CLASSIFICATIONS OF THE FLOWS OF LINEAR ODE

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Abstract. The goal of this paper is to examine characterizations of linear differential equations. We define the flow of an equation and examine some of the flows of linear equations. Then we partition the flows of such equations based on linear, differentiable, and topological equivalence.

Contents

1. Phase Flows and Phase Curves

In general, a typical ordinary differential equation features a spatial variable and a temporal variable, and thus the solution to an ODE will describe the motion of points through space. One important way to capture this idea is through the concept of flow.

Definition 1.1. A *phase flow* on phase space M is a one-parameter group of transformations g^t on M.

That is, for each time $t \in R$, there is a bijection $g^t : M \to M$ and this collection follows the group property. Think of M as being a collection of particles in space. This space is in some state at g^0. After time t, it is in a different state, with the particle at point x now at the point $g^t x$. We think of this process as deterministic, or that the state of the system at a moment uniquely determines the state in the past and the future, which will motivate our necessity that g be a bijection. Motivated by physical intuition, we require that the state at time $t + s$ be the same if we follow the particles to time t and then travel to time s, or if we jump by time $t + s$ immediately. This necessitates that $g^{t+s} = g^s g^t$. Similarly, we can look at the system in reverse, motivating the necessity that $g^{-t} = (g^t)^{-1}$.

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Example 1.2. One of the simplest physical examples is the group of transformations where we define $g^t(x) = x + t$. Physically, this takes the line, and sends it at unit speed to the right. Note that each of these maps is a bijection, that $g^{-t}x = x - t = (g^t)^{-1}x$, and $g^{t+s}x = x + t + s = g^s(x + t) = g^s g^t x$.

It is instructive to view a process both in terms of the state of the whole system at a specific time, like in the definition above, and by tracking the path of a specific particle over all time. This motivates the idea of a phase curve.

Definition 1.3. A *phase curve* of $\{g^t\}$ is the orbit of a point x, which is the set $\{y \in M \,:\, y = g^t x \text{ for some } t \in \mathbb{R}\}$.

Example 1.4. Let g^t be rotation of the plane about the origin by angle $2\pi t$, then the phase curves are the point 0 and circles around the origin.

It is often useful to require extra regularity on a one-parameter transformation group.

Definition 1.5. We say that $\{g^t\}$ is a *one-parameter diffemorphism group* if it is a one-parameter transformation group whose elements are diffeomorphisms and if $g^t x$ depends smoothly on x and t.

This allows us to formalize the concept of the velocity of a particle traveling in the phase space M. Let $\{g^t\}$ be a one-parameter group of diffeomorphisms.

Definition 1.6. The *phase velocity vector of the flow* $\{g^t\}$, denoted $v(x)$, is defined by

$$v(x) = \left.\frac{d}{dt}\right|_{t=0} g^t x.$$

Note that this defines a smooth vector field since $g^t x$ is smooth in x and t. This field represents the velocity at which $g^t x$ leaves x.

Consider the mapping $\phi(t) = g^t x_0$, and let v be the velocity field defined above.

Theorem 1.7. *The above map ϕ is the solution to the following ODE:*

$$\begin{cases} \dot{x} = v(x) \\ x(0) = x_0 \end{cases}$$

Proof. The proof follows from the group property of the transformation group.

$$\dot{\phi}(s) = \left.\frac{d}{dt}\right|_{t=s} g^t x_0 = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} g^{s+\varepsilon} x_0 = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} g^{\varepsilon} g^s x_0 = v(g^s x_0) = v(\phi(s))$$

Furthermore, since $g^0 x_0 = g^{-1+1} x_0 = (g^1)^{-1} g^1 x_0 = x_0$, we have that $\phi(0) = x_0$, as desired. □

Thus we find a clean relationship between the velocity field defined above and the orbit of a point. Namely, that under the action of the phase flow, a point moves in such a way that its velocity at every instant is determined by the value of the phase velocity vector at the point it is currently occupying.

In the above theorem, we associated a phase flow to a differential equation. It is often instructive to reverse this process, associating an ODE to a flow. Consider the differential equation $\dot{x} = v(x)$.

