

# COMPLEXES ASSOCIATED TO A SURFACE AND THE MAPPING CLASS GROUP

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ABSTRACT. We explore simplicial complexes associated to a surface, namely the curve complex, the arc complex, and the pants complex. Studying the combinatorics modeled by these complexes helps elucidate the algebraic properties of the mapping class group of a surface. First, we prove that the group of automorphisms of the curve complex is equal to the mapping class group of a surface. Next, we prove that the group of automorphisms of the pants complex is also equal to the mapping class group of a surface. Finally, as an application of these theorems, we show that all isomorphisms between subgroups of finite index of the mapping class group are induced by conjugation and that the mapping class group is finitely generated.

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## 1. INTRODUCTION

Given a compact orientable surface  $S$ , one of the most fundamental objects associated to it is its group of homeomorphisms, denoted  $\text{Homeo}(S)$ .  $\text{Homeo}(S)$  is the group of topological symmetries of  $S$  and thus is fundamental in questions of dynamics, geometry, and topology. However, the algebraic structure of  $\text{Homeo}(S)$  is complex and difficult to study directly. In light of this, we endow the set of orientation-preserving homeomorphisms of  $S$ , denoted  $\text{Homeo}^+(S)$ , with the compact-open topology and study the group of isotopy classes of elements of  $\text{Homeo}^+(S)$  that fix the boundary components of  $S$  pointwise. This group is known

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as the *mapping class group* of the surface  $S$ , denoted  $\text{Mod}(S)$ . The mapping class group, while still complex, is a more manageable algebraic object to study and provides a means of understanding some aspects of  $\text{Homeo}(S)$ .

To elucidate algebraic properties of the mapping class group, one often studies the actions of this group on structures associated to  $S$ . In this paper, we consider the relationships between the mapping class group of a surface and abstract simplicial complexes associated to the surface. These abstract simplicial complexes, namely the curve complex, the arc complex, and the pants complex, combinatorially encode information about objects associated to  $S$ . For example, the curve complex combinatorially models when two simple closed curves on  $S$  are disjoint. By studying the combinatorial structures of these complexes and their relationships with the mapping class group of  $S$ , we are able to gain information about the algebraic structure of the mapping class group.

This paper is organized as follows. In Section 2, we recall background knowledge and introduce notation that will be used throughout this paper. In Section 3, we define and derive basic properties of three abstract simplicial complexes associated to a surface, namely the curve complex, the arc complex, and the pants complex, denoted  $\mathcal{C}(S)$ ,  $\mathcal{A}(S)$ , and  $\mathcal{P}(S)$ , respectively. In Section 4, we consider the relationship between the automorphisms of the curve complex and the elements of the mapping class group.<sup>1</sup> In particular, we prove the following result originally due to Ivanov [11].

**Theorem 4.1.** *If  $S$  is not one of  $\Sigma_{0,3}, \Sigma_{0,4}, \Sigma_{1,1}, \Sigma_{1,2}, \Sigma_{2,0}$  and is an orientable surface with  $\chi(S) < 0$ , then all automorphism of the curve complex of  $S$  are given by elements of the mapping class group of  $S$ , that is,*

$$\text{Aut } \mathcal{C}(S) = \text{Mod}(S).$$

In Section 5, we use Theorem 4.1 to show a similar result which relates the automorphisms of the pants complex and the elements of the mapping class group. We prove the follow theorem due to Margalit [14].

**Theorem 5.1.** *If  $S$  is not one of  $\Sigma_{0,3}, \Sigma_{0,4}, \Sigma_{1,1}, \Sigma_{1,2}, \Sigma_{2,0}$  and is an orientable surface with  $\chi(S) < 0$ , then the group of automorphisms of the pants complex of  $S$  is isomorphic to the group of automorphisms of the curve complex of  $S$ , that is,*

$$\text{Aut } \mathcal{P}(S) \cong \text{Aut } \mathcal{C}(S) = \text{Mod}(S).$$

The exceptional cases in Theorem 4.1 have nontrivial kernels which are generated by hyperelliptic involutions. Given the reliance of Theorem 5.1 on Theorem 4.1, its exceptional cases also have nontrivial kernels.

In Section 6, we apply the relationship established in Theorem 4.1 and properties of these complexes to derive some algebraic properties of the mapping class group. We first prove a theorem due to Ivanov [10], which shows that all isomorphisms between subgroups of finite index of  $\text{Mod}(S)$  are induced by the symmetries of the surface geometry, that is, induced by conjugation in  $\text{Mod}(S)$ .

**Theorem 6.1.** *Let  $H_1, H_2$  be subgroups of finite index of  $\text{Mod}(S)$ . If  $S$  is an orientable surface of genus at least 2 and  $S$  is not a closed surface of genus 2, then*

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<sup>1</sup>Specifically, we consider the *extended* mapping class group. This group includes isotopy classes of both orientation-preserving and orientation-reversing homeomorphisms of  $S$ . For notational simplicity, in the proofs of Theorem 4.1 and Theorem 5.1, we will denote the extended mapping class group also as  $\text{Mod}(S)$ .

all isomorphisms  $H_1 \rightarrow H_2$  have the form  $x \mapsto gxg^{-1}$  for  $g \in \text{Mod}(S)$ . Moreover, the group of outer automorphisms  $\text{Out}(H_i)$  is finite.

Finally, we conclude Section 6 by giving a proof of the finite generation of the mapping class group. This result is originally due to Dehn and Lickorish.

**Theorem 6.2.** *The mapping class group of an orientable surface  $S$  is generated by finitely many Dehn twists about nonseparating simple closed curves on  $S$ .*

## 2. PRELIMINARIES

In this section, we present the necessary background knowledge and notation used in this paper. First, we review definitions and well-known facts about the geometry of surfaces and the mapping class group. Second, we introduce Dehn twists and recall some of their basic properties. For a more in-depth treatment of this background material and proofs of the propositions in this section, see [1]. Throughout this paper, we assume a familiarity with homotopy and the fundamental group. For an introduction to these ideas, see [7].

A *surface* is a 2-dimensional manifold, possibly with boundary. By the Classification of Surfaces Theorem, every connected, orientable, compact surface can be uniquely identified by three different topological invariants: the *genus*  $g$ , the number of *boundary components*  $b$ , and the number of *marked points*  $n$ . Distinct boundary components are obtained by removing open disks with disjoint closures from the surface. We note that marked points can equivalently be viewed as *punctures* on a surface. Punctures are obtained by removing individual points from the surface. Given this equivalence of marked points and punctures, we will switch between the two when convenient. We assume all surfaces are connected, orientable, and compact (except when punctures are present as a punctured surface is clearly not compact) unless stated otherwise. We will denote a specific surface of genus  $g$  with  $b$  boundary components as  $\Sigma_{g,b}$ . When referring to an arbitrary surface, we will denote it as  $S$ . Finally, we let  $\partial S$  denote the boundary of  $S$ .

A *closed curve* is a continuous map  $S^1 \rightarrow S$ . We say a closed curve is *essential* if it is not homotopic to a curve in a regular neighborhood of a point, puncture, or boundary component. A closed curve is *simple* if the map  $S^1 \rightarrow S$  is an embedding. When considering homotopy classes of closed curves, we will mean the free homotopy classes of curves (homotopy classes without a fixed base point).

Given any two homotopy classes of simple closed curves, it is natural to define their intersection number.

**Definition 2.1.** The *geometric intersection number* of two homotopy classes  $a$  and  $b$  of simple closed curves on  $S$ , denoted  $i(a,b)$ , is the minimum number of intersections among all pairs of representatives of the classes  $a$  and  $b$ , that is,

$$i(a,b) = \min\{|\alpha \cap \beta| : \alpha \in a \text{ and } \beta \in b\}.$$

Using the geometric intersection number, we say two curves  $\alpha$  and  $\beta$  are in *minimal position* if they are representatives of their homotopy classes  $a$  and  $b$  which satisfy  $i(a,b) = |\alpha \cap \beta|$ . To make this more precise, we introduce the bigon, see Figure 1.

**Proposition 2.2.** *Two transverse simple close curves on a surface  $S$  are in minimal position if and only if they do not form a bigon.*

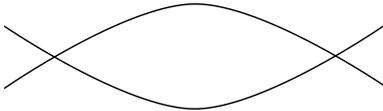


FIGURE 1. A bigon formed by two curves that are not in minimal position.

This proposition implies that the problem of determining if two curves minimally intersect is a purely local problem.

**Definition 2.3.** Two simple closed curves  $\alpha$  and  $\beta$  are *isotopic* if there exists a homotopy

$$H : S^1 \times [0, 1] \rightarrow S$$

from  $\alpha$  to  $\beta$  such that the closed curve  $H(S^1 \times \{t\})$  is simple for each  $t \in [0, 1]$ .

**Proposition 2.4.** *Let  $\alpha$  and  $\beta$  be two essential simple closed curves on a surface  $S$ . Then  $\alpha$  is isotopic to  $\beta$  if and only if  $\alpha$  is homotopic to  $\beta$ .*

Given this proposition, we will refer to homotopy and isotopy classes of curves interchangeably as needed.

Let  $X$  be the set of punctures on a surface  $S$ . A *proper arc* is a map  $\alpha : [0, 1] \rightarrow S$  such that  $\alpha^{-1}(X \cup \partial S) = \{0, 1\}$ . We will assume that all arcs are proper arcs unless stated otherwise. An arc is *simple* if it is an embedding. An arc is *essential* if it is not homotopic to a curve in a regular neighborhood of a point, puncture, or boundary component. We fix the homotopy classes of arcs to be relative to the boundary, that is, each end point of the arc must remain on the boundary or at the marked point throughout the entire homotopy.

For a final note on curves, we recall the *change of coordinates principle*. It roughly states that given any two collections of simple closed curves  $\{\alpha_1, \dots, \alpha_n\}$  and  $\{\beta_1, \dots, \beta_n\}$  on  $S$  that have the same intersection pattern, there exists an orientation-preserving homeomorphism  $f : S \rightarrow S$  such that  $f(\alpha_i) = \beta_i$  for all  $i$ . The idea behind this principle comes from the Classification of Surfaces Theorem. For more details and examples of this principle, see [1].

The primary object that we are interested in studying is the *mapping class group* of a surface  $S$ , denoted  $\text{Mod}(S)$ . Let  $\text{Homeo}^+(S, \partial S)$  denote the set of orientation-preserving homeomorphisms of  $S$  that restrict to the identity on  $\partial S$ . We endow this set with the compact-open topology.

**Definition 2.5.** The *mapping class group* of a surface  $S$  is the group

$$\text{Mod}(S) = \pi_0(\text{Homeo}^+(S, \partial S)).$$

Hence,  $\text{Mod}(S)$  is the group of isotopy classes of elements of  $\text{Homeo}^+(S, \partial S)$  that fix the boundary of  $S$  pointwise. Similarly, if  $\text{Homeo}_0(S, \partial S)$  denotes the connected component of the identity in  $\text{Homeo}^+(S, \partial S)$ , then equivalently we have

$$\text{Mod}(S) = \text{Homeo}^+(S, \partial S) / \text{Homeo}_0(S, \partial S).$$

As with functional notation, elements of  $\text{Mod}(S)$  are applied right to left. Given our definition, punctures on  $S$  are not fixed like the boundary components and are allowed to be permuted by elements of  $\text{Mod}(S)$ .

A Dehn twist about a simple closed curve is a fundamental example of an element of  $\text{Mod}(S)$ . Consider an annulus  $A = S^1 \times [0, 1]$  and embed it in the plane with polar coordinates by  $(\theta, t) \mapsto (\theta, t + 1)$ . Let  $T : A \rightarrow A$  be the map defined by  $T(\theta, t) = (\theta + 2\pi t, t)$ .  $T$  is an orientation-preserving homeomorphism of  $A$  to itself that fixes the boundary of  $A$  pointwise. Let  $S$  be an oriented surface and let  $\alpha$  be a simple closed curve on  $S$ . Denote a regular neighborhood of  $\alpha$  by  $N(\alpha)$ . Let  $\phi : A \rightarrow N(\alpha)$  be an orientation-preserving homeomorphism.

