AN INVESTIGATION INTO COVERS OF SOME FINITE SPACES

ADELE PADGETT

Abstract. Covering spaces are important objects in a variety of areas of mathematics. As the investigation into finite spaces and their properties continues, information about what covers of these spaces look like and how to construct them could become useful. One way of searching for this information is to look for similarities between covers of finite spaces and covers of spaces we are accustomed to working with. In this paper, we will investigate the relationship between the wedge of two circles and the 5-point space weakly homotopy equivalent to it. Later, we will suggest an intuitive way to find covers for any height-2 poset by looking at other wedges of circles.

Contents

1. Introduction

One of the problems mathematicians face as they explore the new territory of finite spaces is how to classify them. A useful way to differentiate between some spaces is to calculate their *fundamental groups*, which give us an understanding of the "holes" in a pointed topological space by taking as elements the equivalence classes of loops from a chosen basepoint in the space. However, it is difficult to intuitively understand what a loop in a finite space would look like, and it can be difficult to calculate the fundamental groups of finite spaces.

One of the reasons covering spaces are so useful is that they are deeply connected to the fundamental group. If a space X is path-connected, locally path-connected, and semi-locally simply connected, there is a *Galois correspondence* between the covers of X and the subgroups of its fundamental group. We will show that any connected finite space is also path-connected and locally

Date: November 30, 2015.

contractible. It follows that any connected finite space is path-connected, locally path-connected, and semi-locally simply connected, so we have the Galois correspondence for finite spaces.

1.1. **Covering Spaces.** Given a space X, a cover is intuitively a larger space $\tilde{X}$ which can be projected neatly into X in such a way that locally $\tilde{X}$ can be regarded as a stack of pancakes.

Definition 1.1. A cover of a space X is a space $\tilde{X}$ and a map $p : \tilde{X} \to X$ such that for each point x in X, there is an open neighborhood U of x where $p^{-1}(U)$ is the union of disjoint open sets in $\tilde{X}$, and p maps each of these sets homeomorphically onto U.

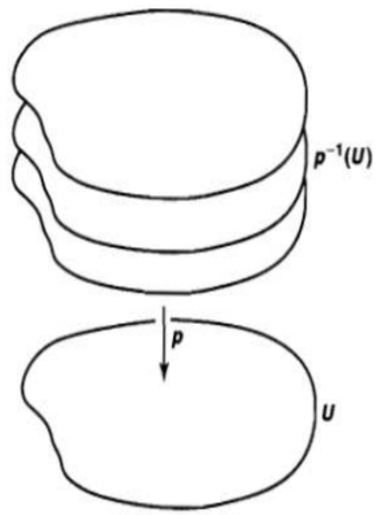
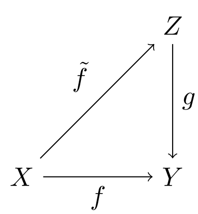


Figure from Munkres's *Topology* [4, p. 336]

We should also mention that a *lift* of $f : X \to Y$ along $g : Z \to Y$ is a map $\tilde{f} : X \to Z$ such that $g \circ \tilde{f} = f$.



Theorem 1.2. *Given a cover $p : \tilde{X} \to X$, a homotopy $f_t : Y \to X$, and a map $\tilde{f}_0 : Y \to \tilde{X}$ lifting f_0, there exists a unique homotopy $\tilde{f}_t : Y \to \tilde{X}$ of $\tilde{f}_0$ that lifts f_t.*

Proofs of this theorem and the ones below can be found in Allen Hatcher's *Algebraic Topology* [1, p. 60, 84-85]. We will only need this theorem to lift paths, not any larger spaces, because none of the objects we will be working with in this paper will have dimension greater than 1.

Now, the Galois correspondence between the covers of a space X and the subgroups of its fundamental group gives us the following:

Theorem 1.3. *Suppose X is path-connected, locally path-connected, and semilocally simply-connected. Then there is a bijection between the set of basepoint-preserving isomorphism classes of path-connected covering spaces $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ and the set of subgroups of $\pi_1(X, x_0)$. This bijection is obtained by associating each subgroup $p_*(\pi_1(\tilde{X}, \tilde{x_0}))$ to the cover $(\tilde{X}, \tilde{x_0})$. If basepoints are ignored, this correspondence gives a bijection between isomorphism classes of path-connected covering spaces $p : \tilde{X} \to X$ and conjugacy classes of subgroups of $\pi_1(X, x_0)$.*

Notice that if K is a subgroup of H, then the space corresponding to K will cover the space corresponding to H. This means that the bijection in the previous theorem is *order reversing*. Also note that the fundamental group of a space X must have a trivial subgroup, which is a subgroup of every other subgroup, so there must be a cover of X that covers every other cover. This is called the *universal cover* of X, and it is unique up to isomorphism. In fact, a space need only be locally path-connected and semi-locally simply connected to have a universal cover, and the proof of the previous theorem actually uses the existence of a universal cover.

