passage through charts.

It is also possible to relax our requirements on Riemann surfaces. The assumption of connectedness can be dispensed by considering only connected components of the space. A famous result by Radó then tells us that every connected Riemann surface is automatically second-countable.

Example 1.2. The Riemann sphere is a canonical example of a Riemann surface, and it can be specified in three conformally equivalent ways. The first way is the one-point compactification of the complex plane, $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. For an atlas, take $\{(\mathbb{C}, z), (\widehat{\mathbb{C}} \setminus \{0\}, 1/z)\}$. Our holomorphic transition map is $z \mapsto 1/z$. We can also identify the Riemann sphere with the real two-sphere $S^2 \subset \mathbb{R}^3$. Our charts are the pairs $(S^2 \setminus (0,0,1), \varphi_+)$ and $(S^2 \setminus (0,0,-1), \varphi_-)$, where $\varphi_\pm$ are the stereographic projections

$$\varphi_+(x_1,x_2,x_3) = \frac{x_1 + ix_2}{1 - x_3}, \quad \varphi_-(x_1,x_2,x_3) = \frac{x_1 - ix_2}{1 + x_3}.$$

If $p = (x_1,x_2,x_3) \in S^2$ is such that $x_3 \neq \pm 1$, then $\varphi_+(p)\varphi_-(p) = 1$. Hence our transition map is still the inversion $z \mapsto 1/z$ sending the punctured complex plane $\mathbb{C}^\times$ to itself. Lastly, we have the one-dimensional complex projective space

$$\mathbb{P}^1 := \{[z,w] : (z,w) \in \mathbb{C}^2 \setminus \{(0,0)\}\}/\sim$$

where the equivalence relation is $(z_1,w_1) \sim (z_2,w_2)$ if and only if there is some $\lambda \in \mathbb{C}^\times$ such that $(z_1,w_1) = (\lambda z_2, \lambda w_2)$. Our charts are (U_1,φ_1) and (U_2,φ_2) where

$$\begin{aligned} U_1 &= \{[z,w] \in \mathbb{P}^1 : w \neq 0\}, \quad \varphi_1([z,w]) = z/w \\ U_2 &= \{[z,w] \in \mathbb{P}^1 : z \neq 0\}, \quad \varphi_2([z,w]) = w/z. \end{aligned}$$

Unsurprisingly the holomorphic transition map is still $z \mapsto 1/z$.

Next we catalog some basic facts concerning analysis on Riemann surfaces.

Definition 1.3. A continuous map $f : X \to Y$ between Riemann surfaces is said to be *holomorphic* if it is holomorphic[1] in charts, that is, for every $p \in X$, if $(U_\alpha, \varphi_\alpha)$ is a chart containing p on X and (V_β, ψ_β) is a chart containing $f(p)$ on Y then $\psi_\beta \circ f \circ \varphi_\alpha^{-1}$ is holomorphic in the usual sense. Similarly, $f : X \to Y$ is *meromorphic* if it is meromorphic in charts.

Theorem 1.4 (Uniqueness Theorem)**.** *Let $f : X \to Y$ and $g : X \to Y$ be holomorphic maps. Then either $f \equiv g$ or $\{p \in X : f(p) = g(p)\}$ is discrete in X.*

Proof. Define the sets

$$\begin{aligned} A &:= \{p \in X : \text{locally at } p,\ f \equiv g\} \\ B &:= \{p \in X : \text{locally at } p,\ f = g \text{ only on a discrete set}\}. \end{aligned}$$

It is clear that both A and B are open subsets of X. Hence if we show $X = A \cup B$ we'll be done as X is connected. If $p \in X$ is such that $f(p) \neq g(p)$ then $p \in B$, since if $f = g$ on more than a discrete set near p, i.e., a set containing a nonisolated point, then then the usual uniqueness theorem applied in charts would show $f = g$ locally,

[1] Throughout this paper we'll be careful to distinguish holomorphic functions (which we'll take to be complex-valued) and maps to codomains other than $\mathbb{C}$ (usually a Riemann surface).

so $f(p) = g(p)$. This is clearly absurd. Similarly, if $f(p) = g(p)$ then the uniqueness theorem applied in charts shows $f = g$ locally whenever $\{p \in X : f(p) = g(p)\}$ is not discrete. Thus we indeed have $X = A \cup B$ and these sets are open and disjoint. Applying connectedness, the claimed result follows. □

Theorem 1.5 (Open Mapping Theorem)**.** *Let $f : X \to Y$ be holomorphic. If f is not constant, then f maps open subsets in X to open subsets of Y and $f(X)$ is open.*

Proof. By the uniqueness theorem if f is locally constant around any point, then f is globally constant. Hence applying the usual open mapping theorem in every chart allows us to conclude. □

Corollary 1.6. *Let X be compact and $f : X \to Y$ holomorphic and nonconstant. Then f is surjective and Y is compact.*

Proof. Since $f(X)$ is both closed (as a compact subset of a Hausorff space) and open by the open mapping theorem, we conclude $f(X) = Y$. □

Corollary 1.7. *Let X be a Riemann surface. The following properties are true:*

1. *If X is compact then every holomorphic function $f : X \to \mathbb{C}$ is constant.*
2. *If X is compact then every meromorphic function $f : X \to \widehat{\mathbb{C}}$ is surjective.*

Proposition 1.8. *Given a holomorphic map f between Riemann surfaces X and Y, there is a unique integer m such that there are local coordinates near p and $f(p)$ with f having the form $z \mapsto z^m$.*

Proof. Choose a chart ψ on Y centered at $f(p)$ and a chart φ on X centered at p. Then $T = \psi \circ f \circ \varphi^{-1}$ is locally holomorphic. Since $T(0) = 0$ by our choice of charts, taking a Taylor expansion shows T has the form $T(w) = w^m S(w)$ where S is holomorphic and nonzero at $w = 0$. We can obviously find a function R holomorphic near 0 such that $R(w)^m = S(w)$. Now $\eta(w) = wR(w)$ has nonvanishing derivative at 0 and is thus invertible by the inverse function theorem. Defining a new chart on X by $\widetilde{\varphi} = \eta \circ \varphi$ gives us new coordinates $z = \eta(w)$ on X, for which

$$(\psi \circ f \circ \widetilde{\varphi}^{-1})(z) = (\psi \circ f \circ \varphi^{-1})(\eta^{-1}(z)) = T(w) = z^m. \qquad \square$$

Definition 1.9. The integer m given by the above proposition is the *multiplicity* of f at p, denoted by $\mathrm{mult}(f, p)$.