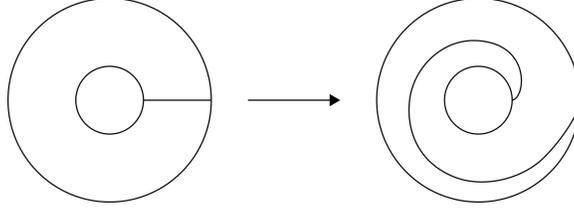


FIGURE 2. Example of a Dehn twist in an annulus.

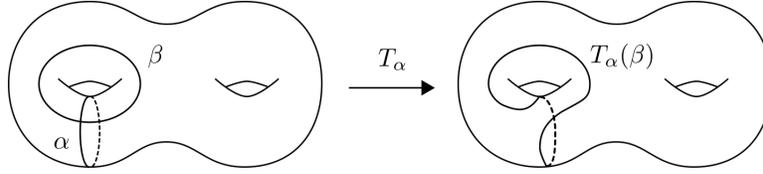


FIGURE 3. Example of a Dehn twist on a surface.

**Definition 2.6.** The *Dehn twist* about a simple closed curve  $\alpha$  on  $S$  is the homeomorphism  $T_\alpha : S \rightarrow S$  given by:

$$T_\alpha(x) = \begin{cases} \phi \circ T \circ \phi^{-1}(x) & \text{if } x \in N(\alpha) \\ x & \text{if } x \in S - N(\alpha) \end{cases}$$

Given a fixed curve  $\alpha$ , the equivalence class  $[T_\alpha]$  depends neither on the choice of  $N(\alpha)$  nor  $\phi$ . Moreover,  $[T_\alpha]$  is well-defined in the isotopy class of  $\alpha$ . Hence, if  $a$  is the isotopy class of  $\alpha$ , then  $T_a$  is a well-defined element of  $\text{Mod}(S)$ .

**Proposition 2.7.** *Let  $a$  be the isotopy class of a simple closed curve  $\alpha$  on a surface  $S$ . If  $\alpha$  is not essential, then the Dehn twist  $T_a$  is a nontrivial element of  $\text{Mod}(S)$ .*

We recall the following properties of Dehn twists.

**Proposition 2.8.** *For any  $f$  in  $\text{Mod}(S)$  and any two isotopy classes  $a$  and  $b$  of simple closed curves on a surface  $S$ , we have*

- (1)  $T_a^j = T_b^k \iff a = b$  and  $j = k$
- (2)  $f T_a^j f^{-1} = T_{f(a)}^j$
- (3)  $f$  commutes with  $T_a \iff f(a) = a$
- (4)  $i(a, b) = 0 \iff T_b^k(a) = a \iff T_a^j T_b^k = T_b^k T_a^j$

The final property, given in the theorem below, is due to Ivanov [10].

**Theorem 2.9.** *Let  $H_1$  and  $H_2$  be subgroups of finite index of  $\text{Mod}(S)$ . If the genus of  $S$  is at least two and  $S$  is not a closed surface of genus two, then all isomorphisms  $\phi : H_1 \rightarrow H_2$  take sufficiently high powers of Dehn twists to powers of Dehn twists.*

### 3. COMPLEXES ASSOCIATED TO A SURFACE

**3.1. The Curve Complex.** The *curve complex* associated to a surface  $S$ , denoted  $\mathcal{C}(S)$ , is an abstract simplicial complex with vertices the isotopy classes of essential simple closed curves on  $S$ . An edge is formed between a pair of isotopy classes  $a$  and  $b$  in  $\mathcal{C}(S)$  if  $a$  and  $b$  are disjoint, that is,  $i(a, b) = 0$ . Similarly, a set of  $k + 1$  vertices define a  $k$ -simplex when all pairwise geometric intersection numbers among all isotopy classes are zero. Hence, if all of the edges of a potential face of the complex are in the complex, then that face must also be in the complex. A simplicial complex that satisfies this condition (as the curve complex does) is called a *flag complex*. The curve complex was first defined by Harvey [4]. An Euler characteristic argument (see Proposition 3.8) shows that the dimension of the curve complex for a surface of genus  $g$  with  $b$  boundary components is  $3g - 4 + b$ . Harer showed the following property of the curve complex [3].

**Theorem 3.1.** *Let  $S$  be a surface of nonzero genus  $g$  with  $b$  boundary components. The curve complex of  $S$  is homotopy equivalent to a wedge sum of spheres of dimension  $2g - 3 + b$ .*

The next result about the curve complex and its proof are due to Lickorish [13].

**Theorem 3.2.** *Let  $S$  be a surface of genus  $g$  with  $b$  boundary components. If  $3g + b \geq 5$ , then the curve complex of  $S$  is connected.*

*Proof.* It suffices to show that given any two isotopy classes  $a$  and  $b$  of simple closed curves on  $S$ , there exists a sequence of vertices  $a = c_1, \dots, c_n = b$  such that  $i(c_i, c_{i+1}) = 0$  for all  $i$ . Note, if  $i(a, b) = 0$ , then we are done. If  $i(a, b) = 1$ , then let  $c$  be the isotopy class containing the boundary of the regular neighborhood of  $\alpha \cup \beta$  for  $\alpha$  in  $a$  and  $\beta$  in  $b$ . Any representative of  $c$  will form the boundary of a torus with one boundary component. If the representatives of  $c$  are nonessential, then  $S \cong \Sigma_{1,1}$  or  $S \cong \Sigma_{1,0}$ ; however, this would violate our hypothesis. Hence,  $i(a, c) = 0 = i(b, c)$ . This implies connectivity in the case of  $i(a, b) = 1$ .

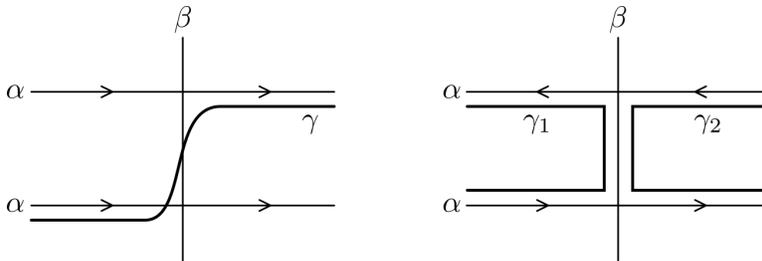


FIGURE 4. Surgered curves in the proof of Theorem 3.2.

Now by way of induction, suppose  $i(a, b) > 1$ . Consider any two consecutive points of intersection of representatives  $\alpha$  and  $\beta$  that are in minimal position on  $S$ . Locally this is shown in Figure 4. If the local curve segments of  $\alpha$  cross over  $\beta$  with

the same orientation each time (see left-hand side of Figure 4), we may select the isotopy class of  $\gamma$  shown in Figure 4 for constructing our path. Since  $|\alpha \cap \gamma| = 1$ , we know  $\gamma$  is essential. Since  $\gamma$  remains to the right of  $\alpha$  except at this local crossing, we know that  $i(a, c) = 1$  and  $i(b, c) < i(a, b)$ . By our inductive hypothesis, there exist paths in  $\mathcal{C}(S)$  connecting  $a$  to  $c$  and  $c$  to  $b$ . Concatenating these paths, we obtain a path connecting  $a$  to  $b$  in  $\mathcal{C}(S)$ .

If the local curve segments of  $\alpha$  cross over  $\beta$  with a different orientation each time (see right-hand side of Figure 4), then consider the curves  $\gamma_1$  and  $\gamma_2$  in Figure 4. Note, neither  $\gamma_1$  nor  $\gamma_2$  is null-homotopic. This would imply that  $\alpha$  and  $\beta$  form a bigon, violating the hypothesis that  $\alpha$  and  $\beta$  are in minimal position. If either  $\gamma_1$  or  $\gamma_2$  is essential, then we may select at least one of the isotopy classes of  $\gamma_1$  or  $\gamma_2$  for constructing our sequence. If both  $\gamma_1$  and  $\gamma_2$  are nonessential, then one side of  $\alpha$  bounds a disk with two boundary components.

In this case, let us consider the curves  $\gamma_3$  and  $\gamma_4$  on the opposite side of  $\alpha$  in similar positions to  $\gamma_1$  and  $\gamma_2$ . Again,  $\gamma_3$  and  $\gamma_4$  can not be null-homotopic. Moreover, either  $\gamma_3$  or  $\gamma_4$  must be essential. If both  $\gamma_3$  and  $\gamma_4$  are nonessential, then  $S \cong \Sigma_{0,4}$ , which violates the hypothesis. Hence, we may select at least one of the isotopy classes of  $\gamma_3$  and  $\gamma_4$  for constructing our sequence. Let the isotopy class of the selected  $\gamma_i$  be  $c$ . Since  $\gamma_i$  does not intersect  $\alpha$  and intersects  $\beta$  at least two times fewer than  $\alpha$ , we have  $i(a, c) < i(a, b)$  and  $i(c, b) < i(a, b)$ . By our inductive hypothesis, there exists paths in  $\mathcal{C}(S)$  connecting  $a$  to  $c$  and  $c$  to  $b$ . Concatenating these paths, we obtain a path connecting  $a$  to  $b$  in  $\mathcal{C}(S)$ .  $\square$

**3.2. The Arc Complex.** The *arc complex* associated to a surface  $S$ , denoted  $\mathcal{A}(S)$ , is an abstract simplicial complex with vertices the isotopy classes of essential simple proper arcs on  $S$ . An edge is formed between two isotopy classes  $a$  and  $b$  if  $a$  and  $b$  are disjoint, that is,  $i(a, b) = 0$ . Similarly, a set of  $k + 1$  vertices define a  $k$ -simplex when all pairwise geometric intersection numbers among all isotopy classes are zero. This excludes shared points of contact at the boundary components or marked points. Clearly,  $\mathcal{A}(S)$  is a flag complex. The following fundamental property of the arc complex is originally due to Harer [2].

**Theorem 3.3.** *Let  $S$  be a compact surface with finitely many boundary components and marked points. If  $\mathcal{A}(S)$  is nonempty, then  $\mathcal{A}(S)$  is contractible.*

The proof we present is a more elementary argument due to Hatcher [6].

*Proof.* Fix some vertex  $v$  in  $\mathcal{A}(S)$  with representative  $\alpha$ . Let  $p$  be an arbitrary point in a simplex of  $\mathcal{A}(S)$  spanned by simplices  $v_1, \dots, v_k$ . Let  $p$  be given in barycentric coordinates, that is,  $p = \sum_i c_i v_i$  such that  $\sum_i c_i = 1$  and  $c_i > 0$  for all  $i$ . We will define a flow from  $p$  into the simplicial star of  $v$ , which itself is contractible. Given that our choice of  $p$  is arbitrary, we may define similar flows for all points in  $\mathcal{A}(S)$ . Using these flows and the contractibility of the simplicial star of  $v$ , we can define a deformation reformation of  $\mathcal{A}(S)$  onto  $v$ .

We realize each  $v_i$  on  $S$  by one of its representative arcs  $\alpha_i$ . We realize  $p$  on  $S$  by thickening each arc  $\alpha_i$  to a width of  $c_i$ , forming an  $\alpha_i$ -band. Via isotopies, we make the union of the  $\alpha_i$ -bands to be a closed interval that is disjoint from the boundary component or marked point that  $\alpha$  starts and/or ends at on  $S$ . In other words, we rearrange the  $\alpha_i$ -bands along  $\alpha$  to look like the left-hand side of Figure 5.

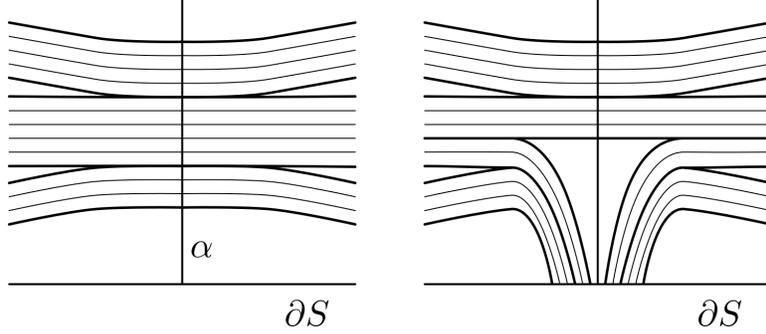


FIGURE 5. Realized flow of  $p$  onto the simplicial star of  $v$ .