Definition 1.8. The *phase flow of the differential equation* $\dot{x} = v(x)$ is the one-parameter diffeomorphism group for which v is the phase velocity vector field. It can be found by solving the differential equation, letting $g^t x_0 = \phi(t)$ where ϕ solves

$$\begin{cases} \dot{x} = v(x) \\ x(0) = x_0 \end{cases}$$

Examining the flow of an ODE is often a a very useful tool one can employ.

2. Equivalence of Flows

Let $\{f^t\}$ and $\{g^t\}$ be flows.

Definition 2.1. We say $\{f^t\}$ and $\{g^t\}$ are equivalent if there exists a bijection $h : \mathbb{R}^n \to \mathbb{R}^n$ such that $h \circ f^t = g^t \circ h$ for any $t \in \mathbb{R}$.

We say the flows are *linearly equivalent* if there exists such an $h \in \mathrm{GL}(\mathbb{R}^n)$ that is a linear isomorphism, *differentiably equivalent* if there exists h that is a diffeomorphism, and *topologically equivalent* if there exists such an h that is a homeomorphism. If flows are linearly equivalent, then they are differentiably equivalent, and if they are differentiably equivalent, then they are topologically equivalent.

Theorem 2.2. *The above relations are equivalence relations.*

Proof. The identity mapping is a linear isomorphism, diffeomorphism, and homeomorphism, so $f^t \sim f^t$ under each relation. If $h \circ f^t = g^t \circ h$, then $f^t \circ h^{-1} = h^{-1} \circ g^t$, so $g^t \sim f^t$ under each relation since h^{-1} is a linear isomorphism, diffeomorphism, or homeomorphism whenever h is. Finally, if $f^t \sim g^t$ and $g^t \sim j^t$, then we have $h \circ f^t = g^t \circ h$ and $k \circ g^t = j^t \circ k$. Thus $h \circ f^t \circ h^{-1} = g^t = k^{-1} \circ j^t \circ k$ or $(kh) \circ f = j \circ (kh)$, and since one can compose linear isomorphisms, diffeomorphisms, and homeomorphisms, we have $f \sim j$. □

Thus we may partition phase flows into disjoint collections based on these three types of equivalences. This allows us to study the equivalence classes and perhaps draw conclusions about the differential equations that generate equivalent flows.

3. The Exponential

In this section we define the exponential of an operator, a useful tool for solving linear equations.

Definition 3.1. Let A be a linear operator. Define the *exponential* e^A by

$$e^A = E + A + \frac{A^2}{2!} + \cdots$$

Where E is the identity operator.

This is the same definition as the exponential of real numbers, and we expect it to behave the same way.

Theorem 3.2. *The series* e^A *converges uniformly on the set* $\{||A|| \leq a\}$ *where* $a \in \mathbb{R}$.

In the following proof, we assume the Weierstrass convergence criterion.

Proof. If $||A|| \leq a$, then

$$||E|| + ||A|| + \frac{||A||^2}{2!} + \cdots \leq 1 + a + \frac{a^2}{2!} + \cdots = e^a$$

which we know converges. □

One important way in which the exponential of an operator will remain the same as the exponential of a number is in how it is differentiated.

Theorem 3.3. $\frac{d}{dt}e^{At} = Ae^{At}$

Proof. The proof follows from differentiating e^At termwise.

$$\frac{d}{dt}\sum_{k=0}^{\infty}\frac{t^k}{k!}A^k = \sum_{k=0}^{\infty}\frac{d}{dt}\frac{t^k}{k!}A^k = A\sum_{k=0}^{\infty}\frac{t^k}{k!}A^k$$

by the uniform convergence of both series involved. □

The above theorem shows us how to solve linear equations.

Theorem 3.4. *The solution of the equation*

$$\begin{cases} \dot{x} = Ax \\ x(0) = x_0 \end{cases}$$

is $\phi(t) = e^{At}x_0$.

Proof. The proof is immediate.

$$\frac{d}{dt}\phi(t) = Ae^{At}x_0 = A\phi(t).$$ □

4. Complexification

Let $(\mathbb{R}^n)^{\mathbb{C}}$ denote the complexification of the real-vector space $\mathbb{R}^n$, defined below.