Definition 1.10. Let $f : X \to Y$ be a nonconstant holomorphic map. A point $p \in X$ is a *ramification point* for f if $\mathrm{mult}(f, p) \geq 2$. A point $y \in Y$ is a *branch point* for f if it is the image of a ramification point for f.

When our Riemann surfaces are compact, holomorphic mappings exhibit several beautiful properties.

Proposition 1.11. *Let $f : X \to Y$ be a nonconstant holomorphic map between compact Riemann surfaces. For each $q \in Y$ define $d_q(f)$ to be the sum of the multiplicities of f at the points of X mapping to q:*

$$d_q = \sum_{p \in f^{-1}(q)} \mathrm{mult}(f, p).$$

Then $d_q(f)$ is constant, independent of q.

This proposition motivates our next definition.

Definition 1.12. Let $f : X \to Y$ be a nonconstant holomorphic map between compact Riemann surfaces. The *degree* of f, denoted $\deg(f)$, is the integer $d_q(f)$ for arbitrary $q \in Y$.

This gives us the following corollary.

Corollary 1.13. *A holomorphic map between compact Riemann surfaces is an isomorphism if and only if it has degree one.*

Proof. The multiplicity of a holomorphic map is nonnegative, so a degree one map must associate to each $q \in Y$ precisely one preimage $p \in X$. This is precisely what it means to be injective. Surjectivity follows by Corollary 1.6. $\square$

Definition 1.14. Let X be a compact Riemann surface. A triangulation of X consists of finitely many triangles W_i such that $X = \bigcup_{i=1}^{n} W_i$. By a triangle, we mean a closed subset of S homeomorphic to a plane triangle Δ, which is a compact subset of $\mathbb{C}$ bounded by three distinct straight lines. Furthermore, we require that any two triangles W_i, W_j be either pairwise disjoint, intersect at a single vertex, or intersect along a common edge.

Definition 1.15. Let X be a compact Riemann surface. Suppose that a triangulation with v vertices, e edges and t triangles of X is given. The *Euler characteristic* of X (with respect to this triangulation) is the integer $\chi(X) = v - e + t$. The *genus* of X, denoted by $g(X)$, is defined by the relation

$$\chi(X) = 2 - 2g(X).$$

It is a natural question to ask at this point whether *every* compact Riemann surface admits a triangulation. The answer is yes, but the proof is nontrivial so we omit it here. An interested reader may find a proof in §2.3.A of [5]. We might also wonder whether χ and g are uniquely defined on a given surface. As it turns out, these quantities are topological invariants (see the appendix of [6]). Intuitively, the genus of a surface can be thought of as the number of handles on the surface, since any such surface is homeomorphic to a sphere with g handles.

Furthermore, given a holomorphic map between two compact Riemann surfaces we can relate the two genera using information about the map.

Theorem 1.16 (Riemann-Hurwitz Formula)**.** *Let $f : X \to Y$ be a nonconstant holomorphic map between compact Riemann surfaces. Then*

$$2g(X) - 2 = \deg(f)(2g(Y) - 2) + \sum_{p \in X} [\mathrm{mult}(f, p) - 1].$$

Proof. Let $n = \deg(f)$. Note that since X is compact, the set of ramification points is finite. Take a triangulation T of Y such that each branch point of f is a vertex. Assume there are v vertices, e edges, and t triangles. By the definition of degree, each point in Y that is not a branch point has n preimages. In fact, using the local structure of f it is easily seen that f determines an n-sheeted covering map from $X \setminus f^{-1}(Q)$ to $Y \setminus Q$, where Q contains all the branch points. This allows us to lift T to a triangulation T' on X, with every ramification point of f as a vertex of T'.

The interior of each triangle t in Y is lifted to interiors of triangles on n sheets in X, so $t' = nt$. Similarly, $e' = ne$. Fixing a vertex $q \in Y$, the number of preimages of q in X is

$$|f^{-1}(q)| = n + \sum_{p\in f^{-1}(q)} [1 - \mathrm{mult}(f,p)].$$

Therefore the total number of preimages of vertices of Y, which is the number v' of vertices of X, is

$$v' = nv - \sum_{p\in X} [\mathrm{mult}(f,p) - 1]$$

as vertices at nonbranch points are lifted to n sheets at which the second term vanishes, and branch points have the correction term since $\mathrm{mult}(f,p) \geq 2$. Therefore

$$\begin{aligned} 2 - 2g(Y) &= v - e + t \\ 2 - 2g(X) &= nv - \sum_{p\in X} [\mathrm{mult}(f,p) - 1] + ne - nt \\ &= n(2 - 2g(Y)) - \sum_{p\in X} [\mathrm{mult}(f,p) - 1], \end{aligned}$$

which is the desired result after changing signs. □

2. Divisors and Meromorphic 1-forms

In this section we switch gears to the theory of divisors and meromorphic forms and functions on compact Riemann surfaces. As we'll eventually see in Sections 4 and 5 with the Riemann-Roch theorem, knowledge about the spaces of these objects will give us topological data for the associated surface.

Definition 2.1. A *holomorphic* 1*-form* on a Riemann surface X is a differential 1-form on X that can be written as $\omega = f dz$, where f is holomorphic and z is a local coordinate. A similar definition applies to meromorphic 1-forms.

We will use the notation $\mathcal{M}(X)$ for the space of meromorphic functions on X and $\mathcal{M}^{(1)}(X)$ for the space of meromorphic 1-forms on X.

Theorem 2.2. *Given points p and q on a compact Riemann surface X there is a meromorphic function $f \in \mathcal{M}(X)$ such that $f(p) = 0$ and $f(q) = \infty$.*

Proof. This result is nontrivial. See [10], page 257. □

Lemma 2.3. *Let ω_1 and ω_2 be two meromorphic 1-forms on a Riemann surface X, with ω_1 not identically zero. Then there is a unique meromorphic function f on X with $\omega_2 = f\omega_1$.*

Proof. Choose a chart $\varphi : U \to V$ on X giving local coordinate z. Write $\omega_i = g_i(z)\, dz$ for meromorphic functions g_i on V. Let $h = g_2/g_1$ be the ratio of these functions, which is well-defined as a function of the common local coordinate z of g_1 and g_2. In particular, h is holomorphic whenever g_2 and g_1 are with $g_1 \neq 0$. At all other points h may have a pole, and it is easy to see that this is a discrete set as X is compact and the set of poles (and for g_1, zeroes) of g_1 and g_2 is discrete. Thus h is also a meromorphic function on V. Now define $f = h \circ \varphi$, a meromorphic function on U.