The thickness of this union of  $\alpha_i$ -bands is  $\omega = \sum_i c_i i(\alpha_i, \alpha)$ . We define a flow by  $\omega t$ , where at  $t = t_0$ , we push the union of the  $\alpha_i$ -bands  $\omega t_0$  along a fixed direction that follows the image of  $\alpha$  on  $S$ . At  $t = 0$ , the realized point in the simplex of  $\mathcal{A}(S)$  is  $p$ . At  $0 < t < 1$ , the arrangement of the  $\alpha_i$ -bands resembles the right-hand side of Figure 5. At  $t = 1$ , the realized point  $p'$  is contained in a simplex composed of arcs disjoint from  $\alpha$  and thus  $p'$  is in the simplicial star of  $v$ . This flow is well-defined on intersections of simplices and is continuous. Thus, it defines a deformation retraction of  $\mathcal{A}(S)$  onto the simplicial star of  $v$ , which itself is contractible.  $\square$

Before deriving the last property of the arc complex that we will need, we first develop a few concepts.

**Definition 3.4.** Let  $X$  be a simplicial complex and let  $\Delta$  be a collection of simplices in  $X$ .

- (1) The *closure* of  $\Delta$ , denoted  $Cl(\Delta)$ , is the smallest simplicial subcomplex of  $X$  that contains each simplex in  $\Delta$ .
- (2) The *simplicial star* of  $\Delta$ , denoted  $St(\Delta)$ , is the set of all simplices in  $X$  that have any faces in  $\Delta$ .
- (3) The *link* of  $\Delta$ , denoted  $L(\Delta)$ , is defined by

$$L(\Delta) = Cl(St(\Delta)) - St(Cl(\Delta)).$$

We note the next definition for use in Section 4.

**Definition 3.5.** The *dual graph* of a vertex  $v$  in a simplicial complex  $X$ , denoted  $L^*(v)$ , is a graph which contains the same set of vertices as  $L(v)$ . Edges in  $L^*(v)$  are formed between pairs of vertices that are not connected by an edge in  $L(v)$ .

**Lemma 3.6.** Let  $X$  be a  $d$ -dimensional simplicial complex which satisfies the following properties:

- (1)  $X$  is connected.
- (2)  $X$  is a flag complex.
- (3) Every simplex of  $X$  is contained in some  $d$ -dimensional simplex.
- (4) If  $\Delta$  is a  $k$ -simplex,  $k \leq d - 2$ , then  $L(\Delta)$  is connected.

If  $\Delta$  and  $\Delta'$  are any two  $d$ -simplices in  $X$ , then there exists a sequence of  $d$ -simplices  $\Delta = \Delta_1, \dots, \Delta_n = \Delta'$  such that  $\Delta_i \cap \Delta_{i+1}$  is a  $(d - 1)$ -simplex.

*Proof.* Let  $\Delta$  and  $\Delta'$  be two  $d$ -dimensional simplices in  $X$ . By property (1),  $X$  is connected. Hence, by property (3), there exists a sequence of maximal simplices  $\Delta = \Delta_1, \dots, \Delta_n = \Delta'$  in  $X$  such that  $\Delta_i \cap \Delta_{i+1}$  is nonempty. We will show that if  $\Delta_i$  and  $\Delta_{i+1}$  share a  $k$ -simplex such that  $k < d - 1$ , then we can expand the sequence between  $\Delta_i$  and  $\Delta_{i+1}$  to  $\Delta_i = \Delta_{i_1}, \dots, \Delta_{i_m} = \Delta_{i+1}$  such that  $\Delta_{i_j} \cap \Delta_{i_{j+1}}$  is a  $(k + 1)$ -simplex. Once we have expanded the sequence between  $\Delta_i$  and  $\Delta_{i+1}$ , we can induct upon this process, apply it to all adjacent simplices in our sequence, and obtain the desired result.

Now we construct the expanded sequence  $\Delta_i = \Delta_{i_1}, \dots, \Delta_{i_m} = \Delta_{i+1}$ . By construction,  $\Delta_i \cap \Delta_{i+1}$  is non-empty and thus contains some simplex  $\tilde{\Delta}$  of  $X$ . If  $\tilde{\Delta}$  is a  $(d - 1)$ -simplex, then we are done. So we suppose that  $\tilde{\Delta}$  has dimension  $k \leq d - 2$ . We consider  $L(\tilde{\Delta})$ . By property (4), we have that the  $L(\tilde{\Delta})$  is connected. We also have that  $\Delta_i \cap L(\tilde{\Delta}) \neq \emptyset$  and  $\Delta_{i+1} \cap L(\tilde{\Delta}) \neq \emptyset$ . Finally, by properties (2) and (3), for all pairs of vertices  $v$  and  $w$  connected by an edge in  $L(\tilde{\Delta})$ , there exists a  $d$ -simplex of  $X$  that contains  $v$ ,  $w$ , and the vertices of  $\tilde{\Delta}$ .

We now begin to expand the sequence between  $\Delta_i$  and  $\Delta_{i+1}$ . Let the vertices  $v_1, \dots, v_k$  in  $L(\tilde{\Delta})$  form a path from  $\Delta_i$  to  $\Delta_{i+1}$ . Without loss of generality, suppose that  $v_1$  is contained in  $\Delta_i$  and  $v_k$  is contained in  $\Delta_{i+1}$ . We set  $\Delta_{i_1} = \Delta_i$ . We may let  $\Delta_{i_2}$  be the  $d$ -simplex containing the  $v_1, v_2$ , and the vertices of  $\tilde{\Delta}$ . Similarly, we let  $\Delta_{i_j}$  be the  $d$ -simplex containing  $v_{j-1}, v_j$ , and the vertices of  $\tilde{\Delta}$ . This gives us a sequence of maximal simplices  $\Delta_i = \Delta_{i_1}, \dots, \Delta_{i_m} = \Delta_{i+1}$  such that  $\Delta_{i_j}$  and  $\Delta_{i_{j+1}}$  share a common face of dimension  $k + 1$ . We replace  $\Delta_i$  and  $\Delta_{i+1}$  in our original sequence with this sequence that we just constructed. We can induct upon this process until we construct the desired sequence with adjacent simplices sharing a  $(d - 1)$ -simplex.  $\square$

We now give the final property of the arc complex, which follows as a corollary of Theorem 3.3 and Lemma 3.6. This corollary will be essential in the proof of Theorem 4.1.

**Corollary 3.7.** *Given a surface  $S$  with finitely many boundary components and marked points, if  $\mathcal{A}(S)$  is nonempty, then any two maximal simplices of  $\mathcal{A}(S)$  are connected by a path of maximal simplices such that any two consecutive simplices in the path share a common codimension-1 face.*

*Proof.* To show this corollary, we will show that the arc complex satisfies the properties listed in Lemma 3.6. After we have shown this, the corollary immediately follows. Property (1) follows trivially from Theorem 3.3. Property (2) follows from the definition of the arc complex.

To show property (4), let  $\Delta$  be a  $k$ -simplex in  $\mathcal{A}(S)$  such that  $k < d - 1$ , where  $d$  is the dimension of  $\mathcal{A}(S)$ . The link  $L(\Delta)$  corresponds to the arc complex of the surface  $S'$  that is obtained by removing representatives of the isotopy classes contained in  $\Delta$ , that is,  $L(\Delta) = \mathcal{A}(S')$ . Hence, if  $\Delta$  contains the vertices  $a_0, \dots, a_k$  that have representatives  $\alpha_0, \dots, \alpha_k$ , then  $S' = S - \cup_i \alpha_i$ . Since  $L(\Delta)$  has dimension greater than zero,  $\mathcal{A}(S')$  has dimension greater than zero. By Theorem 3.3,  $\mathcal{A}(S')$  is contractible and thus connected. Hence,  $L(\Delta)$  is connected.

To show property (3), let  $\Delta$  be a simplex in  $\mathcal{A}(S)$ . If the isotopy classes of arcs of  $\Delta$  do not make up a maximal simplex, then they do not form a triangulation of  $S$ . By constructing a triangulation of  $S$  that uses the isotopy classes of arcs in  $\Delta$ , we obtain a maximal simplex which contains  $\Delta$ .  $\square$

**3.3. The Pants Complex.** A *pair of pants* is a surface homeomorphic to  $\Sigma_{0,3}$ . A *pants decomposition* of a surface  $S$  is a collection of pairwise disjoint essential simple closed curves  $\{\alpha_1, \dots, \alpha_n\}$  such that their complement on  $S$ , that is,  $S - \cup_i \alpha_i$ , is a disjoint union of pairs of pants. An equivalent definition is that a pants decomposition is a maximal collection of pairwise disjoint essential simple closed curves on  $S$  such that no two curves are isotopic. By this definition, pants decompositions of a surface correspond to maximal simplices in the curve complex of the surface. We will denote a specific pants decomposition of  $S$  by  $\{\alpha_1, \dots, \alpha_n\}$ , where the  $\alpha_i$ 's represent the disjoint curves in the decomposition.

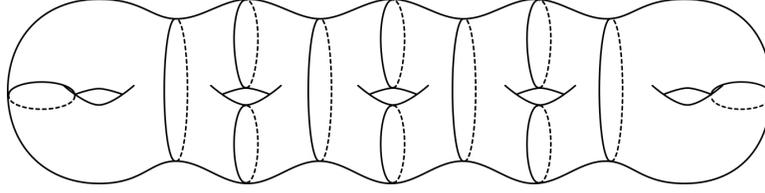


FIGURE 6. A pants decomposition.

**Proposition 3.8.** *For a surface  $S$  of genus  $g$  with  $b$  distinct boundary components, any pants decomposition of  $S$  contains  $3g - 3 + b$  curves and  $2g - 2 + b$  pairs of pants.*

*Proof.* A pair of pants has an Euler characteristic of  $-1$ . Given a surface  $S$  and a set of simple closed curves  $\{\alpha_1, \dots, \alpha_n\}$ , the Euler characteristic of  $S$  is maintained by cutting along any curve  $\alpha_i$ , that is,  $\chi(S) = \chi(S - \cup_i \alpha_i)$ . Let  $m$  be the number of pairs of pants in a pants decomposition. By cutting along all the curves of a pants decomposition, we have

$$m = -m\chi(\Sigma_{0,3}) = -(2 - 2g - b) = 2g - 2 + b.$$

Next, each pair of pants has three boundary components associated to either curves in the pants decomposition or boundary components of  $S$ . Any curve in the pants decomposition is shared by two pairs of pants. Hence, by keeping track of unique boundary components, we find that the number of curves in a pants decomposition is

$$n = \frac{3(2g + b - 2) - b}{2} = 3g - 3 + b.$$

□

The above argument holds if  $S$  has punctures. We simply replace  $b$  with the number of boundary components plus the number of punctures.

Two pants decompositions,  $p$  and  $p'$ , differ by an *elementary move* if  $p'$  is obtained from  $p$  by changing one curve  $\alpha_i$  in  $p = \{\alpha_1, \dots, \alpha_n\}$  to a different curve  $\alpha'_i$  that minimally intersects  $\alpha_i$ . Hence,  $p' = \{\alpha_1, \dots, \alpha'_i, \dots, \alpha_n\}$ . Note,  $\alpha'_i$  minimally intersects  $\alpha_i$  if

- (1)  $\alpha'_i$  does not intersect any other curves in the pants decomposition  $p$ .
- (2) the isotopy class of  $\alpha'_i$  achieves the minimal geometric intersection number among all isotopy classes of curves that satisfy (1).

Elementary moves correspond to switches between curves on subsurfaces of  $S$  that are homeomorphic to  $\Sigma_{1,1}$  or  $\Sigma_{0,4}$ . To see this hold  $n - 1$  curves fixed. Cut along these curves. The resulting subsurfaces are pairs of pants and a subsurface homeomorphic to  $\Sigma_{1,1}$  or  $\Sigma_{0,4}$ , which contains the uncut curve. Clearly, the only elementary moves are switches between curves of minimal intersection on the resulting subsurface.