This theorem is also stated as an *equivalence of categories* in Theorem 1.19, which will give us a general relationship between the covers of any two weakly homotopy equivalent spaces. However, the aim of this paper is to suggest an explicit geometric relationship between covers of weakly equivalent spaces. We will describe this explicit relationship for the wedge of circles and the 5-point space weakly equivalent to it.

The following theorem will provide some intuition about the relationship between posets (which we will see are equivalent to finite spaces) and wedges of circles, if both are considered as graphs.

Theorem 1.4. *For a connected graph X with maximal tree T, $\pi_1(X)$ is a free group with basis the classes of loops $[f_\alpha]$ corresponding to the edges e_α of $X - T$.*

This makes sense intuitively because a tree has no non-trivial loops and can be retracted to a single vertex. Collapsing a maximal tree in a connected graph leaves one vertex with a bouquet of edges, forming a wedge of circles, and the fundamental group of a wedge of κ circles is the free group on κ generators. Because trees are contractible, collapsing a maximal tree is a homotopy equivalence. This means that the fundamental group of X is the same as the fundamental group of the wedge of circles made up of the edges left over when the maximal tree is collapsed. These edges are exactly the edges e_α of $X - T$.

1.2. **Categorical Equivalence.** One of the major goals of this paper is to describe an equivalence between the category of covers of the wedge of circles and the category of covers of a finite space weakly homotopy equivalent to the wedge of circles. Establishing this categorical equivalence will show that the structure of these two categories is basically the same. Before defining what it means for two categories to be equivalent, we must recall some basic definitions in category theory.

Definition 1.5. A *category* consists of a class of objects and a class of morphisms between those objects. Every object has an identity morphism, and morphisms compose associatively with one another. Suppose F is a map between two categories $\mathcal{C}$ and $\mathcal{D}$ that associates an object $F(X)$ in $\mathcal{D}$ to each object X in $\mathcal{C}$ and an arrow $F(f) : F(X) \to F(Y)$ in $\mathcal{D}$ to each arrow $f : X \to Y$ in $\mathcal{C}$. F is a *functor* if

$$F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$$
$$F(g \circ f) = F(g) \circ F(f)$$

for objects X, Y, and Z, and morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{C}$.

For two functors $F : \mathcal{B} \to \mathcal{C}$ and $G : \mathcal{C} \to \mathcal{D}$, the composition of these functors $G \circ F : \mathcal{B} \to \mathcal{D}$ is also a functor.

Definition 1.6. Two categories $\mathcal{C}$ and $\mathcal{D}$ are isomorphic if there are functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that $G \circ F = \mathrm{id}_{\mathcal{C}}$ and $F \circ G = \mathrm{id}_{\mathcal{D}}$.

In other words, two categories are isomorphic if there are inverse functors between them. The notion of categorical equivalence is less strict than isomorphism. For two categories to be equivalent, there only need to be inverses "up to isomorphism." Two functors are inverses "up to isomorphism" if there are *natural isomorphisms* $G \circ F \cong \mathrm{id}_{\mathcal{C}}$ and $F \circ G \cong \mathrm{id}_{\mathcal{D}}$.

Definition 1.7. A natural transformation between two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{D}$ associates a morphism $\eta_X : F(X) \to G(X)$ in $\mathcal{D}$ to each object X in $\mathcal{C}$ such that the following diagram commutes for every morphism $f : X \to Y$ in $\mathcal{C}$:

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ {\scriptstyle \eta_X}\downarrow & & \downarrow{\scriptstyle \eta_Y} \\ G(X) & \xrightarrow[G(f)]{} & G(Y) \end{array}$$

A *natural isomorphism* is a natural transformation for which each of the morphisms η_X is an isomorphism.

Just as a functor is a map between categories that preserves their structure, a natural transformation is a map between functors that preserves their structure.

Definition 1.8. Two categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent if there are two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ with natural isomorphisms $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow GF$ and $\epsilon : FG \Rightarrow \mathrm{id}_{\mathcal{D}}$.

Categorical equivalence is a more useful notion of "sameness" than isomorphism because it is not so strong that it is useless, but it is not so weak that it doesn't say anything.

1.3. **Finite Spaces.** The following definitions and theorems building up the basics of finite spaces can be found in Peter May's book in progress, *Finite Spaces and Larger Contexts* [2, p. 4, 9, 16, 19].

Definition 1.9. A topological space X is an *Alexandroff space* if arbitrary intersections of open sets are open.

Definition 1.10. X is a T_0-*space* if for any two points of X, there is an open neighborhood of one that does not contain the other.

In other words, X is T_0 if its topology distinguishes between its points.

Proposition 1.11. *A finite space is an Alexandroff space.*

Definition 1.12. A preorder on a set X is a reflexive, transitive relation, denoted $\leq$. A partial order on a set X is an antisymmetric preorder. A partially ordered set $(X, \leq)$ is sometimes called a *poset*.