Definition 4.1. $(\mathbb{R}^n)^{\mathbb{C}}$ is the vector space over $\mathbb{C}$, composed of vectors (ξ, η), $\xi, \eta \in \mathbb{R}^n$, denoted by $\xi + i\eta$, and where scalar multiplication is defined by $(a + bi)(\xi + \eta i) = (a\xi - b\eta) + (a\eta + b\xi)i$.

Note that $\dim(\mathbb{R}^n)^{\mathbb{C}} = n$, as if $(e_1, \ldots e_n)$ is a basis for $\mathbb{R}^n$, then $(e_1, \ldots, e_n)$ is a basis for $(\mathbb{R}^n)^{\mathbb{C}}$, where in the second case e_i is the complexified vector having imaginary part 0. If $A : \mathbb{R}^n \to \mathbb{R}^n$ is a real operator, it has a corresponding complexified operator $A^{\mathbb{C}} : (\mathbb{R}^n)^{\mathbb{C}} \to (\mathbb{R}^n)^{\mathbb{C}}$ defined by the equation $A^{\mathbb{C}}(\xi + \eta i) = A\xi + (A\eta)i$.

Theorem 4.2. *Let* $A : \mathbb{C}^n \to \mathbb{C}^n$ *be a complex linear transformation. Then* $\phi(t) = e^{At}z_0$ *solves the complex differential equation*

$$\begin{cases} \dot{z} = Az \\ z(0) = z_0 \end{cases}$$

just as in the real case.

Proof. The differentiation formula for exponentials holds up for complex exponentials, so the proof is the same as in Theorem 3.4. □

Now suppose A has n pairwise distinct eigenvalues λ_k that correspond to eigenvectors ξ_k. Then we know from linear algebra that $\mathbb{C}^n$ decomposes into the direct sum of subspaces $\mathbb{C}\xi_1 \oplus \cdots \oplus \mathbb{C}\xi_n$ with each $\mathbb{C}\xi_k$ invariant under At and e^{At}, where we take At to mean A composed with a scaling by t. Furthermore, in each $\mathbb{C}\xi_k$, e^{At} merely acts as multiplication by $e^{\lambda_k t}$. Thus we can write ϕ above as $\phi(t) = \sum_{i=1}^n c_i e^{\lambda_i t}\xi_i$ where c_i is a complex constant dependent on the initial conditions. We will use this representation of the solution to linear equations later.

Definition 4.3. If $\dot{x} = Ax$ is a linear equation, its complexification is defined by $\dot{z} = A^{\mathbb{C}}z$ where $z \in \mathbb{C}$.

We aim to develop some machinery with the compelexified operator $A^{\mathbb{C}}$ to help inform us about the structure of $\mathbb{R}^n$ under A. To that end, a few lemmas are necessary.

Lemma 4.4. *Let ϕ solve $\dot{z} = A^{\mathbb{C}}z$ with initial condition z_0 and let ψ solve $\dot{z} = A^{\mathbb{C}}z$ with initial condition $\overline{z_0}$. Then $\psi = \overline{\phi}$.*

Proof. Since $\overline{\phi(0)} = \overline{z_0}$, by uniqueness it is only required to show that $\overline{\phi}$ solves the equation. But we know that

$$\frac{d\overline{\phi}}{dt} = \overline{\frac{d\phi}{dt}} = \overline{A^{\mathbb{C}}\phi} = \overline{A^{\mathbb{C}}}\,\overline{\phi} = A^{\mathbb{C}}\overline{\phi}$$

since $A^{\mathbb{C}}$ is real, completing the proof. □

Corollary 4.5. *If z_0 is real, we have that ϕ is real.*

Proof. Since ϕ solves the equation with initial condition z_0 and $\overline{\phi}$ solves the equation with initial conditions $\overline{z_0} = z_0$, we have by uniqueness that $\phi = \overline{\phi}$ and thus is real. □

Lemma 4.6. *Let A be linear. Then ϕ solves $\dot{z} = A^{\mathbb{C}}z$ if and only if its real and imaginary parts solve $\dot{x} = Ax$.*

Proof. Note that if we realify, and since we have that A is linear, solving $\dot{x} + i\dot{y} = A(x + yi) = Ax + iAy$ is equivalent to solving the system $\dot{x} = Ax$ and $\dot{y} = Ay$. The other direction is obvious. □