The *pants complex* associated to a surface  $S$ , denoted  $\mathcal{P}(S)$ , is an abstract simplicial complex with vertices the pants decompositions of  $S$ . An edge joins two vertices when the corresponding pants decompositions differ by an elementary move. 2-cells in the pants complex correspond to specific cycles that form among sets of vertices. We will mostly be concerned with the 1-skeleton of the pants complex, denoted  $\mathcal{P}^1(S)$ . The pants complex was introduced by Hatcher-Thurston [5]. The following fundamental property of the pants complex is due to Hatcher-Lochak-Schneps [8].

**Theorem 3.9.** *Given a surface  $S$ , if the pants complex of  $S$  has dimension greater than zero, then it is simply-connected.*

The connectedness of the pants complex trivially follows from Theorem 3.9. We will use the connectedness of the pants complex in the proof of Theorem 5.1.

#### 4. AUTOMORPHISMS OF THE CURVE COMPLEX

In this section, we give a proof of Theorem 4.1.

**Theorem 4.1.** *If  $S$  is not one of  $\Sigma_{0,3}, \Sigma_{0,4}, \Sigma_{1,1}, \Sigma_{1,2}, \Sigma_{2,0}$  and is an orientable surface with  $\chi(S) < 0$ , then all automorphism of the curve complex of  $S$  are given by elements of the mapping class group of  $S$ , that is,*

$$\text{Aut } \mathcal{C}(S) = \text{Mod}(S).$$

The proof of this theorem was originally given by Ivanov [11]. Before giving the proof, we need the following lemma.

**Lemma 4.2.** *Let  $a_1$  and  $a_2$  be isotopy classes of essential simple closed curves on  $S$ . The geometric intersection number of  $a_1$  and  $a_2$  is equal to one if and only if there exist isotopy classes,  $a_3, a_4, a_5$ , of essential simple closed curves on  $S$  satisfying the following properties:*

- (1) *The complete subgraph of the curve complex restricted to these 5 isotopy classes is given in Figure 7.*
- (2) *If  $a_4$  is the isotopy class of a separating curve and one of the resulting subsurfaces obtained by cutting along a representative of  $a_4$  is a torus with one boundary component, then this subsurface contains representatives of  $a_1$  and  $a_2$ .*

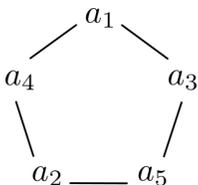


FIGURE 7. Subgraph for property (1) of Lemma 4.2.

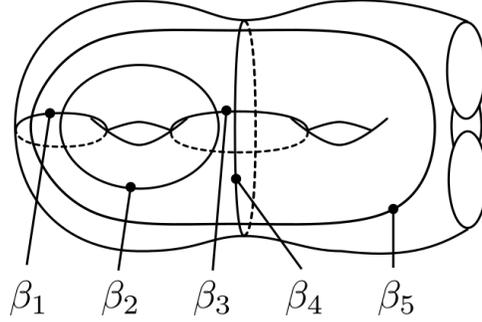


FIGURE 8. A set of curves satisfying properties (1) and (2) from Lemma 4.2.

*Proof.* To show the forward implication, we consider Figure 8. If  $i(a_1, a_2) = 1$ , then by the change of coordinates principle, there exists an element  $\phi$  in  $\text{Mod}(S)$  that takes the configuration of curves in Figure 8, which clearly satisfy the listed properties, to a configuration with isotopy classes  $a_1$  and  $a_2$ . Hence, if  $\phi(b_1) = a_1$  and  $\phi(b_2) = a_2$ , then there exist isotopy classes  $\phi(b_3), \phi(b_4), \phi(b_5)$  that satisfy the listed properties. This shows the forward implication.

To show the reverse implication, let  $a_1, \dots, a_5$  be a set of isotopy classes of essential simple closed curves on  $S$  that satisfy the listed properties and have representatives  $\alpha_1, \dots, \alpha_5$ . We cut  $S$  along  $\alpha_4$  to obtain a torus, denoted here as  $S'$ , with one boundary component, namely  $\alpha_4$ . Given property (1),  $\alpha_3$  and  $\alpha_5$  nontrivially intersect  $\alpha_4$ . Thus, there exist arcs  $\alpha'_3$  and  $\alpha'_5$  on  $S'$  that correspond to the parts of the curves  $\alpha_3$  and  $\alpha_5$ , respectively, where  $\alpha_3$  and  $\alpha_5$  cross  $\alpha_4$ , form a nontrivial curve on  $S'$ , and then cross over  $\alpha_4$  in the opposite direction. The start and end points of  $\alpha'_3$  and  $\alpha'_5$  are these points of crossings. We shrink the boundary of  $S'$  to a point, producing a surface with two free loops  $\alpha_1$  and  $\alpha_2$  and two fixed loops  $\alpha'_3$  and  $\alpha'_5$ . We view the shrunk boundary component as a marked point on the torus. Recall, any loop on a torus may be written as a vector in  $\mathbb{Z}^2$  via the relationship  $\mathbb{Z}^2 \cong H_1(T^2, \mathbb{Z})$ . Without loss of generality, we take  $\alpha_1 = (1, 0)$ . Let  $\alpha_2 = (p_2, q_2)$ ,  $\alpha_3 = (p_3, q_3)$ , and  $\alpha_5 = (p_5, q_5)$ . Since  $a_1, \dots, a_5$  are isotopy classes of simple closed curves, we have  $\gcd(p_i, q_i) = 1$ .

Now we compute. By definition,

$$i(\alpha_1, \alpha'_3) = |(1)q_3 - (0)p_3| = |q_3|.$$

But by property (1), we have  $i(\alpha_1, \alpha'_3) = 0$ . Hence,  $q_3 = 0$ . By definition,

$$i(\alpha'_3, \alpha'_5) = |p_3q_5 - p_5q_3| = |p_3q_5|.$$

But by construction,  $i(\alpha'_3, \alpha'_5) = 1$ . Hence,  $|p_3q_5| = 1$ . Since  $p_3$  and  $q_5$  are in  $\mathbb{Z}$ , we have  $p_3 = 1 = q_5$ . By definition,

$$i(\alpha_2, \alpha'_5) = |p_2q_5 - p_5q_2|.$$

But by property (1),  $i(\alpha_2, \alpha'_5) = 0$ . Hence,  $p_2q_5 = p_5q_2$ . Recalling that  $\gcd(p_i, q_i) = 1$ , we have  $p_2 = p_5$  and  $q_2 = q_5$ . Finally, it follows by definition and substitution that

$$i(\alpha_1, \alpha_2) = |(1)q_2 - (0)p_2| = |q_2| = |q_5| = 1.$$

□

We now give the proof of Theorem 4.1.

*Proof. Step 1: Automorphisms of  $\mathcal{C}(S)$  preserve linked collections of curves.* In this first step, we show that any automorphism of  $\mathcal{C}(S)$  preserves the pairwise intersection numbers of *specific* sets of curves on  $S$ . Specifically, we will show that any automorphism of  $\mathcal{C}(S)$  preserves the pairwise intersection numbers of curves in Figure 9.

Let  $a_1$  and  $a_2$  be two isotopy classes of essential simple closed curves on  $S$ . Since  $i(a_1, a_2) = 0$  if and only if the vertices  $a_1$  and  $a_2$  are connected by an edge in  $\mathcal{C}(S)$ , it follows that property (1) in Lemma 4.2 can be recognized from the structure of  $\mathcal{C}(S)$ . Property (2) in Lemma 4.2 can also be recognized from the structure of  $\mathcal{C}(S)$ . To see this, let  $a$  be a vertex in  $\mathcal{C}(S)$  with some representative curve  $\alpha$ . Let  $L(a)$  be the link of  $a$  in  $\mathcal{C}(S)$ . Now consider the dual graph  $L^*(a)$  (recall Definition 3.5). Let  $S'$  denote the surface obtained by cutting along  $\alpha$  on  $S$ , that is,  $S' = S - \alpha$ . Note, the connected components of  $S'$  correspond to the connected components of  $L^*(a)$ . Hence,  $\mathcal{C}(S)$  contains sufficient information to identify the subsurfaces of  $S - \alpha$  and their corresponding isotopy classes of curves. Thus, from  $\mathcal{C}(S)$ , we can construct  $\mathcal{C}(S')$  such that  $S'$  is one of the subsurfaces of  $S - \alpha$ .

Using this construction,  $\mathcal{C}(S)$  also contains sufficient information to identify the topological types of the subsurfaces. Recall that the dimension of the curve complex of a surface homeomorphic to  $\Sigma_{g,b}$  is  $3g - 4 + b$ . By Theorem 3.1, the curve complex of a surface is homotopy equivalent to a wedge sum of spheres of dimension  $2g - 3 + b$ . Applying these two facts to  $\mathcal{C}(S')$ , we can identify the genus and number of boundary components of the subsurface  $S'$ , and thus its topological type. It follows that property (2) can be recognized by the structure of  $\mathcal{C}(S)$ .

Given that the structure of  $\mathcal{C}(S)$  is preserved by any automorphism of  $\mathcal{C}(S)$ , we can deduce that properties (1) and (2) of Lemma 4.2 are preserved by any automorphism of  $\mathcal{C}(S)$ . Hence, Lemma 4.2 implies that the property of two isotopy classes of simple closed curves having geometric intersection number 1 may be recognized by using the structure of  $\mathcal{C}(S)$  and is preserved by any automorphism of  $\mathcal{C}(S)$ . We have now completed step 1.

*Step 2: Producing an element of the mapping class group.* In this step, we show that given any automorphism  $A$  of  $\mathcal{C}(S)$ , there exist an element of  $\text{Mod}(S)$  that agrees with  $A$  on a set of isotopy classes of simple closed curves having geometric intersection number 1.

We let a *chain* be a sequence of curves  $\alpha_1, \dots, \alpha_k$  with  $i(\alpha_i, \alpha_{i+1}) = 1$  and  $i(\alpha_i, \alpha_j) = 0$  for  $|i - j| > 1$ . By the argument above, any automorphism of  $\mathcal{C}(S)$  maps the sequence of curves in a chain to another sequence of curves with the same configuration. By the change of coordinates principle, for any automorphism  $A$  of  $\mathcal{C}(S)$ , there exists an element of  $\text{Mod}(S)$  that agrees with  $A$  on a chain. More generally, the argument above implies that for any automorphism  $A$  of  $\mathcal{C}(S)$ , there exists an element of  $\text{Mod}(S)$  that agrees with  $A$  on a set of connected isotopy classes of curves  $\{a_1, \dots, a_n\}$  such that if  $a_i$  and  $a_j$  are not disjoint, then  $i(a_i, a_j) = 1$ . Hence, any automorphism of  $\mathcal{C}(S)$  must agree with some element of  $\text{Mod}(S)$  on the set of isotopy classes of curves in Figure 9.

*Step 3: Inducing an element of  $\text{Aut } \mathcal{A}(S)$ .* In this step, we show that any element of  $\text{Aut } \mathcal{C}(S)$  induces an automorphism of  $\mathcal{A}(S)$ .

To show that each automorphism of  $\mathcal{C}(S)$  agrees with an element of  $\text{Mod}(S)$  on every curve of  $S$ , we will utilize the arc complex associated to  $S$ . We claim that

if any automorphism  $A$  of  $\mathcal{A}(S)$  agrees with an element of  $\text{Mod}(S)$  for at least one maximal simplex in  $\mathcal{A}(S)$ , then  $A$  agrees with every maximal simplex of  $\mathcal{A}(S)$ . To see this let  $A$  fix the maximal simplex  $\Delta$  and let  $\Delta'$  be an arbitrary maximal simplex in  $\mathcal{A}(S)$ . By Corollary 3.7, there exists a sequence of maximal simplices  $\Delta = \Delta_1, \dots, \Delta_n = \Delta'$ . Since  $\Delta_1 \cap \Delta_2$  is a codimension-1 simplex,  $A$  fixes all but one vertex in  $\Delta_2$ . However, since maximal simplices correspond to triangulations of  $S$ , if all but one isotopy class of arcs is fixed, then there exist only two possible isotopy classes of arcs that can complete the triangulation. Recall,  $A$  fixes one of these two possible isotopy classes of arcs in  $\Delta_1$ . Given that  $\Delta_1 \neq \Delta_2$ ,  $A$  must fix the other possible isotopy class of arcs in  $\Delta_2$ . Hence,  $A$  fixes  $\Delta_2$ . Inducting upon this argument, we see that  $A$  fixes  $\Delta'$  and hence all of  $\mathcal{A}(S)$ .