From any partial order on a set X, we can get a topology on X by taking the sets $U_x = \{y : y \leq x\}$ as basis elements. Notice that U_x is the smallest open set containing x in this topology. Now, if we take the intersection U of some set of open sets $\{U_i\}_{i \in I}$, then for any x in U, $U_x \subset U_i$ for each i, so U must be the union of the open sets U_x. This means that X with this topology is an Alexandroff space.

Theorem 1.13. *The Alexandroff space topologies on a set X are in bijective correspondence with the preorders on X, and the topology $\mathcal{T}$ corresponding to $\leq$ is T_0 if and only if the relation $\leq$ is a partial order. Also, maps between Alexandroff spaces are continuous if and only if they are order-preserving.*

This means that the category of finite spaces and the category of posets are isomorphic. We will use Hasse diagrams to depict posets. The arrows in our diagrams will point to the larger points. Before we can prove that the Galois correspondence applies to finite spaces, we need two more facts.

Lemma 1.14. *An Alexandroff space is connected if and only if it is path-connected*

Lemma 1.15. *If an Alexandroff space X contains a point y such that the only open (or closed) subset of X containing y is X, then X is contractible.*

In particular, this means that for any point x in an Alexandroff space X, U_x is contractible. Note that being locally contractible is a stronger condition than being semi-locally simply connected because if X is a locally contractible space, then for any point x in X, we can choose a contractible set U_x containing x. Then $\pi_1(U_x, x)$ is trivial, so its image in $\pi_1(X, x)$ is the trivial subgroup.

Using the previous two lemmas, we can now prove that the Galois correspondence applies to Alexandroff spaces.

Proposition 1.16. *For any connected finite space X, there is a Galois correspondence between the covers of X and the subgroups of its fundamental group.*

Proof. Since X is connected, it is path-connected, and X is locally path-connected because its connected components and path components coincide. By the previous lemma, X is locally contractible and hence semi-locally simply connected, so theorem 1.3 applies. □

1.3.1. *Formalizing Our Claim.* To prove that the categories of covers of weakly equivalent spaces are equivalent, we will define a functor $E : \mathcal{O}(\pi_1(X, x_0)) \to \mathrm{Cov}(X)$ from the *orbit category* of the fundamental group of a space X to $\mathrm{Cov}(X)$. This functor will be the same as mapping from the category of subgroups of the fundamental group of X to the category of isomorphism classes of path-connected covers of X, which is just the Galois correspondence. Any two weakly homotopy equivalent spaces X and Y have isomorphic orbit categories, and we achieve the desired equivalence between $\mathrm{Cov}(X)$ and $\mathrm{Cov}(Y)$ because categorical equivalence is an equivalence relation.

Before defining the orbit category, we need to define the constructions that will be its objects and morphisms. A *left action* of a group G on a set X is a function $G \times S \to S$ such that $ex = x$ for all x in X and $(gh)x = g(hx)$ for all g and h in G and x in X. An action is *transitive* if for all x and y in X, there is an element g in G such that $gx = y$. If H is a subgroup of G, the set G/H of cosets gH is a *transitive G-set.* An *equivariant map* is a map $\alpha : G/H \to G/K$ such that $\alpha(gx) = g\alpha(x)$. This means that α commutes with the action of G. Also, if there is an equivariant map as above, then H is subconjugate to K, meaning that there is an element g in G such that $g^{-1}Hg$ is a subgroup of K. Finally, the *orbit* generated by x is $\{gx : g \in G\}$.

Definition 1.17. The category $\mathcal{O}(G)$ has as objects the canonical orbit G-sets G/H, and as morphisms G-equivariant maps.

Proofs of the following two theorems can be found in Peter May's *A Concise Course in Algebraic Topology* [3, p. 60, 84-85].

Theorem 1.18. *The category $\mathcal{O}(G)$ is isomorphic to the category $\mathcal{G}$ whose objects are the subgroups of G and whose morphisms are the distinct subconjugacy relations $\gamma^{-1}H\gamma \subset K$ for γ in G.*

This means that we can think of the functor E between the category of covers of a space X and the orbit category of its fundamental group as building off the Galois correspondence between the covers of X and the subgroups of its fundamental group.

The following theorem will give us the equivalences we need to prove the desired result.

Theorem 1.19. *Choose a basepoint b in B. There is a functor*

$$E : \mathcal{O}(\pi_1(B, b)) \to \mathrm{Cov}(B)$$

that is an equivalence of categories. Let $G = \pi_1(B, b)$. For each subgroup H of G, the cover $p : E(G/H) \to B$ has a canonical basepoint e in its fiber over b such that

$$p_*(\pi_1(E(G/H), e)) = H.$$

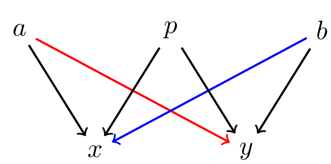
Also, $F_b \cong G/H$ as a G-set, and for a G-map $\alpha : G/H \to G/K$ in $\mathcal{O}(G)$, the restriction of $E(\alpha) : E(G/H) \to E(G/K)$ to fibers over b coincides with α.