A final easy lemma of linear algebra:

Lemma 4.7. *If λ is an eigenvalue of $A^{\mathbb{C}}$ corresponding to eigenvector ξ, then $\overline{\lambda}$ is an eigenvalue corresponding to eigenvector $\overline{\xi}$.*

Proof. Since $A^{\mathbb{C}}\xi = \lambda\xi$, $\overline{A^{\mathbb{C}}\xi} = \overline{\lambda\xi}$, or $A^{\mathbb{C}}\overline{\xi} = \overline{\lambda}\,\overline{\xi}$ □

The preceding lemma allows the following representation of $\mathbb{R}^n$:

Theorem 4.8. *Let $A : \mathbb{R}^n \to \mathbb{R}^n$ have n pairwise distinct eigenvalues, k real and $2m$ complex. Then $\mathbb{R}^n$ decomposes into the direct product of k one dimensional subspaces and m two dimensional subspaces that are invariant under A.*

Proof. We know there are k A-invariant one dimensional subspaces.

Let λ be a complex-eigenvalue to eigenvector ξ under the complexified operator $A^{\mathbb{C}}$. From the above, we know that $\overline{\lambda}$ is an eigenvalue to eigenvector $\overline{\xi}$. Thus $\operatorname{span}(\xi, \overline{\xi})$ is invariant under $A^{\mathbb{C}}$, and since $\mathbb{R}^n$ is invariant under $A^{\mathbb{C}}$, we have that

$\text{span}(\xi, \overline{\xi}) \cap \mathbb{R}^n$ is invariant under $A^{\mathbb{C}}$. The goal is to show that the preceding intersection defines a real plane $\mathbb{R}^2 \subset \mathbb{R}^n$.

Note that the real and imaginary parts of ξ are in this intersection, since, letting $x = \text{re}(\xi)$ and $y = \text{im}(\xi)$, we have that

$$x = \frac{\xi + \overline{\xi}}{2} \ , \ y = \frac{\xi - \overline{\xi}}{2i}$$

showing that they are in $\text{span}(\xi, \overline{\xi})$, and they are real as

$$\overline{x} = \overline{\frac{\xi + \overline{\xi}}{2}} = \frac{\overline{\xi} + \xi}{2} = x \ , \ \overline{y} = \frac{\overline{\xi - \overline{\xi}}}{-2i} = \frac{\overline{\xi} - \xi}{-2i} = y$$

It is clear that x and y are linearly independent under $\mathbb{C}$, since $\xi = x + iy$ and $\overline{\xi} = x - iy$ are linearly independent under $\mathbb{C}$, since they're eigenvectors. Thus they span $\mathbb{C}^2$. But if they are $\mathbb{C}$ linearly independent they are $\mathbb{R}$ linearly independent, and thus they span under $\mathbb{R}$ a real plane $\mathbb{R}^2$, which is exactly equal to $\text{span}(x, y) \cap \mathbb{R}^n$ and thus is invariant under $A^{\mathbb{C}}$ and thus A.

The eigenvalues of A restricted to the plane $\mathbb{R}^2$ defined above are λ and $\overline{\lambda}$, since complexification does not change eigenvalues and $(\mathbb{R}^2)^{\mathbb{C}}$ contains $\overline{\xi}$ and ξ.

That there are m such planes is thus clear, since they correspond to the eigenvalues.

Each subspace is linearly independent because the eigenvectors are in each subspace in $(\mathbb{R}^n)^{\mathbb{C}}$, with one eigenvector in each one dimensional subspace and two in each two dimensional subspace.

This completes the proof. □

5. Linear and Differentiable Equivalence of Linear Equations

In this section, we classify linear equations based on their various types of equivalence.

Theorem 5.1. *Let $A, B : \mathbb{R}^n \to \mathbb{R}^n$ be linear operators with simple eigenvalues (that is each eigenvalue has algebraic multiplicity 1). Then the differential equations*

$$\dot{x} = Ax \text{ and } \dot{y} = By$$

are linearly equivalent if and only if the eigenvalues of A and B are the same.

Here we of course mean that the equations are linearly equivalent if the associated flows are linearly equivalent.