It is easy to check that f is well-defined, independent of the choice of coordinate chart. This is the desired function. □

Definition 2.4. For a Riemann surface X, a *divisor* is a function $D : X \to \mathbb{Z}$ whose support is a discrete subset of X.

The divisors on X clearly form a group under pointwise addition, which elides a convenient notation as a formal sum

$$D = \sum_{p \in X} D(p) \cdot p,$$

where the set of points $p \in X$ with $D(p) \neq 0$ is discrete. The support of D is clearly finite if X is also compact, so the group of divisors for a compact Riemann surface is a free abelian group whose elements are finite formal linear combinations of points of X. We denote the space of divisors on X by $\mathrm{Div}(X)$.

Definition 2.5. The *degree* of a divisor D on a compact Riemann surface is defined as

$$\deg(D) = \sum_{p \in X} D(p).$$

Definition 2.6. To each nontrivial meromorphic function f on X we associate a *principal divisor* given by

$$(f) = \sum_{p \in X} \operatorname{ord}(f, p) \cdot p.$$

Similarly, for a meromorphic 1-form ω on X we associate a *canonical divisor* given by

$$(\omega) = \sum_{p \in X} \operatorname{ord}(\omega, p) \cdot p$$

where $\operatorname{ord}(\omega, p) = \operatorname{ord}(g, p)$ with $\omega = g\,dz$ being the local representation of ω at p.

Proposition 2.7. *If f is a meromorphic function on a compact Riemann surface X, then $\deg((f)) = 0$. In other words, the degree of any principal divisor on X is 0.*

Definition 2.8. Define a relation $\sim$ between two divisors D_1 and D_2 on a Riemann surface such that $D_1 \sim D_2$ if and only if there is a meromorphic function f satisfying $D_1 = D_2 + (f)$.

Let f, g be meromorphic functions and ω a meromorphic 1-form. We clearly have $(f \cdot g) = (f) + (g)$ and $(f/g) = (f) - (g)$. We also have the formula

$$(f\omega) = (f) + (w)$$

when f is a nonzero meromorphic function and ω is a nonzero meromorphic 1-form on X. By Proposition 2.7 and the additivity of degree, $\deg((fw)) = \deg((w))$. In particular, all canonical divisors on X have the same degree by Lemma 2.3.

Theorem 2.9. *If K is a canonical divisor on a compact Riemann surface X, then* $\deg(K) = -\chi = 2g - 2$.

Proof. Since all canonical divisors have the same degree, consider the meromorphic 1-form $K = df$, where f is a nonconstant meromorphic function whose existence is guaranteed by Theorem 2.2. We view this function as a holomorphic map from X

to the Riemann sphere $\widehat{\mathbb{C}}$. The Riemann sphere has genus 0, so by the Riemann-Hurwitz formula we may conclude once we show

$$\deg(K) = \sum_{p \in X} \operatorname{ord}(df, p) = 2n - \sum_{p \in X} [\operatorname{mult}(f, p) - 1]$$

where $n = \deg(f)$. By application of a Möbius transformation to rotate $\widehat{\mathbb{C}}$ we may assume ∞ is not a branch point. Since X is compact there is a finite number of ramification points for f, say r of them. Let $m_1, \dots, m_r$ be the corresponding multiplicities of f. If f has multiplicity m_i at a point, then df has a zero of order $m_i - 1$ at this point (since the derivative of z^{m_i} is $m_i z^{m_i - 1}$). Furthermore, as f had n preimages of ∞, which are each simple poles, df has n poles of order two. At all other points the order of df is zero, so

$$\sum_{p \in X} \operatorname{ord}(df, p) = 2n - \sum_{i=1}^{r} (m_i - 1),$$

which proves the claim. □

3. Spaces of Meromorphic Functions and 1-forms

We define a partial ordering on the divisors of X as follows: $D \geq 0$ if $D(p) \geq 0$ for all p, $D > 0$ if $D \geq 0$ and $D \neq 0$, and $D_1 \geq D_2$ if $D_1 - D_2 \geq 0$ (and similarly for $>$).

Definition 3.1. The *space of meromorphic functions with poles bounded by D*, denoted $L(D)$, is the set of meromorphic functions

$$L(D) = \{f \in \mathcal{M}(X) : (f) \geq -D\}.$$

$L(D)$ is a vector space over $\mathbb{C}$, and we define its dimension to be $l(D)$.

Lemma 3.2. *Let X be a compact Riemann surface. If D is a divisor on X with $\deg(D) < 0$, then $L(D) = \{0\}$.*

Proof. Suppose that $f \in L(D)$ and f is not identically zero. Consider the divisor $E = (f) + D$. Since $f \in L(D)$, $E \geq 0$, so certainly $\deg(E) \geq 0$. However since $\deg((f)) = 0$, we have $\deg(E) = \deg(D) < 0$. This contradiction proves the result. □

Definition 3.3. The *space of meromorphic 1-forms with poles bounded by D*, denoted $I(D)$, is the set of meromorphic 1-forms

$$I(D) = \{\omega \in \mathcal{M}^{(1)}(X) : (\omega) \geq -D\}.$$

This is also a vector space over $\mathbb{C}$, and we denote its dimension by $i(D)$.

Theorem 3.4. *The multiplication map μ_ω gives an isomorphism $\mu_\omega : L(K - D) \to I(D)$. Thus $i(D) = l(K - D)$.*

Proof. The map is obviously linear and injective. For surjectivity, choose a 1-form $\omega' \in I(D)$, so that $(\omega') - D \geq 0$. By Lemma 2.3 there is a meromorphic function f such that $\omega' = f\omega$. Note that

$$K + (f) - D = (\omega) + (f) - D = (f\omega) - D = (\omega') - D \geq 0,$$

so $f \in L(K - D)$. We conclude $\mu_\omega(f) = \omega'$. □

Lemma 3.5. *Let X be a Riemann surface, and let D be a divisor on X, and let p be a point of X. Then either $L(D-p) = L(D)$ or $L(D-p)$ has codimension one in $L(D)$.*

Proof. Choose a local coordinate z centered at p, and let $n = -D(p)$. Then every function f in $L(D)$ has a Laurent series at p of the form cz^n+ higher order terms. Define a map $\alpha : L(D) \to \mathbb{C}$ by sending f to the coefficient of the z^n term in its Laurent series. Clearly α is a linear map, and the kernel of α is exactly $L(D-p)$. If α is identically the zero map, then $L(D-p) = L(D)$. Otherwise α is onto, and so $L(D-p)$ has codimension one in $L(D)$. $\square$

Theorem 3.6. *Let X be a compact Riemann surface and let D be a divisor on X. Then the space of functions $L(D)$ is a finite-dimensional complex vector space. Indeed, given the decomposition $D = P - N$, we have the dimensional estimate $l(D) \leq 1 + \deg(P)$.*

Proof. Note that the statement is true for $D = 0$: on a compact Riemann surface, $L(0)$ conists of only the constant functions and therefore has dimension one. We go by induction on the degree of the positive part P of D. If $\deg(P) = 0$, then $P = 0$, so that $l(P) = 1$; since $D \leq P$ we see that $L(D) \subset L(P)$, so that $l(D) \leq l(P) = 1 = 1 + \deg(P)$ as required.