Given any finite space weakly homotopy equivalent to a well-known space, we can get an equivalence between the isomorphism classes of covers of the finite space and of the well-known space. However, by taking this route to prove the categorical equivalence, we lose the geometric intuition behind our investigation into the connection between covers of weakly homotopy equivalent spaces. To recover the intuition motivating this high level categorical proof, we will consider the wedge of two circles and the space W, depicted below, which is a finite space weakly homotopy equivalent to $S^1 \vee S^1$. We will describe how to construct two other functors, $\mathrm{Thin} : \mathrm{Cov}(W) \to \mathrm{Cov}(S^1 \vee S^1)$ and $\mathrm{Thick} : \mathrm{Cov}(S^1 \vee S^1) \to \mathrm{Cov}(W)$, and use them to move between examples in $\mathrm{Cov}(W)$ and $\mathrm{Cov}(S^1 \vee S^1)$ to demonstrate an explicit equivalence between these two categories.

2. Finding an explicit equivalence

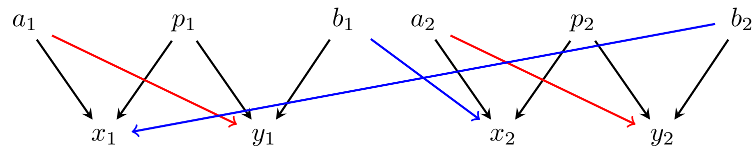
Although $\mathrm{Cov}(W)$ and $\mathrm{Cov}(S^1 \vee S^1)$ are equivalent by Theorem 1.19, it is a worthwhile exercise to reformulate the equivalence in a more intuitive way. We will suggest two pseudo-inverse functors, a *thinning map* and a *thickening map* that are constructed according to the method discussed informally below.

2.1. **Covers of the 5-Point Space.** Consider the following 5-point space, W, which is weakly homotopy equivalent to the wedge of two circles.



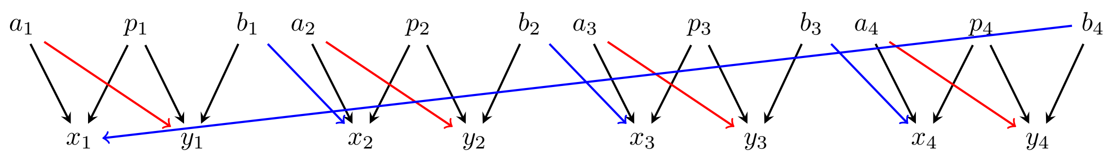
If we consider the two zigzags $p < x > a < y > p$ and $p < x > b < y > p$, then the first is a loop containing the red edge, and the second is a loop containing the blue edge.

We can construct a 2-fold cover of W by connecting two copies of the space as follows.

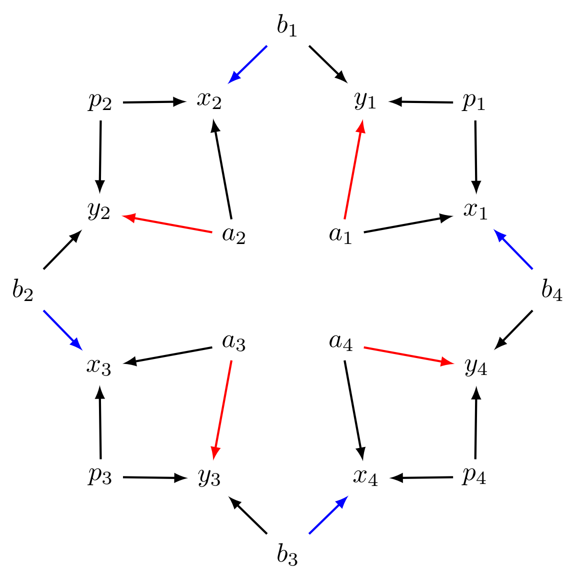


It is easy to imagine simply picking up one of the two W shapes, shifting it over so that its points match up with the points in the other W shape, and flattening it to get the original space.

Similarly, we can construct a 3-fold cover by taking three copies of W without the blue zigzag $b < x$, and then connecting b_i to x_{i+1} for $i = 1, 2$ and b_3 to x_1. A 4-fold cover, shown below, can be constructed in the same way by taking four copies of W without the blue zigzag and connecting them with the blue zigzags $b_i < x_{i+1}$ for $i = 1, \cdots, 3$ and $b_4 < x_1$. It is easy to see that an n-fold cover of this space may be constructed by stringing together n of these W-shaped "beads" using the blue edges.