Any isotopy class of arcs in  $\mathcal{A}(S)$  can be realized as a curve or a pair of curves on  $S$ . Let  $c$  be an isotopy class of essential simple arcs in  $\mathcal{A}(S)$  with representative  $\gamma$ . If  $\gamma$  connects two distinct boundary components,  $B_1$  and  $B_2$  of  $S$ , then we obtain an isotopy class  $a$  of simple closed curves in  $\mathcal{C}(S)$  by letting the representative of  $a$  be the boundary of the regular neighborhood of  $B_1 \cup \gamma \cup B_2$ . If  $\gamma$  connects a boundary component  $B$  to itself, then we may obtain two isotopy classes of simple closed curves in  $\mathcal{C}(S)$  by letting the respective representatives be the boundary components of the regular neighborhood of  $\gamma \cup B$ . One of these two curves may be trivial on  $S$ . Nevertheless, we still make a note of the trivial curve. By noting this triviality, we can distinguish arcs connecting one as opposed to two distinct boundary components.

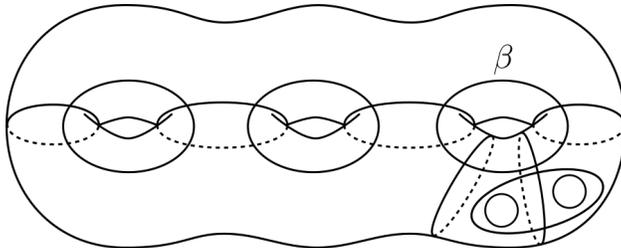


FIGURE 9. Fixed set of curves from the proof of Theorem 4.1.

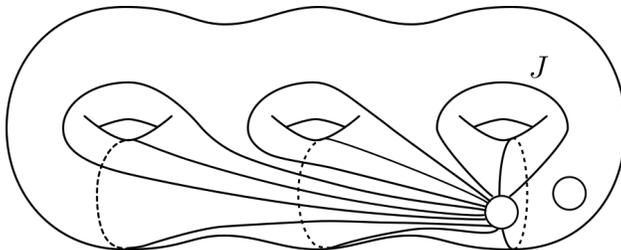


FIGURE 10. Fixed set of arcs from the proof of Theorem 4.1.

It was shown by Korkmaz [12] that we can assign an automorphism  $G_*$  of  $\mathcal{A}(S)$  to every automorphism  $G$  of  $\mathcal{C}(S)$  that preserves linked collections of curves as in Figure 9.

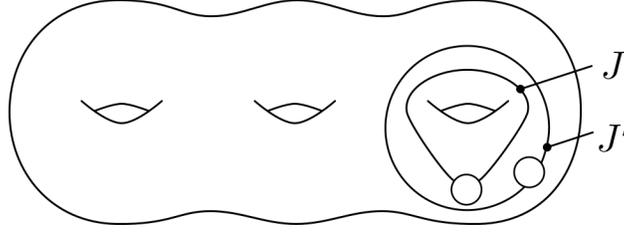


FIGURE 11. Two arcs from the proof of Theorem 4.1.

*Step 4: Showing every automorphism of  $\mathcal{C}(S)$  agrees with an element of  $\text{Mod}(S)$ .*

Consider Figure 9 and Figure 10. Let  $G$  be an element of  $\text{Aut } \mathcal{C}(S)$ . By the above argument,  $G$  agrees with some element  $g$  of  $\text{Mod}(S)$  on the set of isotopy classes of curves in Figure 9. Fix  $G$  to equal  $g^{-1} \circ G$  instead. Hence,  $G$  now fixes the isotopy classes of curves in Figure 9. Let  $G_*$  be the element of  $\text{Aut } \mathcal{A}(S)$  that is the associated automorphism obtained from  $G$ .

Consider one of the isotopy classes  $c$  of arcs in Figure 10 that is not the isotopy class containing the arc  $J$ . Consider all isotopy classes of curves in Figure 9 that are disjoint from  $c$  and cut  $S$  along their representative curves. This produces a subsurface  $D$  that is homeomorphic to a disk with two holes, containing the isotopy class  $c$  connected to its boundary. This subsurface can only contain one isotopy class of arcs connecting  $\partial D$  to  $\partial D$ . Hence,  $c$  must be fixed by  $G_*$ .

Now consider the isotopy class of arcs that has the representative  $J$  in Figure 10. Performing the same procedure above on  $J$  yields a disk with three holes. However, the above argument is insufficient to show that  $G_*$  fixes  $J$ . The other isotopy classes of arcs on this subsurface either contain  $J'$  as a representative (see Figure 11) or are the isotopy classes of arcs connecting the second boundary component with itself that are of the same form as  $J$  and  $J'$  of the first boundary component. Among these possible arcs,  $J$  is the only arc which has a coding associated to the curve  $\beta$  and which has a coding that intersects any other fixed arc from the previous argument with geometric intersection number 1. Given that  $G_*$  fixes these properties,  $G_*$  fixes  $J$  and thus the isotopy class containing  $J$ . By this argument,  $G_*$  fixes  $2g$  arcs on  $S$ .

We now show that  $G_*$  fixes a set of arcs that correspond to a maximal simplex in  $\mathcal{A}(S)$ . We cut our surface  $S$  along each fixed arc in Figure 10. This produces a polygon with  $4g$  sides corresponding to the boundary components of  $S$  containing the end points of the set of arcs fixed by  $G_*$ . In the center of this polygon is a hole corresponding to the second boundary component of  $S$ . We connect this hole to each edge with  $4g$  disjoint arcs. This produces a triangulation of  $S$  and thus a maximal simplex in  $\mathcal{A}(S)$ . Given that the sides of the polygon are fixed by  $G_*$ , these added arcs are fixed by  $G_*$ . Hence,  $G_*$  fixes a maximal simplex of  $\mathcal{A}(S)$ . By the argument above,  $G_*$  fixes  $\mathcal{A}(S)$  and is thus the identity. It follows that  $G$  must be the identity. Hence, the original automorphism agrees with an element of  $\text{Mod}(S)$  for all vertices in  $\mathcal{C}(S)$ .

This dispatches the case where  $S$  has two boundary components. When  $S$  has more than two boundary components, a very similar combinatorial argument follows and shows that  $G_*$  fixes a set of disjoint arcs that form a triangulation of  $S$ . This

proves Theorem 4.1 for when  $S$  has more than one boundary component. The case when  $S$  has fewer than two boundary components requires an additional argument.

Suppose  $S$  has fewer than two boundary components. We consider  $L^*(a)$ . By the argument above,  $L^*(a)$  is connected if and only if  $a$  is an isotopy class of a nonseparating curve. Given that the connectivity of  $L^*(a)$  is preserved by any automorphism of  $\mathcal{C}(S)$ , it follows that nonseparating curves are taken to nonseparating curves by any automorphism of  $\mathcal{C}(S)$ . All nonseparating curves on  $S$  are part of the same orbit of  $\text{Homeo}(S)$ . Thus, we can assume an automorphism  $A$  of  $\mathcal{C}(S)$  fixes at least one nonseparating curve.  $A$  induces an automorphism of  $L(a)$  and thus  $\mathcal{C}(S')$ , where  $S' = S - \alpha$  for some representative  $\alpha$  of  $a$ . This induced automorphism is simply a restriction of an automorphism of  $\mathcal{C}(S)$ . Also,  $S'$  has at least two boundary components. By the argument above, this automorphism of  $\mathcal{C}(S')$  corresponds to an element of  $\text{Mod}(S)$ . Considering all nonseparating isotopy classes on  $S$ , one can deduce that the nonrestricted automorphisms of  $\mathcal{C}(S)$  agree with elements of  $\text{Mod}(S)$ .  $\square$

## 5. AUTOMORPHISMS OF THE PANTS COMPLEX

In this section, we give a proof of Theorem 5.1 due to Margalit [14].

**Theorem 5.1.** *If  $S$  is not one of  $\Sigma_{0,3}, \Sigma_{0,4}, \Sigma_{1,1}, \Sigma_{1,2}, \Sigma_{2,0}$  and is an orientable surface with  $\chi(S) < 0$ , then the group of automorphisms of the pants complex of  $S$  is isomorphic to the group of automorphisms of the curve complex of  $S$ , that is,*

$$\text{Aut } \mathcal{P}(S) \cong \text{Aut } \mathcal{C}(S) = \text{Mod}(S).$$

*Proof.* Given a pants decomposition  $\{\alpha_1, \dots, \alpha_n\}$  of  $S$ , if we fix  $\{\alpha_2, \dots, \alpha_n\}$  and consider isotopy classes of simple close curves that only intersect  $\alpha_1$  and do so minimally, then we obtain a set of pants decompositions differing by elementary moves, that is, a subgraph of  $\mathcal{P}^1(S)$ . To see this, consider the collection of subsurfaces produced by cutting  $S$  along the  $n - 1$  fixed curves, that is,  $S - \cup_{i>1} \alpha_i$ . There is a unique subsurface in this collection that is isomorphic to either  $\Sigma_{1,1}$  or  $\Sigma_{0,4}$  and contains the curve  $\alpha_1$ . Considering the isotopy classes of curves that minimally intersect the isotopy class of  $\alpha_1$  on this subsurface, we obtain the subgraph of  $\mathcal{P}^1(S)$ .

The isomorphism type of this graph is known as a *Farey graph*. Thus, we call this subgraph of  $\mathcal{P}^1(S)$  an *abstract Farey graph*, denoted by  $F = \{\star, \alpha_2, \dots, \alpha_n\}$ . Similarly, a *marked abstract Farey graph* is a graph  $F$  with a distinguished vertex  $X$ . This distinguished vertex is a vertex in  $\mathcal{P}(S)$ . We denote this as the pair  $(F, X) = (\{\star, \alpha_2, \dots, \alpha_n\}, \{\alpha_1, \alpha_2, \dots, \alpha_n\})$ . Finally, we say two marked abstract Farey graphs *differ by an elementary move* if their associated marked points, which are pants decompositions, differ by an elementary move as defined in Section 3.3.

Using this construction, we will establish a natural association between marked abstract Farey graphs and vertices of  $\mathcal{C}(S)$ . Once this association is established, we will use it to define a map  $\phi : \text{Aut } \mathcal{P}^1(S) \rightarrow \text{Aut } \mathcal{C}(S)$ . The main work of this proof is showing that this map is well-defined and in fact an isomorphism. With this in mind, we give the following lemma.

**Lemma 5.2.** *There is a natural surjective map from the set of marked abstract Farey graphs in  $\mathcal{P}^1(S)$  to the set of vertices of  $\mathcal{C}(S)$ .*

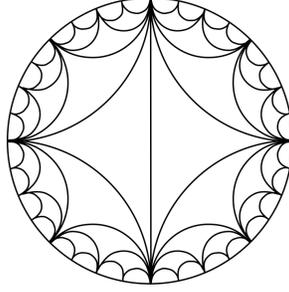


FIGURE 12. A Farey graph.

*Proof.* Let  $(F, X) = (\{\star, \alpha_2, \dots, \alpha_n\}, \{\alpha_1, \alpha_2, \dots, \alpha_n\})$  be a marked abstract Farey graph in  $\mathcal{P}^1(S)$ . Like the argument above, by considering  $S - \cup_{i>1} \alpha_i$ , we obtain a subsurface  $S'$  of  $S$  such that  $\alpha_1$  is on  $S'$ . Hence, the marked vertex  $X$  of  $F$  distinguishes a specific curve on this subsurface. More specifically, it distinguishes an essential simple closed curve on  $S$  and thus a unique vertex of  $\mathcal{C}(S)$ . This establishes the natural map from marked abstract Farey graphs to vertices of  $\mathcal{C}(S)$ .