Assume then that the statement is true for divisors whose positive part has degree $k-1$, and let us prove it for a divisor whose positive part has degree $k \geq 1$. Fix such a divisor D, and write $D = P - N$ as above, with $\deg(P) = k$. Choose a point p in the support of P, so that $P(p) \geq 1$. Consider the divisor $D - p$; its positive part is $P - p$, which has degree $k - 1$. Hence the induction hypothesis applies, and we have that $l(D-p) \leq \deg(P-p) + 1 = \deg(P)$. Now we apply the codimension statement above and conclude that $l(D) \leq 1 + l(D-p)$. Hence $l(D) \leq \deg(P) + 1$, as claimed. $\square$

4. Riemann-Roch Theorem

In this section we present an argument for the famous Riemann-Roch theorem for compact Riemann surfaces.

Definition 4.1. Let X be a Riemann surface, and fix some local coordinate z at p. If we have the Laurent expansion

$$f = \sum_{k=-\infty}^{\infty} c_k z^k$$

in some neighborhood about p, then the *Laurent principal part* is the tail end of the series

$$\sum_{k=-\infty}^{-1} c_k z^k.$$

We now turn to the problem of finding meromorphic functions with prescribed poles and principal parts on a Riemann surface, otherwise known as the Mittag-Leffler problem. As it turns out, the Mittag-Leffler problem is always solvable over $\mathbb{C}$ as well as any noncompact Riemann surface. For the case where X is compact, we have

Theorem 4.2. *Given a set of points $\{p_1, \ldots, p_n\}$ on a compact Riemann surface X and a set of principal parts $\{f_1, \ldots, f_n\}$, there is a meromorphic function f that has principal part f_i at each p_i and no other poles if and only if $\sum_{i=1}^n \operatorname{Res}(f_i\omega, p_i) = 0$ for all holomorphic 1-forms ω on X.*

Proof. See [1], page 57 or [8], page 173. □

Theorem 4.3. *On a compact Riemann surface X of genus g there are g linearly independent holomorphic 1-forms, so $l(K) = i(0) = g$.*

Proof. See [1], page 53 or [11], page 252 or [10], page 284. □

We now have all the tools we need to prove one of the most important theorems in the study of Riemann surfaces, the Riemann-Roch theorem. We opt for an analysis-flavored approach that closely follows [7], though one can use cohomology and sheaves just as well.

Theorem 4.4 (Riemann-Roch)**.** *Let K be a canonical divisor on X. Then for any divisor D,*

$$l(D) = \deg(D) + 1 - g + l(K - D)$$

Proof. First we prove the statement for the case where $D \geq 0$. We may dispense with the case where $D = 0$ as this is just the formula $l(0) = 1$ by Theorem 4.3, which is true since $L(0)$ is just the space of holomorphic functions and hence the space of constants $\mathbb{C}$. Now consider a divisor $D > 0$ with

$$D = \sum_{i=1}^{n} n_i p_i$$

where $n_i \geq 0$ for each i.

Let V be the set of tuples $(f_1, \ldots, f_n)$ of Laurent principal parts of the form

$$f_i = \frac{c_{m_i}}{z^{m_i}} + \cdots + \frac{c_{-1}}{z}.$$

Obviously V is a vector space over $\mathbb{C}$ with dimension $\deg(D)$.

Define the map $\varphi : L(D) \to V$ that sends $f \in L(D)$ to the tuple of Laurent principal parts of f at the points p_i. Also, consider the kernel of φ, which is the set of functions in $L(D)$ that are sent to 0. Since $D > 0$, a function f with $(f) \geq -D$ that is in the kernel has no Laurent principal parts at the points p_i and can have no other poles. This implies that such an f is holomorphic and hence constant since X is compact, as before.

Now if we let $\operatorname{Im}(\varphi) = W$, we have

$$l(D) = \dim(\operatorname{Ker}(\varphi)) + \dim(\operatorname{Im}(\varphi)) = 1 + \dim(W)$$

by the rank-nullity theorem. To compute the dimension of W, observe that W is the set of tuples of Laurent principal parts $(f_1, \ldots, f_n)$ such that there is a $f \in L(D)$ whose tail is f_i at p_i. According to Theorem 4.2, such f exists if and only if for all holomorphic 1-forms ω on our Riemann surface $\sum_{i=1}^n \operatorname{Res}(f_i\omega, p_i) = 0$. For each holomorphic 1-form ω we will consider the linear map $\lambda_\omega : V \to \mathbb{C}$ given by

$$(f_1, \ldots, f_n) \mapsto \sum_{i=1}^{n} \operatorname{Res}(f_i\omega, p_i).$$

Now $W = \bigcap \mathrm{Ker}(\lambda_\omega)$ is the intersection of kernels of λ_ω for all holomorphic 1-forms ω. It follows that

$$\dim(W) = \dim\left(\bigcap \mathrm{Ker}(\lambda_\omega)\right) = \dim(V) - \dim \operatorname{span}(\{\lambda_\omega\})$$

We already know that $\dim(V) = \deg(D)$ so $\dim(W) = \deg(D) - \dim \operatorname{span}(\{\lambda_\omega\})$

Next, we need to find $\dim \operatorname{span}(\{\lambda_\omega\})$. We know that the dimension of this span is at most the the genus g, since there are g linearly independent holomorphic 1-forms on a compact Riemann surface by Theorem 4.3. If the Laurent principal part f_i at p_i is sent to zero we need to multiply it by an ω which has $\mathrm{ord}(\omega, p_i) \geq m_i$. This is true if and only if $(\omega) \geq D$, i.e., if and only if ω is in $I(D)$. Thus $\lambda_\omega = 0$ if and only if ω is in $I(D)$, and so $\dim \operatorname{span}(\{\lambda_\omega\}) = g - i(D) = g - l(K - D)$. This proves $l(D) = \deg(D) + 1 - g + l(K - D)$ where D is a nonnegative divisor, as desired.