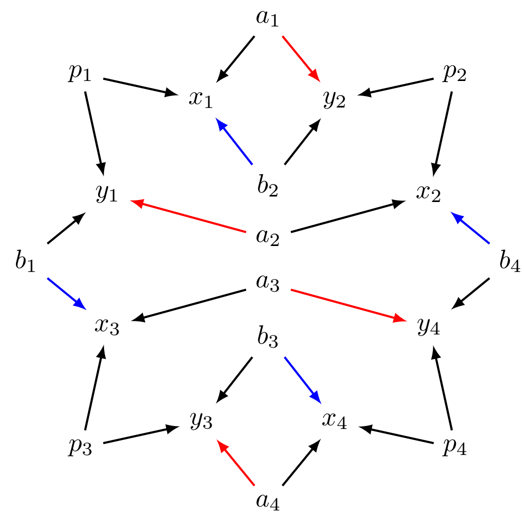


The symmetry of this 4-fold cover is clearer if it is drawn planar:

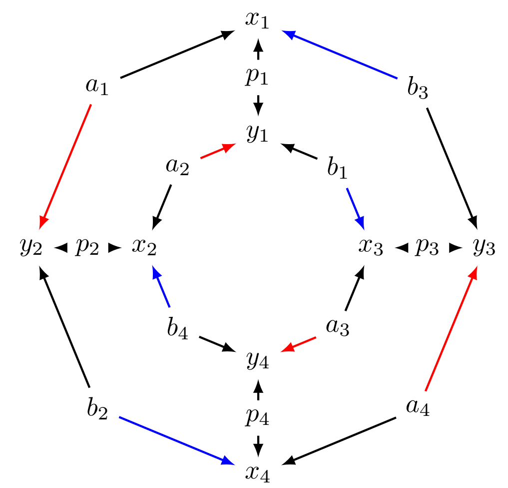


A

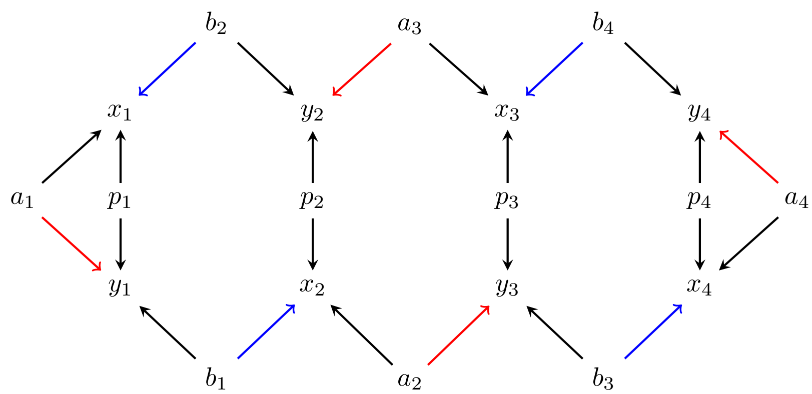
Three more 4-fold covers follow:



B



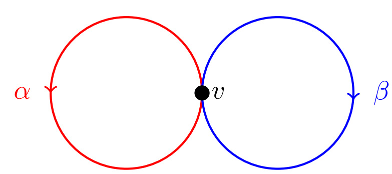
C



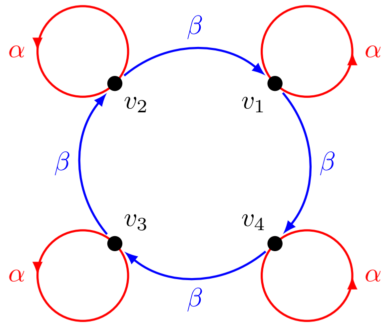
D

Now that we have found several 4-fold covers and a way to construct an n-fold cover of W for any n, it makes sense to ask whether there is a simple way to construct *all* covers of this space. To address this question, we will look at the wedge of two circles.

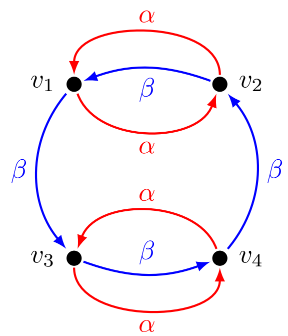
2.2. **Finding a Relationship to the Wedge of Circles.** Depicted below is the wedge of two circles. Note that we have colored the edges to distinguish the two generators α and β and have given them orientations.



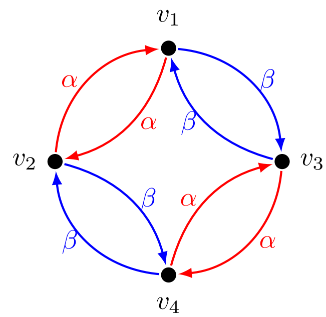
Four 4-fold covers of the wedge of two circles are shows below.