Proof. We first claim that the flows are linearly equivalent if and only if $B = hAh^{-1}$ for some $h \in \text{GL}(\mathbb{R}^n)$. Suppose the above holds. Let f^t be the flow of the first equation and g^t the flow of the second equation. We claim $h \circ f^t = g^t \circ h$. This is clear as

$$\frac{d}{dt}(h \circ f^t) = h \circ \dot{x} = hAx = hAh^{-1}hx = Bhx = \frac{d}{dt}(g^t \circ h)$$

since letting $y = hx$ we get $\dot{y} = h\dot{x} = hAx = hAh^{-1}y = By$.

Now if the flows are equivalent then we have $\dot{y} = h\dot{x} = hAx = hAh^{-1}y$ for each y so $B = hAh^{-1}$.

Now we show that all the eigenvalues are the same if and only if $B = hAh^{-1}$ which will complete the proof.

Suppose $B = hAh^{-1}$. It is a fact from linear algebra that all the eigenvalues are the same.

Suppose all the eigenvalues are the same. Since they are simple, $\mathbb{R}^n$ decomposes into the same number of one-dimensional and two-dimensional subspaces invariant under both A and B, as shown in Theorem 4.8. Since these subspaces are linearly equivalent, and we have that A and B are linearly equivalent in these subspaces, which completes the proof. □

Now we turn to a more elementary result regarding the differentiable equivalence of linear equations.

Theorem 5.2. *The systems $\dot{x} = Ax$ and $\dot{y} = By$ are differentiably equivalent if and only if they are linearly equivalent.*

Proof. If they are linearly equivalent, then it is clear that they are differentiably equivalent.

Now suppose they are differentiably equivalent, or there exists a diffeomorphism h such that $he^{At} = e^{Bt}h$. Now 0 is a fixed point for system A since A is linear. Thus $c := h(0)$ is a fixed point of the flow B. Let d be defined as translation by c, i.e. $d(x) = x - c$. Note that $d(y)$ also solves the system since $\dot{d(y)} = \dot{y} = By = B(y - c) = Bd$.

But then if we have $h_1 = d \circ h$ then we have that $h_1(0) = 0$, and that $h_1 \circ e^{At} = e^{Bt} \circ h_1$. Let Dh_1 be the derivative of h_1 at 0. Then we have that $Dh_1 \circ e^{At} = e^{Bt} \circ Dh_1$. This is linear, and thus we have proved the theorem. □

6. Topological Equivalence of Linear Equations

Consider two linear equations $\dot{x} = Ax$ and $\dot{y} = By$. For every eigenvalue λ of either A or B, we suppose that $\mathrm{re}(\lambda) \neq 0$. Let $m_+(A)$ be the number of eigenvalues of A with positive real part, and $m_-(A)$ be the number of eigenvalues of A with negative real part.

The goal of this section is to prove the following topological characterization of linear equations:

Theorem 6.1. *If, as above, two linear systems have no eigenvalues with real part 0, then they are topologically equivalent if and only if $m_+(A) = m_+(B)$ and $m_-(A) = m_-(B)$.*

Before we can prove the above theorem, we must develop some machinery and prove a few lemmas. The first lemma is trivial:

Lemma 6.2. *The direct products of topologically equivalent systems are topologically equivalent.*

Proof. Indeed, suppose $h_1 \circ f_1^t = g_1^t \circ h_1$, and $h_2 \circ f_2^t = g_2^t \circ h_2$. Then the direct product of the systems gives rise to the flows $(f_1^t x, f_2^t x)$ and $(g_1^t x, g_2^t x)$. These flows are clearly related by the homeomorphism $h(x, y) = (h_1(x), h_2(y)$. □

The next lemma is a theorem from linear algebra:

Lemma 6.3. *If $A : \mathbb{R}^n \to \mathbb{R}^n$ has no purely imaginary eigenvalues (as required above), then $\mathbb{R}^n$ decomposes into the direct product of two spaces $\mathbb{R}^{m-}$ and $\mathbb{R}^{m+}$, each invariant under A, and with the restriction of A to $\mathbb{R}^{m-}$ having only eigenvalues with negative real part and the restriction of A to $\mathbb{R}^{m+}$ having only eigenvalues with positive real part.*