Next, we prove the inequality $l(D) - l(K - D) \geq 1 + \deg(D) - g$. We will show this by first proving $l(D - p) - i(D - p) \geq (l(D) - i(D)) - 1$ for a point $p \in X$. It is obvious that $l(D) \geq l(D - p) \geq l(D) - 1$ and $i(D) + 1 \geq i(D - p) \geq i(D)$. Thus the worst case scenario occurs when $l(D - p) = l(D) - 1$ and $i(D) + 1 = i(D - p)$ since this would imply $l(D - p) - i(D - p) = (l(D) - i(D)) - 2$ (in all other cases we achieve or surpass the desired -1 lower bound). But this is impossible: take $f \in L(D) \setminus L(D - p)$ and $\omega \in I(D - p) \setminus I(D)$. We know that f and ω exist since $l(D - p) = l(D) - 1$ and $i(D) + 1 = i(D - p)$. Then $(f) \geq -D$, $(f) < p - D$, $(\omega) \geq D - p$, and $(\omega) < D$. Therefore $-D(p) + 1 > \mathrm{ord}(f, p) \geq -D(p)$, meaning $\mathrm{ord}(f, p) = -D(p)$.

Similarly, $D(p) > \mathrm{ord}(\omega, p) \geq D(p) - 1$, which implies $\mathrm{ord}(\omega, p) = D(p) - 1$. Thus, $\mathrm{ord}(f\omega, p) = -D(p) + D(p) - 1 = -1$. In addition, for all $q \neq p$ we have $\mathrm{ord}(f\omega, q) \geq 0$, since $(f) \geq -D$ and $(f\omega) \geq -p$, which implies that the poles may be at p only. Indeed, $\mathrm{ord}(f\omega, p) = -1$.

But now we have both $\sum_{i=1}^{n} \mathrm{Res}(f\omega, p_i) = 0$ and $\mathrm{Res}(f\omega, p) = c_{-1} \neq 0$ where c_{-1} is the coefficient of z^{-1} in the Laurent expansion at p (this is because we have a pole of exactly first order at p). Hence p is the only point at which we have a nonzero residue, which proves that the worst case scenario mentioned above can never happen.

Thus, $l(D - p) - i(D - p) \geq (l(D) - i(D)) - 1$, so for all divisors D we obtain the inequality

$$l(D) - i(D) \geq \deg(D) + 1 - g.$$

For the last part of the proof we substitute $K - D$ for D in the inequality proven above. Since $l(K - D) = i(D)$ we have $i(D) - l(D) \geq \deg(K) - \deg(D) + 1 - g$. Since $\deg(K) = -\chi = 2g - 2$ by Theorem 2.9, we obtain $l(D) - i(D) = \deg(D) + 1 - g$. Combining this with the inequality above the theorem is proved for every divisor D on the Riemann surface X. $\square$

Before concluding this section we consider the problem of embedding compact Riemann surfaces of genus 0 in projective space, which will illustrate the Riemann-Roch theorem's usefulness in such situations. We consider the problem of embedding for a general Riemann surface in the next section.

Corollary 4.5. *Suppose X is a compact Riemann surface of genus 0. Then X is analytically isomorphic to $\mathbb{P}^1$.*

Proof. Choose a point p on X and let $D = p \in \mathrm{Div}(X)$. We have

$$0 \leq i(D) \leq \dim \mathcal{H}^{(1)}(X) = g = 0.$$

Thus $i(D) = 0$, and by the Riemann-Roch theorem, we get

$$l(D) = d - g + i(D) + 1 = 1 - 0 + 0 + 1 = 2.$$

Therefore $L(p)$ necessarily contains two linearly independent functions, one of which may be chosen to be the constant function $f_1 = 1$. Since any holomorphic function on X must be a constant multiple of f_1, any other function must therefore be a meromorphic function which has p as a pole of order 1; call such a function f_2. Then $f_2^{-1}(\infty) = p$, and it follows that the holomorphic mapping $f_2 : X \to \mathbb{P}^1$ has degree 1. This then implies f_2 is an analytic isomorphism from X to $\mathbb{P}^1$. $\square$

5. Embedding into Projective Space

Definition 5.1. The *complete linear system of* D, denoted by $|D|$, is the set of all nonnegative divisors $E \geq 0$ which are linearly equivalent to D:

$$|D| = \{E \in \mathrm{Div}(X) : E \sim D,\, E \geq 0\}.$$

Definition 5.2. Given a divisor D on a compact Riemann surface X we say that the system $|D|$ is *free* if $l(D - p) = l(D) - 1$ for every $p \in \mathrm{supp}D$.

We associate with a divisor D on a compact Riemann surface the map $\varphi_D : S \to \mathbb{P}^n$ given by

$$p \mapsto [f_0(p), f_1(p), \ldots, f_n(p)]$$

where $f_0, \ldots, f_n$ are nontrivial meromorphic functions that constitute a basis for $L(D)$.

Lemma 5.3. *Let X be a compact Riemann surface and let D be a divisor on X where $|D|$ is a free system. Fix a point $p \in X$. Then there is a basis $f_0, f_1, \ldots, f_n$ for $L(D)$ such that* $\mathrm{ord}(f_0, p) = -D(p)$ *and* $\mathrm{ord}(f_i, p) > -D(p)$ *for $i \geq 1$.*

Proof. Consider the codimension one subspace $L(D-p)$ of $L(D)$ and let $f_1, \ldots, f_n$ be a basis for $L(D - p)$. Extend this to a basis for $L(D)$ by adding a function f_0 from $L(D) \setminus L(D - p)$. Then for every $i \geq 1$ we have

$$\mathrm{ord}(f_i, p) \geq -D(p) + 1 \geq -D(p).$$

Note further that $\mathrm{ord}(f_0, p) > -D(p)$, then $f_0 \in L(D - p)$, a contradiction. $\square$

Lemma 5.4. *For fixed distinct points p and q on X, $\varphi_D(p) = \varphi_D(q)$ if and only if $L(D - p - q) = L(D - p) = L(D - q)$. Hence φ_D is injective if and only if $l(D - p - q) = l(D) - 2$.*

Proof. If suffices to check whether $\varphi_D(p) = \varphi_D(q)$ using the basis for $L(D)$ given in the previous lemma. In this basis $\varphi_D(p) = [1, 0, \ldots, 0]$, so the injectivity fails exactly when $\varphi_D(q) = [1, 0, \ldots, 0]$ as well. This is equivalent to having $\mathrm{ord}(f_0, q) < \mathrm{ord}(f_i, q)$ for each $i \geq 1$ by the construction of the map. Since q is not a base point of $|D|$, this happens if and only if $\mathrm{ord}(f_0, q) = -D(q)$ and $\mathrm{ord}(f_i, q) > -D(q)$ for each $i \geq 1$, which occurs if and only if $\{f_1, \ldots, f_n\}$ is a basis for $L(D-q)$. However this basis was chosen such that $\{f_1, \ldots, f_n\}$ was a basis for $L(D-q)$. We conclude that the function spaces coincide precisely when injectivity fails.