To see that this map is surjective, we fix a vertex  $v$  in  $\mathcal{C}(S)$  and let  $\alpha$  be an essential simple closed curve on  $S$  that is a representative of  $v$ . Now pick some pants decomposition of  $S$  that contains  $\alpha$ . This pants decomposition corresponds to a vertex  $X = \{\alpha, \alpha_2, \dots, \alpha_n\}$  in  $\mathcal{P}^1(S)$ . Like before, by considering the subsurface in the collection of subsurfaces  $S' = S - \cup_{i>1} \alpha_i$  that contains  $\alpha$  and the set of all isotopy classes of curves on  $S'$  that minimally intersect  $\alpha$ , we obtain a subgraph of  $\mathcal{P}^1(S)$  that is an abstract Farey graph,  $F = \{\star, \alpha_2, \dots, \alpha_n\}$ . Hence,  $(F, X)$  corresponds to the vertex  $v$ .  $\square$

Given Lemma 5.2, we can define a map  $\phi : \text{Aut } \mathcal{P}^1(S) \rightarrow \text{Aut } \mathcal{C}(S)$ . Let  $A$  be an automorphism of  $\mathcal{P}^1(S)$ . We will define  $\phi : \text{Aut } \mathcal{P}^1(S) \rightarrow \text{Aut } \mathcal{C}(S)$  according to what  $\phi(A)$  does to each vertex of  $\mathcal{C}(S)$ . Let  $v$  be a vertex of  $\mathcal{C}(S)$  and let  $(F, X)$  be a marked abstract Farey graph associated to it. We define  $\phi(A)(v)$  to be the unique vertex corresponding to the marked abstract Farey graph  $(A(F), A(X))$ .

To show that this map is well-defined, we introduce some needed concepts. First, a *circuit* is a subgraph of  $\mathcal{P}^1(S)$  that is homeomorphic to a circle. Note, any edge in  $\mathcal{P}^1(S)$  lies in a unique abstract Farey graph. A sequence of consecutive vertices  $X_1, \dots, X_k$  in a circuit of  $\mathcal{P}^1(S)$  is *alternating* if the unique abstract Farey graph containing the edge  $X_{i-1}X_i$  is different from the unique abstract Farey graph containing the edge  $X_iX_{i+1}$  for  $1 < i < k$ . A circuit is an *alternating circuit* if any sequence of three vertices is alternating. A *small circuit* is a circuit with six or fewer vertices. Finally, a *2-curve small circuit* is a small circuit with the property that all vertices contain the same set of  $n - 2$  fixed curves, that is, the vertices have the form  $\{\star, \star, \alpha_3, \dots, \alpha_n\}$ .

**Lemma 5.3.** *Any small circuit which is not a 2-curve small circuit is an alternating circuit with six vertices.*

*Proof.* Let  $\mathcal{L}$  be a small circuit which is not a 2-curve small circuit. Given that  $\mathcal{L}$  is not a 2-curve small circuit, it must have three edges corresponding to the elementary moves

$$\alpha_1 \xrightarrow{m_1} \alpha'_1 \quad \alpha_2 \xrightarrow{m_2} \alpha'_2 \quad \alpha_3 \xrightarrow{m_3} \alpha'_3$$

Since  $\mathcal{L}$  is closed, there exist edges that correspond to the elementary moves

$$\star \xrightarrow{m'_1} \alpha_1 \quad \star \xrightarrow{m'_2} \alpha_2 \quad \star \xrightarrow{m'_3} \alpha_3$$

Note, we have that  $\alpha_i \neq \alpha_j$  for all  $i$  and  $j$ . Since if  $\alpha_i = \alpha'_j$ , then  $i(\alpha_i, \alpha_j) > 0$  (they differ by an elementary move). But by construction, we have that  $i(\alpha_i, \alpha'_i) > 0$ . Hence, we will not be able to construct a pants decomposition with the curve  $\alpha_i$ , which is a contradiction. Now, by way of contradiction and without loss of generality, suppose that  $\alpha'_2 \xrightarrow{m'_1} \alpha_1$  is an edge in the circuit. We have that  $i(\alpha'_2, \alpha_1) > 0$ . By our original construction,  $i(\alpha_i, \alpha'_i) > 0$  for all  $i$ . If we look at all possible pants decompositions associated to the vertices connected by the edge  $\alpha'_2 \xrightarrow{m'_1} \alpha_1$ , we have

$$\begin{aligned} \{\alpha'_2, \alpha'_1, \alpha_3, \dots, \alpha_n\} &\xrightarrow{m'_1} \{\alpha_1, \alpha'_1, \alpha_3, \dots, \alpha_n\} \\ \{\alpha'_2, \alpha'_1, \alpha'_3, \dots, \alpha_n\} &\xrightarrow{m'_1} \{\alpha_1, \alpha'_1, \alpha'_3, \dots, \alpha_n\} \\ \{\alpha'_2, \alpha_3, \alpha'_3, \dots, \alpha_n\} &\xrightarrow{m'_1} \{\alpha_1, \alpha_3, \alpha'_3, \dots, \alpha_n\} \end{aligned}$$

None of these form a pants decomposition of  $S$ . Thus, we have a contradiction and all edges  $m'_i$  must be of the form  $\alpha'_i \xrightarrow{m'_i} \alpha_i$ . It follows that  $\mathcal{L}$  is an alternating circuit with six vertices.  $\square$

Alternating sequences in  $\mathcal{P}^1(S)$  are combinatorially defined and thus preserved by any automorphism of  $\mathcal{P}^1(S)$ . Combining this fact with Lemma 5.3, we have the following result.

**Corollary 5.4.** *If  $A$  is an element of  $\text{Aut } \mathcal{P}^1(S)$  and  $\mathcal{L}$  is a small circuit which is not an alternating circuit with six vertices, then  $A(\mathcal{L})$  is a 2-curve small circuit.*

We also note the follow fact.

**Lemma 5.5.** *Given two marked abstract Farey graphs  $(F, X)$  and  $(F', X')$  that correspond to the same vertex  $v$  in  $\mathcal{C}(S)$  and differ by an elementary move, there exists a 2-curve small circuit such that four of its vertices  $W, X, X', Y$  make up an alternating sequence of the form:*

$$W \rightarrow X \rightarrow X' \rightarrow Y \text{ for } W, X \in F \text{ and } X', Y \in F'.$$

The proof of this statement follows from a straightforward combinatorial argument that reduces to looking at the possible subsurfaces of  $S$  that contain the curves  $\alpha_i$  in  $X$  and  $\alpha'_i$  in  $X'$  that correspond to the elementary move

$$X = \{\alpha_1, \dots, \alpha_i, \dots, \alpha_n\} \rightarrow \{\alpha_1, \dots, \alpha'_i, \dots, \alpha_n\} = X'.$$

For full details, see [14].

Finally, we can show the following result.

**Lemma 5.6.** *The map  $\phi : \text{Aut } \mathcal{P}^1(S) \rightarrow \text{Aut } \mathcal{C}(S)$  is well-defined.*

*Proof.* Let  $v$  be a vertex in  $\mathcal{C}(S)$  corresponding to the curve  $\alpha_1$  on  $S$ . Let  $(F, X)$  and  $(F', X')$  be two marked abstract Farey graphs that correspond to the vertex  $v$  and differ by an elementary move. This is a special case; however, as we will show, by the connectedness of  $\mathcal{P}(S)$ , it is sufficient to prove the claim. Finally, fix some

element  $A$  in  $\text{Aut } \mathcal{P}^1(S)$ . We will now show that the marked abstract Farey graphs  $(A(F), A(X))$  and  $(A(F'), A(X'))$  correspond to the same vertex  $w$  in  $\mathcal{C}(S)$ .

By Lemma 5.5, there exist a 2-curve small circuit with a sequence of alternating vertices

$$W \rightarrow X \rightarrow X' \rightarrow Y \text{ such that } W, X \in F \text{ and } X', Y \in F'.$$

By Corollary 5.4, we have that

$$A(W) \rightarrow A(X) \rightarrow A(X') \rightarrow A(Y)$$

is an alternating sequence of vertices that is part of a 2-curve small circuit with  $A(W), A(X)$  in  $A(F)$  and  $A(X'), A(Y)$  in  $A(F')$ . Let  $w$  be the vertex in  $\mathcal{C}(S)$  that corresponds to the marked abstract Farey graph  $(A(F), A(X))$  and let  $w$  have the representative curve  $\beta_1$  on  $S$ .

Suppose that the elementary move from  $A(W)$  to  $A(X)$  is of the form

$$\{\beta'_1, \beta_2, \dots, \beta_n\} \rightarrow \{\beta_1, \beta_2, \dots, \beta_n\}.$$

Without loss of generality, suppose that the elementary move from  $A(X)$  to  $A(X')$  is of the form

$$\{\beta_1, \beta_2, \dots, \beta_n\} \rightarrow \{\beta_1, \beta'_2, \dots, \beta_n\}.$$

Since  $A(W) \rightarrow A(X) \rightarrow A(X') \rightarrow A(Y)$  is an alternating sequence in a 2-curve small circuit, the elementary move from  $A(X')$  to  $A(Y)$  must be of the form

$$\{\beta_1, \beta'_2, \dots, \beta_n\} \rightarrow \{\beta''_1, \beta'_2, \dots, \beta_n\}.$$

Combining these results, we see that

$$\begin{aligned} (A(F), A(X)) &= (\{\star, \beta_2, \dots, \beta_n\}, \{\beta_1, \beta_2, \dots, \beta_n\}) \\ (A(F'), A(X')) &= (\{\star, \beta'_2, \dots, \beta_n\}, \{\beta_1, \beta'_2, \dots, \beta_n\}) \end{aligned}$$

By the defined correspondence between marked abstract Farey graphs and vertices of  $\mathcal{C}(S)$ , we see that  $(A(F), A(X))$  and  $(A(F'), A(X'))$  correspond to the same vertex  $w$  in  $\mathcal{C}(S)$ .

Now we show the general case. By Theorem 3.9,  $\mathcal{P}^1(S)$  is connected. Thus, for any two marked abstract Farey graphs  $(F, X)$  and  $(F', X')$  that correspond to the same vertex  $v$  in  $\mathcal{C}(S)$  but are not necessarily connected by a single elementary move, there exists a sequence of marked abstract Farey graphs  $F = F_1, \dots, F_m = F'$  such that  $(F, X)$  corresponds to the vertex  $v$  in  $\mathcal{C}(S)$  and  $(F_i, X_i)$  and  $(F_{i+1}, X_{i+1})$  are connected by an elementary move for  $1 \leq i < m$ . Applying the above result to each pair  $(A(F_i), A(X_i))$  and  $(A(F_{i+1}), A(X_{i+1}))$  in the sequence  $A(F) = A(F_1), \dots, A(F_m) = A(F')$ , we find that  $(A(F), A(X))$  and  $(A(F'), A(X'))$  correspond to the same vertex in  $\mathcal{C}(S)$ . Hence,  $\phi$  is well-defined.  $\square$

We now need to show that  $\phi(A)$  extends to an automorphism of  $\mathcal{C}(S)$ . To do this, it suffices to show that  $\phi(A)$  preserves all edges in  $\mathcal{C}(S)$ . Note, there exists an edge between two vertices  $a$  and  $b$  in  $\mathcal{C}(S)$  if and only if their associated isotopy classes of curves are disjoint on  $S$ . The isotopy classes of curves are disjoint on  $S$  if and only if there exists a pants decomposition of the form  $X = \{\alpha, \beta, \alpha_3, \dots, \alpha_n\}$ , where  $\alpha$  and  $\beta$  are representatives of  $a$  and  $b$ , respectively. Letting  $F_a$  and  $F_b$  be two abstract Farey graphs of the form  $\{\star, \beta, \alpha_3, \dots, \alpha_n\}$  and  $\{\alpha, \star, \alpha_3, \dots, \alpha_n\}$ , respectively, we see that these two abstract Farey graphs have a common vertex of intersection, namely  $X$ . Since the intersection of abstract Farey graphs is preserved by all automorphisms of  $\mathcal{P}^1(S)$ , following the logic above, we find that the edge

between  $a$  and  $b$  is preserved by  $\phi(A)$ . Since the edge was arbitrary,  $\phi(A)$  extends to an automorphism of  $\mathcal{C}(S)$ .