We may thus without loss of generality take the case $m_-(A) = m_-(B) = 0$, considering $-A$ and $-B$ in $\mathbb{R}^{m_-}$. Thus, the problem reduces to the following lemma:

Lemma 6.4. *Let $A : \mathbb{R}^n \to \mathbb{R}^n$ such that $m_+(A) = n$. Then the system $\dot{x} = Ax$ is topologically equivalent to the system $\dot{x} = x$.*

In order to prove the lemma, we require the construction of a special quadratic form:

Theorem 6.5. *There exists a positive-definite quadratic form on $\mathbb{R}^n$ denoted r^2, such that the derivative of r^2 in the direction of the vector field Ax is positive, for $x \neq 0$*

It will be easier to prove the above theorem for the complex case:

Theorem 6.6. *There exists a positive-definite quadratic form on $\mathbb{C}^n$ denoted r^2, such that the derivative of r^2 in the direction of the vector field $A^{\mathbb{C}}z$ is positive definite, for $z \neq 0$, or $L_{A^{\mathbb{C}}z} r^2 > 0$, where L denotes the derivative operator.*

If we apply the inequality in the above theorem where $z \in \mathbb{R}^n$, we obtain the real case, Theorem 6.5.

Proof. We will take $r^2 = \langle z, z\rangle$ in an appropriate complex basis. Note that this definition is positive definite for any choice of basis, so we must only check that it has a positive derivative in the vector field $A^{\mathbb{C}}z$. Computation shows that the derivative is in fact $\langle z, Az\rangle + \langle Az, z\rangle = 2\,\mathrm{re}\langle Az, z\rangle$.

It is a fact from linear algebra that there is some basis $\{\xi_i\}$ in which A is upper triangular. Denote the matrix representing A in the basis by a_{jk}. Note that $a_{ii} = \lambda_i$. Fix $\varepsilon > 0$. We claim there is some basis such that $|a_{jk}| < \varepsilon$ for all a_{jk} above the diagonal, i.e. $k > j$. Now replace $\{\xi_i\}$ with the basis $\{c^i\xi_i\}$. Let a'_{jk} denote the matrix representation of A in this new basis. Note that $a'_{jk} = a_{jk}c^{k-j}$. It is clear that we can pick c small enough such that $|a'_{jk}| < \varepsilon$ for all a'_{jk} above the diagonal, i.e. $k > j$.

Note that we can break down $2\,\mathrm{re}\langle Az, z\rangle$ into two quadratic forms, $2\,\mathrm{re}\langle Az, z\rangle = P + Q$ where $P = 2\,\mathrm{re}\sum_{k=l} a'_{kl} z_k \overline{z_l} = 2\,\mathrm{re}\sum_{i=1}^{n} \lambda_i |z_i|^2$ and $Q = 2\,\mathrm{re}\sum_{k<l} a'_{kl} z_k \overline{z_l}$. It is clear that P is positive definite since all the λ_i are positive. Thus it remains to show that $P + Q$ is positive definite. Note that P is positive definite, and continuous, thus for the unit sphere $|z|^2 = 1$ which is compact , we have that P attains a lower bound $\alpha > 0$. Now since $|a'_{kl}| < \varepsilon$, we have that on the unit sphere that $Q(z) \le n^2\varepsilon||z|| = n^2\varepsilon$. Thus picking $\varepsilon < \alpha/2n^2$, we have that $(P + Q)(z) \ge \alpha - n^2(\alpha/2n^2) = \alpha/2$. This shows that $P + Q$ is positive on the unit sphere, which due to linearity, shows $P + Q$ positive everywhere with $z \neq 0$. Thus we have the derivative $L_{A^{\mathbb{C}}z} r^2 > 0$, as desired. □

The above function r^2 is called the Lyapunov function. We will use its existence to prove **Lemma 6.4.**:

Proof. We must construct the required homeomorphism h such that $h \circ f^t = g^t \circ h$, where f^t is the flow of the equation $\dot{x} = Ax$ and g^t is the flow of the equation $\dot{x} = x$.