Furthermore, as $|D|$ is base-point-free, $l(D - p) = l(D - q) = l(D) - 1$, meaning $l(D - p - q)$ must be either $l(D) - 1$ or $l(D) - 2$. By the first statement we know

$L(D-p-q)$ is a proper subspace of $L(D-p)$ when φ is injective, and thus has dimension $l(D)-2$. Conversely if the dimension always drops by two, we have the set of proper inclusions $L(D-p-q) \subset L(D-p) \subset L(D)$, where each of these spaces is distinct for every distinct p and q. The conclusion that φ_D is injective follows, and we are done. □

We have established a criterion that yields an injective map to $\mathbb{P}^n$, but we need to find an additional assumption that makes this map a holomorphic embedding. To do this we want one of the functions f_i (with $i \geq 1$) to vanish at p but not to order two—so the zero of f_i at p is simple. Then when we differentiate the ith coordinate the inverse function theorem shows φ_D is regular and hence an embedding. This scenario occurs exactly when f_i has order $-D(p)+1$, with f_0 attaining the minimum order $-D(p)$. Thus f_i is a function in $L(D-p)$ but $L(D-2p)$. A necessary and sufficient condition to ensure the existence of such a function is for $l(D-p-q) = l(D)-2$. This is the same requirement we had for injectivity, so we now have

Lemma 5.5. *For a compact Riemann surface X and base-point-free $|D|$, φ_D is an embedding if and only if*

$$l(D-p-q) = l(D) - 2$$

for any $p, q \in X$.

Now we are finally ready to make use of the Riemann-Roch formula

$$l(D) - l(K-D) = \deg(D) - g + 1.$$

This formula gives us a way to compute the values of $l(D-p-q)$ and $l(D)-2$, as well as verify that $|D|$ is free.

Proposition 5.6. *If D is a divisor of degree at least $2g-1$, then $l(D) = \deg(D) - g + 1$.*

Proof. The degree of a canonical divisor K is $\deg(K) = 2g-2$, meaning $K-D$ has degree at most -1. From Theorem 3.6, $l(K-D)$ is bounded by $\deg(K-D)+1$, so $l(K-D) = 0$ in the Riemann-Roch formula. □

Theorem 5.7 (Embedding Theorem)**.** *Every compact Riemann surface X of genus g can be embedded in $\mathbb{P}^n$ for some n.*

Proof. Pick a divisor D on X such that $\deg(D) \geq 2g+1$. Then D, $D-p$, and $D-p-q$ satisfy the proposition where p and q are arbitrary points from the support of D. Therefore

$$l(D) = \deg(D) - g + 1$$
$$l(D-p) = \deg(D-p) - g + 1$$
$$l(D-p-q) = \deg(D-p-q) - g + 1$$

Since $\deg(D-p-q) = \deg(D-p) - 1 = \deg(D) - 2$, $|D|$ is a free system and Lemma 5.5 applies. This completes the proof. □

6. Algebraic Curves

Definition 6.1. A subset $C \subset \mathbb{P}^n$ is a (projective) *algebraic variety* if there are homogeneous polynomials $\{F_\alpha\}$ such that $C = \{p \in \mathbb{P}^n : F_\alpha(p) = 0 \text{ for every } \alpha\}$. Algebraic curves are algebraic varieties of dimension 1. An *analytic variety* is a set that at every point is locally a zero set of a holomorphic function.

For our purposes we will always mean *projective* algebraic curves whenever we use the term. Before proving Chow's theorem, we record (without proof) an important result concerning complex manifolds due to Remmert and Stein. An interested reader can consult [3] or [4] for proof.

Theorem 6.2 (Proper Mapping Theorem)**.** *If M and N are complex manifolds, $f : M \to N$ is a holomorphic map, and $V \subset M$ is an analytic variety such that $f|_V$ is proper, then $f(V)$ is an analytic subvariety of N.*

Theorem 6.3 (Chow's Theorem)**.** *Any analytic subvariety of projective space is algebraic. Furthermore, holomorphic maps between analytic subvarieties induce regular rational morphisms between algebraic varieties.*

Proof. We prove the first statement, and recommend the reader see [3] for a proof of the second. Suppose that we have a complex analytic variety $X \in \mathbb{P}^n$, and let $\sigma : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ be the projection $(z_1, \ldots, z_{n+1}) \mapsto [z_1, \ldots, z_{n+1}]$. It is not hard to show that $\sigma^{-1}(X)$ is a complex analytic subvariety of $\mathbb{C}^{n+1} \setminus \{0\}$ using the proper mapping theorem. Furthermore V is a complex cone, so if $z = (z_1, \ldots, z_{n+1}) \in V$, then $tz \in V$ for all $t \in \mathbb{C}$.

Thus it suffices to show that if a complex analytic subvariety $V \subset \mathbb{C}^{n+1}$ is a complex cone, then it is given by the vanishing of finitely many homogeneous polynomials. Take a finite set of defining functions of V near the the origin such that $B \cap V = \{z \in V : f_1(z) = \cdots = f_k(z) = 0\}$ for $B = B(0, \epsilon)$. We can suppose that ϵ is small enough that the power series for f_j converges in B for all j. Expand f_j in a power series near the origin and group together homogenous terms as $f_j = \sum_{m=0}^{\infty} f_{jm}$, where f_{jm} is a homogeneous polynomial of degree m. For $t \in \mathbb{C}$ we write

$$f_j(tz) = \sum_{m=0}^{\infty} f_{jm}(tz) = \sum_{m=0}^{\infty} t^m f_{jm}(z).$$

For a fixed $z \in V$ we know that $f_j(tz) = 0$ for all $|t| < 1$, hence we have a power series in one variable that is identically zero, and so all coefficients are zero. Therefore f_{jm} vanishes on $V \cap B$ and thus on all of V. It follows that V is defined by a family of homogeneous polynomials. Since the ring of polynomials is Noetherian we need only finitely many. □