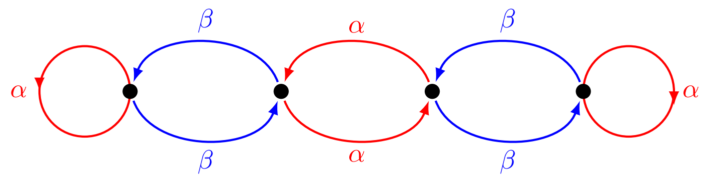
1



2



3



4

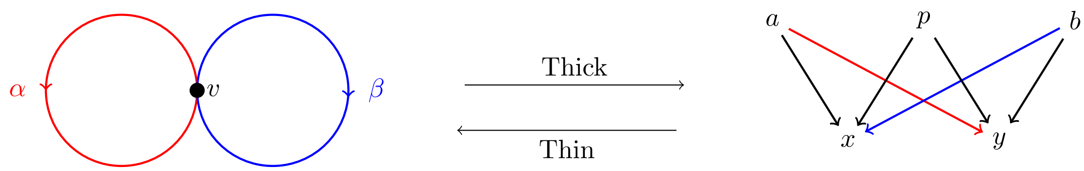
There are some immediate visual parallels between these covers and those of W in the previous section. On one hand, looking at the covers of the wedge of circles as graphs, each vertex has one red edge and one blue edge going in, and one red edge and one blue edge going out. On the other hand, each zigzag $a_i < x_i > p_i < y_i > b_i$ is connected to one red edge and one blue edge pointing in, and one red edge and one blue edge pointing out.

If we think of collapsing the points in W along $a < x > p < y > b$, then $a < y$ and $b < x$ would correspond to the red and blue generators of the wedge of two circles. This fits with theorem 1.4 because $a < x > p < y > b$ is a maximal tree in W, and W has the same fundamental group as the wedge of two circles. Similarly, we can collapse the black edges in covers A, B, C, and D above to get 1, 2, 3, and 4, respectively.

Now, if we think of the wedge of two circles and its covers as graphs, it becomes clear that for every node, we will have a zigzag $a_i < x_i > p_i < y_i > b_i$ in the corresponding poset cover of W. We already know that generator α corresponds to the zigzag containing $a < y$ in W, β corresponds to the zigzag containing $b < x$, and the direction of each generator is preserved by which point is reached first in the zigzag. Therefore, given a cover of the wedge of two circles, we need only turn each point in the cover into five points, endow these points with the appropriate ordering, and connect the colored edges to the correct points to create an analogous cover of W. We shall turn to the formalism of category theory to show how this correspondence between the covers of W and the covers of $S^1 \vee S^1$ can be made more precise.

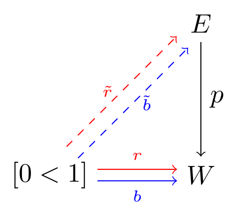
2.3. **Constructing Thinning and Thickening Functors.** In this section, we will not be so concerned with proofs and rigorous definitions, as making these two functors precise is a quite lengthy and tedious process, and our aim is to give geometric intuition for what is happening in these categories.

Our *thinning map* will collapse the black edges in a cover of W, leaving only red and blue edges and forming a corresponding cover of the wedge of circles, and our *thickening map* will turn each vertex in a cover of $S^1 \vee S^1$ into five points connected by four black edges.



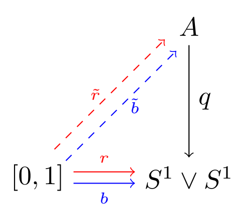
For this geometric method to work, we will have to color the edges of $S^1 \vee S^1$ and W to keep track of them, and these colors will lift to color the edges of the covers of $S^1 \vee S^1$ and W.

Definition 2.1. Let W be the 5-point space weakly homotopy equivalent to the wedge of two circles with points a, p, b, x, and y. Of the six edges $a < x$, $a < y$, $p < x$, $p < y$, $b < x$, and $b < y$ in W, one is labeled by $r : \{0, 1\} \to W$, one by $b : \{0, 1\} \to W$, and the rest are simply included into W, just as we have marked the red and blue zigzags above. The objects in the category $\mathrm{Cov}(W)$ are covers of W with points $\{a_i, p_i, b_i, x_i, y_i : i \in \mathscr{I}\}$ for some index set $\mathscr{I}$ and edges labeled by $\tilde{r}$ and $\tilde{b}$:



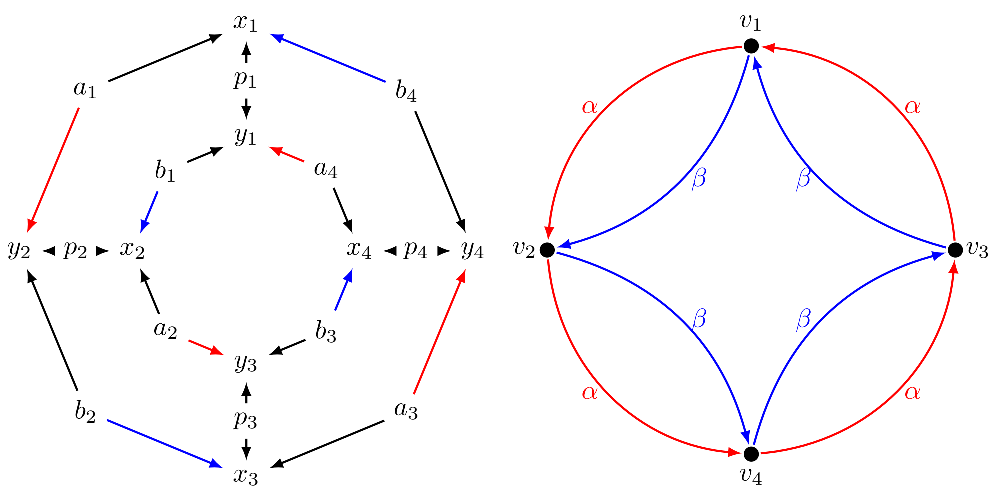
The lifts $\tilde{r}$ and $\tilde{b}$ are not unique. The number of red edges in E equals the number of blue edges in E, which equals the degree of the cover.