To see that  $\phi$  is a homomorphism, fix elements  $A$  and  $B$  in  $\text{Aut } \mathcal{P}^1(S)$  and a vertex  $v$  in  $\mathcal{C}(S)$ . Let  $(F_v, X_v)$  be the marked abstract Farey graph corresponding to  $v$ . By definition, we have

$$\begin{aligned}\phi(AB)(v) &\text{ corresponds to } (AB(F_v), AB(X_v)) \\ \phi(B)(v) &\text{ corresponds to } (B(F_v), B(X_v)) \\ \phi(A)(w) &\text{ corresponds to } (A(F_w), A(X_w))\end{aligned}$$

for some vertex  $w$  in  $\mathcal{C}(S)$ . We may chose  $(B(F_v), B(X_v)) = (F_w, X_w)$ . Hence,

$$\phi(A)\phi(B)(v) \text{ corresponds to } (AB(F_v), AB(X_v))$$

and  $\phi$  is a homomorphism.

Next, we note that the surjectivity of  $\phi$  follows from the straightforward computation of showing that the following diagram commutes:

$$\begin{array}{ccc} \text{Mod}(S) & \longrightarrow & \text{Mod}(S) \\ \downarrow & & \downarrow \\ \text{Aut } \mathcal{P}^1(S) & \longrightarrow & \text{Aut } \mathcal{C}(S) \end{array}$$

To show injectivity, suppose  $\phi(A)$  is the identity of  $\text{Aut } \mathcal{C}(S)$ . Let  $X$  be a pants decomposition of the form  $\{\alpha_1, \dots, \alpha_n\}$  such that each  $\alpha_i$  represents a vertex  $v_i$  in  $\mathcal{C}(S)$ . Denote the marked abstract Farey graphs corresponding to each  $v_i$  by  $(F_{v_i}, X) = (\{\alpha_1, \dots, \alpha_{i-1}, \star, \alpha_{i+1}, \dots, \alpha_n\}, \{\alpha_1, \dots, \alpha_n\})$ . Note,  $(A(F_{v_i}), A(X))$  must be marked abstract Farey graphs corresponding to  $\phi(A)(v_i) = v_i$  for all  $i$ . All these graphs intersect at a vertex, which must be  $X$ . Thus,  $A(X) = X$  for any vertex  $X$  in  $\mathcal{P}^1(S)$ . This shows that  $\phi$  is an isomorphism. It follows that

$$\text{Aut } \mathcal{P}^1(S) \cong \text{Aut } \mathcal{C}(S) = \text{Mod}(S).$$

Via a combinatorial argument, one can show that  $\text{Aut } \mathcal{P}^1(S) \cong \text{Aut } \mathcal{P}(S)$ . For details, see [14]. This concludes the proof.  $\square$

## 6. APPLICATIONS OF COMPLEXES

**6.1. Isomorphisms Between Subgroups of Finite Index of  $\text{Mod}(S)$ .** In this section, we first show an application of Theorem 4.1. Using this theorem and facts about Dehn twists given above, we are able to derive the following algebraic property of  $\text{Mod}(S)$ .

**Theorem 6.1.** *Let  $H_1, H_2$  be subgroups of finite index of  $\text{Mod}(S)$ . If  $S$  is an orientable surface of genus at least 2 and  $S$  is not a closed surface of genus 2, then all isomorphism  $H_1 \rightarrow H_2$  have the form  $x \mapsto gxg^{-1}$  for  $g \in \text{Mod}(S)$ . Moreover, the group of outer automorphisms  $\text{Out}(H_i)$  is finite.*

*Proof.* By Proposition 2.8, powers of Dehn twists commute if and only if their corresponding isotopy classes,  $a$  and  $b$ , of simple closed curves satisfy  $i(a, b) = 0$ . By definition,  $i(a, b) = 0$  if and only if there exists an edge connecting  $a$  and  $b$  in  $\mathcal{C}(S)$ . Given that any isomorphism between subgroups sends commuting elements to commuting elements, any isomorphism between subgroups of  $\text{Mod}(S)$  must also preserve all edges in  $\mathcal{C}(S)$ . Hence, every isomorphism  $\phi : H_1 \rightarrow H_2$  induces an

automorphism of  $\mathcal{C}(S)$ . By Theorem 4.1, these automorphisms are induced by elements of  $\text{Mod}(S)$ .

Let  $\phi : H_1 \rightarrow H_2$  be an isomorphism between subgroups of finite index and let the automorphism of  $\mathcal{C}(S)$  induced by the isomorphism be represented by the element  $g$  in  $\text{Mod}(S)$ . Since  $\phi$  induces an automorphism, it takes the isotopy class  $a$  of simple closed curves to the isotopy class  $g(a)$  of simple closed curves. By Theorem 2.9, for sufficiently large  $j \in \mathbb{Z}$ , there exists a  $k \in \mathbb{Z}$ , dependent on the isotopy class  $a$  of simple closed curves, such that  $\phi(T_a^j) = T_{g(a)}^k$ . Fixing an element  $f$  in  $H_1$  and using Proposition 2.8, we compute

$$\begin{aligned} T_{g(f(a))}^k &= \phi(T_{f(a)}^j) \\ &= \phi(fT_a^j f^{-1}) \\ &= \phi(f)\phi(T_a^j)\phi(f^{-1}) \\ &= \phi(f)\phi(T_a^j)\phi(f)^{-1} \\ &= \phi(f)T_{g(a)}^k\phi(f)^{-1} \\ &= T_{\phi(f)(g(a))}^k \end{aligned}$$

Hence,  $\phi(f)(g(a)) = g(f(a))$  for any isotopy class  $a$  of simple closed curves on  $S$ . Given that  $a$  was arbitrary, we may let  $a = g^{-1}(b)$  for some isotopy class  $b$  of simple closed curves on  $S$ . Thus,  $\phi(f)(b) = g \circ f \circ g^{-1}(b)$ . This shows the first statement. The second statement follows from the first by letting  $H_1 = H = H_2$  and recalling that  $H$  is a finite-index subgroup.  $\square$

**6.2. Finite Generation of  $\text{Mod}(S)$ .** In the remainder of this section, we give a proof of Theorem 6.2.

**Theorem 6.2.** *The mapping class group of an orientable surface  $S$  is generated by finitely many Dehn twists about nonseparating simple closed curves on  $S$ .*

Before giving the proof, we must develop some new concepts. To begin, we note that any element  $f$  of  $\text{Mod}(S)$  is determined by where it sends curves on  $S$ . Given some isotopy class  $a$  of nonseparating simple closed curves on  $S$ , we want to show that there exists a product of Dehn twists which takes  $f(a)$  to  $a$ . Such a product  $T_{a_1} \cdots T_{a_k}(f(a)) = a$  would imply that  $f^{-1} = T_{a_1} \cdots T_{a_k}$  when restricted to the isotopy class  $a$ . By performing this procedure inductively on a sufficiently large family of curves, we can reconstruct  $f$  in terms of Dehn twists and thus show the result.

By noting Figure 13 and applying the change of coordinates principle, if  $i(f(a), a) = 1$ , then the Dehn twist  $T_{f(a)}T_a$  takes  $f(a)$  to  $a$ . However, when  $i(f(a), a) \neq 1$ , we need to construct a chain of isotopy classes of nonseparating curves  $f(a) = a_1, \dots, a_m = a$  such that  $i(a_i, a_{i+1}) = 1$  for  $1 \leq i < m$ . This will provide a product of Dehn twists that takes  $f(a)$  to  $a$  as desired. To show the main result holds for an arbitrary surface, we will need to perform a double induction on both the number of punctures and the genus. We start with the following definition.

**Definition 6.3.**  $\mathcal{N}(S)$  is the 1-dimensional simplicial complex with vertices the isotopy classes of nonseparating simple closed curves on  $S$  and edges corresponding to pairs of vertices  $a$  and  $b$  with  $i(a, b) = 1$ .

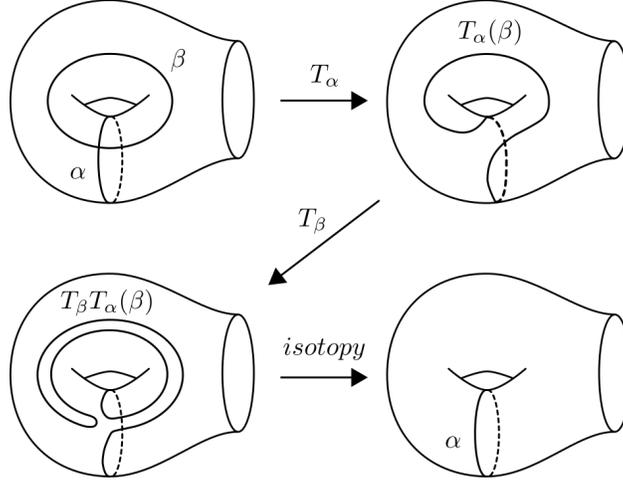


FIGURE 13. Illustration of  $T_\beta T_\alpha(\beta) = \alpha$  when  $i(\alpha, \beta) = 1$ .

**Proposition 6.4.**  $\mathcal{N}(S)$  is connected.

*Proof.* In this proof, we make the assumption that  $S$  has a genus of at least two. When we use the proposition to prove Theorem 6.2, we will also make the same assumption. This is because in Theorem 6.2, the case where  $S$  has a genus of 1 is handled as a specific basecase. We first show that any two isotopy classes of nonseparating essential simple closed curves on  $S$  are connected by a path of vertices in  $\mathcal{C}(S)$ . Let  $a$  and  $b$  be two isotopy classes of nonseparating essential simple closed curves on  $S$ . By Theorem 3.2, there exists a path  $a = c_1, \dots, c_m = b$  in  $\mathcal{C}(S)$  from  $a$  to  $b$ . We will show by induction that this path can be modified to contain only isotopy classes of nonseparating curves.

By way of induction let  $m \geq 1$  and suppose  $c_i$  is an isotopy class of a separating curve. We cut  $S$  along a representative of  $c_i$  and consider the two subsurfaces we obtain, denoted  $S'$  and  $S''$ . Since  $i(c_i, c_{i+1}) = 0 = i(c_i, c_{i-1})$  by construction, we have that  $c_{i-1}$  and  $c_{i+1}$  lie on either  $S'$  or  $S''$ . If  $c_{i+1}$  and  $c_{i-1}$  both lie on  $S'$ , then we may select some isotopy class  $d$  of nonseparating curves on  $S''$ . Replacing  $c_i$  in the original path, we have  $a = c_1, \dots, c_{i-1}, d, c_{i+1}, \dots, c_m = b$ . If  $c_{i-1}$  is on  $S'$  and  $c_{i+1}$  is on  $S''$ , then we may simply remove  $c_i$  from the original path to obtain  $a = c_1, \dots, c_{i-1}, c_{i+1}, \dots, c_m = b$ . Continuing this process for all  $i$ , we can obtain a path between any two vertices that is composed of isotopy classes of nonseparating curves.

In the special case where  $c_{i+1}$  and  $c_{i-1}$  both lie on  $S'$  and  $S''$  is a surface of genus zero, then by the inductive hypothesis, we can find a path connecting  $c_{i+1}$  and  $c_{i-1}$  composed of isotopy classes of nonseparating curves. Replacing  $c_i$  with this path gives us the desired result.