Conversely, a smooth algebraic curve lying in $\mathbb{P}^n$ is a compact Riemann surface. The fact that it is described by polynomials gives it a holomorphic structure and it is compact as a closed subset of the compact space $\mathbb{P}^n$. Combining the results from the embedding theorem and Chow's theorem as well as this construction proves

Theorem 6.4. *Every compact Riemann surface is isomorphic to a smooth algebraic curve.*

Any morphism between smooth algebraic curves is given by a regular rational mapping (i.e., the quotient of a polynomial mapping, such that the denominator doesn't vanish). These mappings are holomorphic in charts, which allows us to conclude that the mapping which assigns to every smooth algebraic curve its corresponding compact Riemann surface is a full functor. We have already shown that this functor is essentially surjective, so we need only verify that it is faithful—that is, injective on the morphisms. This is accomplished easily enough by considering a connected set of points on a curve in the image of regular maps $f^* = g^*$, which we lift to holomorphic mappings to conclude $f = g$ via the uniqueness theorem. Indeed, the following extension of Theorem 6.4 is true:

Theorem 6.5. *The functor described above establishes an equivalence between the categories of compact Riemann surfaces and smooth algebraic curves.*

See [2] for further discussion.

7. Function Fields

In this section we extend the results of the previous one by constructing a functor which takes compact Riemann surfaces to their associated fields of meromorphic functions. We will make use of the notation for $(\,\cdot\,)$ for algebraic ideals in this section—not to be confused with a divisor.

Let K be an algebraically closed field.

Theorem 7.1 (Hilbert's Nullstellensatz)**.** *Suppose $F(x,y), G(x,y) \in K[x,y]$ where F is irreducible. If the curve $C_G = \{G = 0\}$ vanishes at all points of the curve $C_F = \{F = 0\}$, then F divides G.*

Consider an irreducible polynomial $F(x,y) \in K[x,y]$ given by the equation

$$F(x,y) = p_0(x)y^n + p_1(x)y^{n-1} + \cdots p_n(x) = q_0(y)x^m + q_1(y)x^{m-1} + \cdots q_m(y)$$

where p_i and q_i are respectively polynomials in x and y. We then denote by $C_F^{(x)}$ and $C_F^{(y)}$ the sets

$$C_F^{(x)} = \{(x,y) \in \mathbb{C}^2 : F(x,y) = 0, F_y(x,y) \neq 0, p_0(x) \neq 0\}$$
$$C_F^{(y)} = \{(x,y) \in \mathbb{C}^2 : F(x,y) = 0, F_x(x,y) \neq 0, q_0(y) \neq 0\}.$$

The proofs of the next several theorems are quite lengthy, so we remark on them briefly and suggest the reader consult the cited references for details.

Theorem 7.2. *$C_F^{(x)}$ and $C_F^{(y)}$ are Riemann surfaces on which the coordinate projections $\pi_x : (x,y) \mapsto x$ and $\pi_y : (x,y) \mapsto y$ are holomorphic. There is a unique compact Riemann surface C_F that contains $C_F^{(x)}$ and $C_F^{(y)}$, and the branching points of the projections lie in $C_F \setminus C_F^{(x)}$ and $C_F \setminus C_F^{(x)}$, respectively.*

The holomorphic structure of each set is due to the implicit function theorem. Connectedness and the existence of a unique extension rely on Bezout's theorems and facts about covering maps. Please see [2] for a detailed proof.

Theorem 7.3. *Let C be a Riemann surface and $\mathcal{M}(C)$ be its field of meromorphic functions. Then $\mathcal{M}(C)$ is a finitely generated field over $\mathbb{C}$ with transcendence degree 1. Thus $\mathcal{M}(C)$ is isomorphic to $\mathbb{C}(f,g)$ where there is some polynomial F such that $F(f,g) = 0$.*

Proof. See [8], page 174. $\square$

Theorem 7.4. *Let C be a Riemann surface and identify $\mathcal{M}(C) = \mathbb{C}(f,g)$ per Theorem 7.3. Let $F(x,y)$ be an irreducible polynomial such that $F(f,g) \equiv 0$. Then the map $\psi : C \to C_F$ defined by $p \mapsto (f(p), g(p))$ is an isomorphism.*

Proof. We start by showing that ψ is a well-defined map. Note that F and F_y have only finitely may zeros in common per Bezout's theorem, so there are only finitely many points in $\widehat{\mathbb{C}}$ which do not lie in the image of the projection π_x. Let $Z = \{a_1, \dots, a_r, \infty\}$ be this finite set, so that $\pi_x(C_F^{(x)}) = \widehat{\mathbb{C}} \setminus Z$. Putting $C^0 = C \setminus f^{-1}(Z)$ we obtain the the following commutative diagram:

$$\begin{tikzcd} C^0 \arrow[r, "\psi"] \arrow[dr, "f"'] & C_F^{(x)} \arrow[d, "\pi_x"] \\ & \widehat{\mathbb{C}} \setminus Z \end{tikzcd}$$

Observe that if $f(p) = a \in \widehat{\mathbb{C}} \setminus Z$ then $g(p)$ must have been one of the n distinct roots of $F(a, y)$. Thus $\psi(p)$ is a well-defined point of $C_F^{(x)}$ for every $p \in C^0$. Due to Theorem 7.2, in order to extend ψ to all of C we need only show that $\psi : C^0 \to C_F^{(x)}$ is a holomorphic covering of finite degree, as such maps possess unique extensions to a compact Riemann surface. This follows from the fact that f and π_x are covering maps. Indeed, if $f^{-1}(V_a) = \bigsqcup U_i$ and $\pi_x^{-1}(V_a) = \bigsqcup W_j$, then $\psi^{-1}(W_j)$ can only be a disjoint union of a number of the open sets U_i.