Similarly, the generators of the wedge of circles are labeled with maps $r : [0, 1] \to S^1 \vee S^1$ and $b : [0, 1] \to S^1 \vee S^1$, and we will call the point of intersection v. The objects in $\mathrm{Cov}(S^1 \vee S^1)$ are covers of the wedge of circles labeled by the lifts $\tilde{r}$ and $\tilde{b}$ with points of intersection $p^{-1}(v)$.



Again, there are the same number of red and blue lifts in A, and this is equal to the degree of the cover. The maps r and b into W or $S^1 \vee S^1$ are the same for all the objects in each category of covers. We will call an edge or zigzag (depending on the context) *red* if it is a lift of r and *blue* if it is a lift of b.

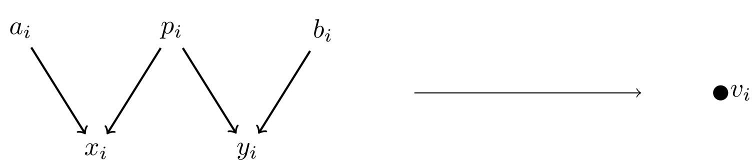
Now we would like to show how the thinning and thickening functors work with a particular example. We will consider W, $S^1 \vee S^1$, and their covers as directed graphs when working through the example. Pick a cover $p : E \to W$ and a cover $q : E' \to S^1 \vee S^1$ that we think will match up.



E E'

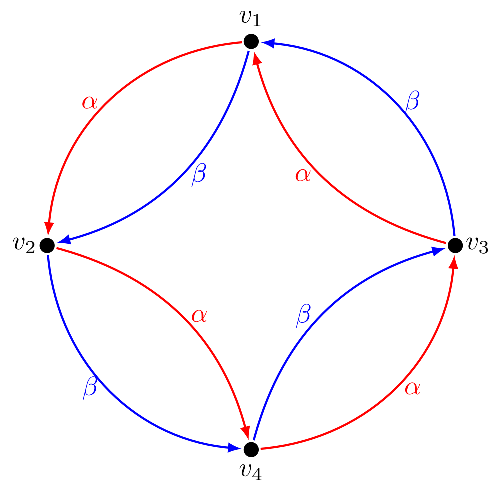
These two covers look similar to cover C in section 2.1 and cover 3 in section 2.2, but notice that the points are labeled differently in E, and the edges are colored differently in E'.

First we will thin $p : E \to W$ to get what we claim will be a cover of $S^1 \vee S^1$. The idea is to collapse the points a_i, x_i, p_i, y_i, and b_i to a single vertex v_i, and to throw out the edges forming the W between them. Then only colored edges will remain.



Formally, we could do this by defining the vertex set of the graph given by $\text{Thin}(p)$ using an equivalence relation that identifies two points if they are connected by a black edge, and by taking the edge set given by $\text{Thin}(p)$ to include only the edges that are red or blue in E. The source of each red edge $a_i \to y_j$ in E will map to v_i, and the target will map to v_j. This will ensure that the direction of the edge is preserved. Similarly, each blue edge $b_i \to x_j$ in E will have its source mapped to v_i and its target to v_j.

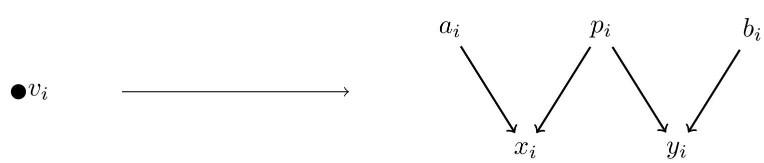
If we follow these instructions, we get that $\text{Thin}(p)$ maps the following space down to $S^1 \vee S^1$.



E'

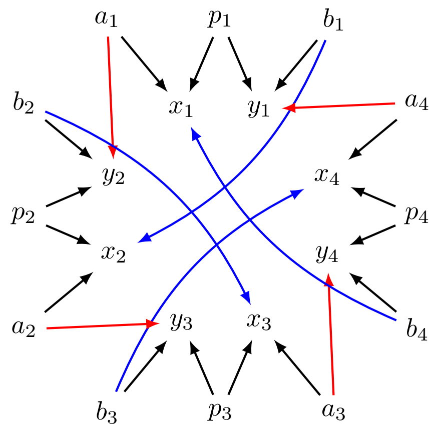
First we must check that this is indeed a cover of $S^1 \vee S^1$. Since we have transferred our work to the category of directed graphs, our computations are combinatorial instead of topological. The graph above has four vertices, and each is the source of one red and one blue edge and the target of one red and one blue edge, so it is a cover of $S^1 \vee S^1$. Furthermore, it is isomorphic to E': the two edges between v_1 and v_3 have switched places and the two edges between v_2 and v_4 have also switched places.