Now we show the connectivity of  $\mathcal{N}(S)$ . By the above argument, given any two isotopy classes  $a$  and  $b$  of nonseparating curves in  $\mathcal{N}(S)$ , there exists a path  $a = c_1, \dots, c_m = b$  in  $\mathcal{C}(S)$  from  $a$  to  $b$  that is composed of isotopy classes of nonseparating curves. We modify this path to obtain a path in  $\mathcal{N}(S)$ . By the change of coordinates principle, for all  $c_i$  in the initial path there exists an isotopy

class  $d_i$  of nonseparating curves such that  $i(c_i, d_i) = 1 = i(c_{i+1}, d_i)$ . We construct our path from  $a$  to  $b$  in  $\mathcal{N}(S)$  as  $a = c_1, d_1, \dots, d_{n-1}, c_n = b$ . This proves the connectivity of  $\mathcal{N}(S)$ .  $\square$

Another piece of technology that we will need to show Theorem 6.2 is the Birman exact sequence. First, given a surface  $S$ , denote by  $(S, x)$  the surface obtained from  $S$  by placing a marked point  $x$  on the interior of  $S$ . Viewing the point  $x$  as a puncture, there exists an inclusive map  $S - x \rightarrow S$ . This inclusive map induces a natural homomorphism

$$Forget : \text{Mod}(S, x) \longrightarrow \text{Mod}(S)$$

which is cleverly named the forgetful map. This map is realized by forgetting that the point  $x$  was marked. Now suppose the element  $f$  in  $\text{Mod}(S, x)$  is in the kernel of  $Forget$  and has a representative homeomorphism  $\phi$ . Note,  $\phi$  can also be thought of as a homeomorphism  $\bar{\phi}$  of  $S$ . Since  $Forget(f) = id$ , there exists an isotopy from  $\bar{\phi}$  to the identity map of  $S$ . Through this isotopy, the image of  $x$  traces out a loop  $\alpha$  on  $S$  with base point  $x$ . Note,  $\alpha : [0, 1] \rightarrow S$  is an isotopy of points from  $x$  to  $x$ , which extends to an isotopy of the entire surface. Letting  $\phi_\alpha$  denote the homeomorphism of  $S$  at the end of this isotopy, we can regard it as a homeomorphism of  $(S, x)$  and thus an element of  $Push(\alpha)$  in  $\text{Mod}(S, x)$ .  $Push(\alpha)$  is the homeomorphism that results when one places their finger on  $x$  and traces out the loop  $\alpha$ . However, when  $\alpha$  is being traced out,  $S$  is also dragged along the path of  $\alpha$ , see Figure 14. The map  $Push(\alpha)$  can be explicitly written in terms of Dehn twists.

**Proposition 6.5.** *Let  $\alpha$  be a simple loop on a surface  $S$  representing an element of  $\pi_1(S, x)$ . Then*

$$Push(\alpha) = T_a T_b^{-1}$$

where  $a$  and  $b$  are the isotopy classes of simple closed curves on  $(S, x)$  obtained by pushing  $\alpha$  off itself to the left and right, respectively.

Note, the isotopy classes  $a$  and  $b$  are nonseparating on  $(S, x)$  if and only if  $\alpha$  is nonseparating on  $S$ .

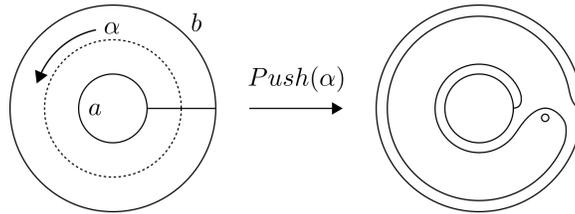


FIGURE 14. The push map in an annulus.

Given these definitions, we have the follow result due to Birman.

**Theorem 6.6. (Birman exact sequence)** *Let  $S$  be a surface with  $\chi(S) < 0$ , possibly with punctures or boundary components. Let  $(S, x)$  denote the surface obtained from  $S$  by marking the point  $x$  on the interior of  $S$ . Then the following sequence is exact:*

$$1 \longrightarrow \pi_1(S, x) \xrightarrow{Push} \text{Mod}(S, x) \xrightarrow{Forget} \text{Mod}(S) \longrightarrow 1.$$

Another concept that we will need is the pure mapping class group.

**Definition 6.7.** The *pure mapping class group*, denoted  $\text{PMod}(S)$ , is the subgroup of  $\text{Mod}(S)$  consisting of the elements that fix each puncture of  $S$  individually.

The action of  $\text{Mod}(S)$  on the punctures of  $S$  gives us the short exact sequence

$$1 \longrightarrow \text{PMod}(S) \longrightarrow \text{Mod}(S) \longrightarrow \Sigma_n \longrightarrow 1$$

where  $\Sigma_n$  denotes the symmetric group on the set of  $n$  elements. In light of this definition, we note that the Birman exact sequence can be rephrased for any subgroup of  $\text{Mod}(S)$ . This gives us the exact sequence

$$1 \longrightarrow \pi_1(S_g^n) \longrightarrow \text{PMod}(S_g^{n+1}) \longrightarrow \text{PMod}(S_{g,n}) \longrightarrow 1$$

where  $S_g^n$  denotes a surface of genus  $g$  with  $n$  punctures/marked points.

Given that any homeomorphism takes nonseparating curves to nonseparating curves and preserves the geometric intersection number,  $\text{Mod}(S)$  acts on  $\mathcal{N}(S)$ . To show the finite generation of  $\text{Mod}(S)$ , we will use this action on  $\mathcal{N}(S)$ . Given this, we need the following lemma.

**Lemma 6.8.** *Suppose a group  $G$  acts by simplicial automorphisms on a connected, 1-dimensional simplicial complex  $X$ . Suppose  $G$  acts transitively on both the vertices in  $X$  and the pairs of vertices connected by edges in  $X$ . Let  $v$  and  $w$  be two vertices in  $X$  and select some element  $h$  in  $G$  such that  $h(w) = v$ . Then  $G$  is generated by the element  $h$  together with the stabilizer of  $v$  in  $G$ .*

*Proof.* Let  $H$  be the subgroup of  $G$  generated by  $h$  together with the stabilizer of  $v$ . Fix some element  $g$  in  $G$ . By the connectivity of  $X$ , there exists a sequence of vertices  $v = v_1, \dots, v_k = g(v)$  such that adjacent vertices are connected by an edge. Since  $G$  acts transitively on the vertices of  $X$ , for each  $i$  there exists a  $g_i$  in  $G$  such that  $g_i(v) = v_i$ . Let  $g_0 = id$  and  $g_k = g$ . Clearly,  $g_0$  is an element of  $H$ .

By way of induction, suppose that  $g_i$  is in  $H$ . We apply  $g_i^{-1}$  to the vertices  $v_i = g_i(v)$  and  $v_{i+1} = g_{i+1}(v)$ . This gives us an edge between  $v$  and  $g_i^{-1}g_{i+1}(v)$ . Since  $G$  acts transitively on the edges of  $X$ , there exists an element  $r$  in  $G$  that takes  $(v, g_i^{-1}g_{i+1}(v))$  to  $(v, w)$ . Note,  $r$  is in the stabilizer of  $v$  and  $rg_i^{-1}g_{i+1}(v) = w$ . Hence,  $hrg_i^{-1}g_{i+1}(v) = v$ . This implies that  $hrg_i^{-1}g_{i+1}$  is in the stabilizer of  $v$  and thus an element of  $H$ . Since  $h, r, g_i$  (by inductive hypothesis) are in  $H$ , it follows that  $g_{i+1}$  is in  $H$ . Therefore,  $g$  is an element of  $H$ .  $\square$

We can now give the proof of Theorem 6.2. To do this, we prove the more general statement below. Theorem 6.2 follows as a special case.

**Theorem 6.9.** *Let  $S_g^n$  be a surface of genus  $g \geq 1$  with  $n \geq 0$  punctures. Then the group  $\text{PMod}(S_g^n)$  is finitely generated by Dehn twists about nonseparating simple closed curves on  $S_g^n$ .*

*Proof.* We will use double induction on the number of puncture of  $S$  and the genus of  $S$ . We note that the base cases for  $S_1^0$  and  $S_1^1$  hold but require slightly more specific arguments, see [1].

By way of induction on the number of punctures, suppose that  $\text{PMod}(S_g^n)$  is finitely generated by Dehn twists about nonseparating simple closed curves on  $S_g^n$ . We consider the Birman exact sequence

$$1 \longrightarrow \pi_1(S_g^n) \longrightarrow \text{PMod}(S_g^{n+1}) \longrightarrow \text{PMod}(S_g^n) \longrightarrow 1.$$

Since  $g \geq 1$ ,  $\pi_1(S_g^n)$  is finitely generated by the isotopy classes of nonseparating loops. By Proposition 6.5, the image of each loop in  $\pi_1(S_g^n)$  is a product of two Dehn twists about nonseparating simple closed curves. Given any nonseparating curve  $\alpha$  on  $S_g^n$ , there exists a nonseparating curve on  $S_g^{n+1}$  that maps to  $\alpha$  via the forgetful map. Hence, the Dehn twist  $T_\alpha$  in  $\text{PMod}(S_g^n)$  has a preimage in  $\text{PMod}(S_g^{n+1})$  which is a Dehn twist about a nonseparating simple closed curve on  $S_g^{n+1}$ . Combining these facts with the Birman exact sequence, we have proved the inductive step for the number of punctures.

By induction on genus, let  $g \geq 2$  and suppose  $\text{PMod}(S_{g-1}^n)$  is finitely generated by Dehn twists about nonseparating simple closed curves. By the change of coordinates principle,  $\text{Mod}(S_g)$  acts transitively on vertices and edges in  $\mathcal{N}(S_g)$ . Also, by Proposition 6.4,  $\mathcal{N}(S_g)$  is connected. Hence, Lemma 6.8 applies to the action of  $\text{Mod}(S_g)$  on  $\mathcal{N}(S_g)$ .

Let  $a$  and  $b$  be isotopy classes of nonseparating simple closed curves on  $S_g$  and let  $b$  be an isotopy class satisfying  $i(a, b) = 1$ . Let  $\text{Mod}(S, a)$  denote the stabilizer of  $a$  in  $\text{Mod}(S_g)$ . By Figure 13 and an argument with the change of coordinates principle,  $T_b T_a(b) = a$ . By Lemma 6.8,  $\text{Mod}(S_g)$  is generated by  $T_a$ ,  $T_b$ , and  $\text{Mod}(S_g, a)$ .

Let  $\text{Mod}(S, \vec{a})$  denote the subgroup of  $\text{Mod}(S_g, a)$  that preserves the orientation of  $a$ . We have the short exact sequence

$$1 \longrightarrow \text{Mod}(S_g, \vec{a}) \longrightarrow \text{Mod}(S_g, a) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1.$$

Using a change of coordinates principle argument, one can show that  $T_b T_a^2 T_b$  switches the orientation of  $a$ . Hence, it represents the coset of  $\text{Mod}(S_g, \vec{a})$  in  $\text{Mod}(S_g, a)$ . We note that

$$1 \longrightarrow \langle T_a \rangle \longrightarrow \text{Mod}(S_g, \vec{a}) \longrightarrow \text{PMod}(S_g - \alpha) \longrightarrow 1$$

is a short exact sequence, where  $S_g - \alpha$  is the surface obtained by deleting the representative  $\alpha$  of  $a$  from  $S_g$ . For a proof of this statement, see [1]. Note,  $S_g - \alpha$  is homeomorphic to  $S_{g-1,2}$ . By the inductive hypothesis,  $\text{PMod}(S - \alpha)$  is finitely generated by Dehn twists about nonseparating simple closed curves. Since each twist has a preimage in  $\text{Mod}(S_g, \vec{a})$ , it is also finitely generated by Dehn twists about nonseparating simple closed curves. This completes the inductive step on the genus and completes the proof of the main result.  $\square$

As an aside, we note that it was shown by Humphries that there exists a set of  $2g + 1$  nonseparating curves that suffice to generate  $\text{Mod}(S)$  [9]. These are known as *Humphries generators* and are shown in Figure 15.

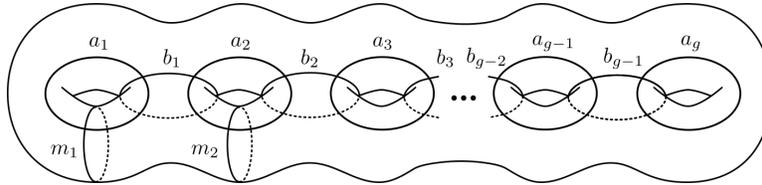


FIGURE 15. The set of Humphries generators for a surface.

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