It remains to be shown that ψ has degree 1, which will prove ψ is an isomorphism by Corollary 1.13. Suppose not; then the fibers of all but finitely many points $q = (a, b) \in C_F^{(x)}$ would contain at least two points q_1, q_2. Now let ψ be an arbitrary meromorphic function. As $\mathcal{M}(C)$ is generated by f and g, φ can be expressed as a rational function in f and g, say

$$\psi = \frac{\sum a_{ij} f^i h^j}{\sum b_{ij} f^i h^j}.$$

Then

$$\psi(q_1) = \frac{\sum a_{ij} a^i b^j}{\sum b_{ij} a^i b^j} = \psi(q_2),$$

so for any such pair q_1 and q_2 every meromorphic function takes the same values. In particular, no meromorphic function can have a zero at q_1 and a pole at q_2, a contradiction to Theorem 2.2. $\square$

Corollary 7.5. *Let (F) denote the ideal of $\mathbb{C}[x, y]$ generated by F. Then*

(1) *The correspondence determined by $x \mapsto f$, $y \mapsto g$ defines a $\mathbb{C}$-isomorphism from the quotient field of $\mathbb{C}[x, y]/(F)$ to $\mathcal{M}(C)$.*
(2) *The correspondence determined by $x \mapsto \pi_x$, $y \mapsto \pi_y$ defines a $\mathbb{C}$-isomorphism from the quotient field of $\mathbb{C}[x, y]/(F)$ to $\mathcal{M}(C_F)$. In particular, $\mathcal{M}(C_F) = \mathbb{C}(\pi_x, \pi_y)$.*

Proof. Since $F(f, g) = 0 \in \mathcal{M}(C)$, the mapping $x \mapsto f$ and $y \mapsto g$ defines a homomorphism of $\mathbb{C}$-algebras $\rho : \mathbb{C}[x, y]/(F) \to \mathcal{M}(C)$. It therefore suffices to show that the kernel of ρ is the ideal (F). If $G(x, y) \in \operatorname{Ker}(\rho)$, then $G(f, g) = 0 \in \mathcal{M}(C)$, which means $G(x, y)$ vanishes identically on the curve defined by $F(x, y) = 0$. Invoking Hilbert's Nullstellensatz, $G \in (F)$ and we may conclude. The second statement follows from Theorem 7.4. $\square$

This rule that associates to each Riemann surface C its function field $\mathcal{M}(C)$ and each holomorphism of Riemann surfaces $f : C_1 \to C_2$ the $\mathbb{C}$-algebra homomorphism

$f^* : \mathcal{M}(C_1) \to \mathcal{M}(C_2)$ defined by $f^*(\varphi) = \varphi \circ f$ is a functor to the category of function fields. In fact, we have the following remarkable result:

Theorem 7.6. *The functor described above establishes an equivalence between the categories of compact Riemann surfaces and function fields of transcendence degree 1.*

Proof. It is enough to prove the following two statements: (1) the functor is faithful, i.e., if $f, g \in \mathrm{Hom}(C_1, C_2)$ satisfy $f^* = g^*$, then $f = g$ and (2) the functor is full and essentially surjective, i.e., $\varphi : \mathcal{M}_1 \to \mathcal{M}_2$ is a $\mathbb{C}$-algebra homomorphism between $\mathcal{M}_1$ and $\mathcal{M}_2$, then there are Riemann surfaces C_1, C_2 with $f \in \mathrm{Hom}(C_1, C_2)$ such that the following diagram commutes

$$\begin{tikzcd} \mathcal{M}(C_2) \arrow[r, "f^*"] \arrow[d, "\alpha_2"'] & \mathcal{M}(C_1) \arrow[d, "\alpha_1"] \\ \mathcal{M}_2 \arrow[r, "\varphi"'] & \mathcal{M}_1 \end{tikzcd}$$

Take $f, g \in \mathrm{Hom}(C_1, C_2)$ together with $z \in C_1$ such that $f(z) = p$ and $g(z) = q$ with $p \neq q$. Lemma 2.2 states that there is a $\varphi \in \mathcal{M}_2$ such that $\varphi(p) \neq \varphi(q)$. Therefore

$$(f^*\varphi)(x) = \varphi(f(x)) \neq \varphi(g(x)) = (g^*\varphi)(x).$$

Hence the functor is a surjection on the morphisms of the two categories.

For the second part, let $\varphi : \mathcal{M}_2 \to \mathcal{M}_1$ be a $\mathbb{C}$-algebra homomorphism of fields (which is necessarily injective). Let f_1, g_1 be generators for $\mathcal{M}_1$ such that g_1 is algebraic over $\mathbb{C}(g_1)$; pick generators f_2, g_2 for $\mathcal{M}_2$ in the same manner. Now for polynomials $F(x, y)$ and $G(x, y)$ with the condition $F(f_1, g_1) = 0 = G(f_2, g_2)$, the following diagram commutes

$$\begin{tikzcd} \mathcal{M}(C_2) \arrow[r, "\widetilde{\varphi}"] \arrow[d, "\alpha_2"'] & \mathcal{M}(C_1) \arrow[d, "\alpha_1"] \\ \mathcal{M}_2 \arrow[r, "\varphi"'] & \mathcal{M}_1 \end{tikzcd}$$

where the α_i are isomorphisms from Corollary 7.5 given by sending the coordinate projections π_x, π_y of the Riemann surfaces C_F (respectively C_G) to the generators f_1, g_1 (respectively f_2, g_2) of $\mathcal{M}_1$ (respectively $\mathcal{M}_2$). We also have $\widetilde{\varphi} = \alpha_1^{-1} \circ \varphi \circ \alpha_2$.

Let $(\alpha_1^{-1} \circ \varphi)(f_2) = R_1(\pi_x, \pi_y) \in \mathcal{M}(C_F)$ and $(\alpha_1^{-1} \circ \varphi)(g_2) = R_2(\pi_x, \pi_y) \in \mathcal{M}(C_F)$ where R_1 and R_2 are rational functions. Note that we have

$$0 = G(f_2, g_2) \in \mathcal{M}(C_G)$$

and therefore

$$\begin{aligned} 0 &= (\alpha^{-1} \circ \varphi)(G(f_2, g_2)) \\ &= G\big((\alpha_1^{-1} \circ \varphi)(f_2), (\alpha_1^{-1} \circ \varphi)(g_2)\big) \\ &= G(R_1(\pi_x, \pi_y), R_2(\pi_x, \pi_y)) \in \mathcal{M}(C_F). \end{aligned}$$

By Theorem 7.4, we know that the function

$$f(x, y) = (R_1(x, y), R_2(x, y))$$

defines a morphism between the Riemann surfaces C_F and C_G. To finish the proof we need to show $f^* = \widetilde{\varphi}$. Note that f^* and $\widetilde{\varphi}$ agree on the generators π_x, π_y of $\mathcal{M}(C_2)$, i.e., we have the following:

$$f^*(\pi_x) = R_1(\pi_x, \pi_y) = (\alpha_1^{-1} \circ \varphi)(f_2) = (\alpha_1^{-1} \circ \varphi)(\alpha_2(\pi_x)) = \widetilde{\varphi}(\pi_x)$$

which proves our claim. □

At long last we have established the threefold categorical equivalence we sat out to prove. The significance of this equivalence is quite remarkable, as it allows us to use tools from both analysis and algebra to study essentially the same set of objects, and many statements that are difficult to prove for one category are much more salient to tools from the other.

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