First, we define the ellipsoid $S = \{x \in \mathbb{R}^n : r^2(x) = 1\}$, where r^2 is the Lyapunov function constructed above. Let $x_0 \in S$. We define $h(f^t x_0) = g^t x_0$ for all $x_0 \in S$, all $t \in R$. And we define $h(0) = 0$. We must check three things:

(1) $h(f^t x_0) = g^t x_0$ for all $x_0 \in S$, all $t \in R$, defines a well-defined function.
(2) h is continuous, one-to-one and onto, and with continuous inverse h^{-1}.
(3) $h \circ f^t = g^t \circ h$.

To this end, we consider the real valued function of a real variable $\rho(t) = \ln r^2(\phi(t))$ where ϕ is a non-zero solution of $\dot{x} = Ax$. We claim that $\rho(t)$ is a diffeomorphism because $\alpha \leq \frac{d\rho}{dt} \leq \beta$. Note that $r^2(\phi(t)) \neq 0$ by uniqueness, thus we can take the derivative

$$\frac{d\rho}{dt} = \frac{L_{Ax} r^2}{r^2}.$$

However since the derivative is a positive definite quadratic form by the above section, we have that $\alpha r^2 \leq L_{Ax} r^2 \leq \beta r^2$, which shows the desired bound.

From this construction follow a number of things.

Firstly, each point $x \neq 0$ can be represented as $x = f^t x_0$ for some $x_0 \in S$ and some $t \in \mathbb{R}$. Consider the solution $\phi(t)$ of $\dot{x} = Ax$ where $\phi(0) = x$. By the above, for some τ we have $r^2(\phi(\tau)) = 1$. Note that $x_0 = \phi(\tau) \in S$. Thus we have $x = f^{-\tau} x_0$.

Secondly, such a representation is unique. Since ρ is a diffeomorphism, each phase curve emanating from a point $x_0 \in S$ can only touch S in one place. Thus we have constructed a bijection

$$F : S \times \mathbb{R} \to \mathbb{R}^n - \{0\}, \ \ F(x_0, t) = f^t x_0.$$

Since ϕ depends continuously on initial conditions, we have that F and F^{-1} are continuous.

Furthermore, we can perform a similar construction with g^t, since $\dfrac{d\rho}{dt} = 2$ for g (where ρ is constructed similarly but with ϕ solving the associated equation to g^t), creating a bijection

$$G : S \times \mathbb{R} \to \mathbb{R}^n - \{0\}, \ \ G(x_0, t) = g^t x_0.$$

By the definition of h, we have that $h = G \circ F^{-1}$ everywhere but at 0, proving that h is both a bijection and a homeomorphism.

Thus it remains to check continuity of h and h^{-1} at 0. This follows from the construction of ρ above. Let $||x|| \leq 1$. Then we have the estimate

$$(r^2(x))^{\frac{2}{\alpha}} \leq r^2(h(x)) \leq (r^2(x))^{\frac{2}{\beta}}$$

and sending x to 0 we obtain that all of the terms go to 0, confirming continuity of h at 0 since $r^2 h$ only goes to 0 if h does. Similarly if we substitute in $h^{-1}x$ instead of x and send it to 0 we get that $\lim_{x\to 0} r^2(h^{-1}x) \leq 0$ and $\lim_{x\to 0} r^2(h^{-1}x) \geq 0$, proving continuity of the inverse.

Finally it remains to show that $h \circ f^t = g^t \circ h$. Let $x \neq 0$. Then we obtain that $h \circ f^t x = h \circ f^t(f^s(x_0)) = h(f^{t+s} x_0) = g^{t+s} x_0 = g^t(g^s x_0) = g^t(h(x_0)) = g^t \circ h$. It is clear that for $x = 0$, $h \circ f^t = g^t \circ h$. This completes the proof. □

Finally, we combine the above lemmas. Let two systems have no purely imaginary eigenvalues. Then if $m_-(A) = m_-(B)$ and $m_+(A) = m_+(B)$, both systems are equivalent to the multidimensional saddle:

$$\dot{x_1} = -x_1, \ \dot{x_2} = x_2 \qquad x_1 \in \mathbb{R}^{m_-(A)}, \ x_2 \in \mathbb{R}^{m_+(A)}$$

This completes the proof of the main theorem of the section.

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