Now we will apply Thick to q to get what we claim will be a cover of W isomorphic to E. Here, we wish to expand each vertex v_i to five points a_i, x_i, p_i, y_i with edges between them as follows:

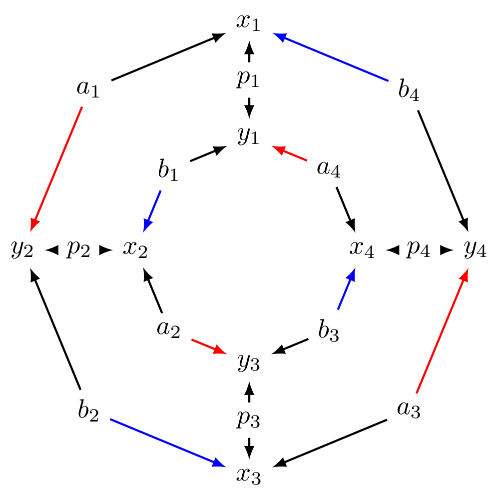


If V is the vertex set of E', we could do this formally by taking the vertex set of the graph given by Thick(q) to be $\{a, p, b, x, y\} \times V$. Then we would have five points $\{a_i, p_i, b_i, x_i, y_i\}$ for every vertex in E. To define the edge set given by Thick(q), we would need to take the edge set of E', call it D, and add in four edges in for every vertex in E'. If we call the four black edges in W e_1, e_2, e_3, and e_4, then the edge set given by Thick(q) would be $D \sqcup \big(\{e_1, \ldots, e_4\} \times V\big)$. Now, we have to be careful about where we connect up our red and blue edges. If there is a red edge $v_i \rightarrow v_j$ in E', then Thick(q) would take this edge to a red edge with a_i as its source and y_j as its target. Similarly, a blue edge $v_i \rightarrow v_j$ in E' would map to a blue edge starting at b_i and ending at x_j.

Following these rules, Thick(q) is a map sending the following space down to W.



Although the cover may look messy when arranged like this, it is easy to see that each black W is the source of one blue and one red edge and the target of one blue and one red edge. This means that it is indeed a cover of W. Also, each black W has a red and a blue edge coming in from another black W, and a red and a blue edge going out to a different black W. This is exactly true of E, and it is not too difficult to see how the cover above can be unwound to form E as we depicted it above.



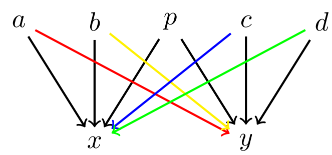
We have shown that $\text{Thin}(p) \cong q$ and $\text{Thick}(q) \cong p$. The process of using the functors Thick and Thin to move between corresponding covers of W and $S^1 \vee S^1$ will be the same for other covers of these two spaces, and if the functors are carefully defined, they will be pseudo-inverses. This means that we could get natural isomorphisms

$$\eta : (\text{Thin} \circ \text{Thick}) \Rightarrow \text{id}_{\text{Cov}(S^1 \vee S^1)}$$

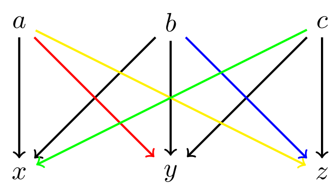
$$\epsilon : \text{id}_{\text{Cov}(W)} \Rightarrow (\text{Thick} \circ \text{Thin}).$$

that give the desired equivalence of categories.

2.4. **Extension to Other Finite Spaces.** By replacing W and $S^1 \vee S^1$ with any other spaces that are weakly homotopy equivalent, we get an equivalence between the categories of their coverings spaces by Theorem 1.19, but the more intuitive, explicit definitions of Thick and Thin also extend to some other spaces. We can get a finite space weakly homotopy equivalent to a wedge of any finite number of circles if we simply add a point of height 2 to the 4-point circle. For example, the following poset has two additional points, c and d, and it is weakly homotopic to wedge of four circles. The zigzags corresponding to the four generators are colored.



However, there are multiple ways to form a poset weakly homotopy equivalent to a wedge of certain numbers circles. For example, the following 6-point space also corresponds to the wedge of four circles.



By theorem 1.4, if X is a graph containing a subtree T, then X is homotopy equivalent to X/T. If we consider the previous two posets as graphs, the black edges form maximal trees, and the graphs retract to wedges of four colored circles.

We claim that thinning and thickening functors can be defined using maximal trees in any height-2 poset W' to associate its covers with the covers of the appropriate wedge of circles $\bigvee_{\kappa} S^1$. Proving that $\mathrm{Cov}(W')$ and $\mathrm{Cov}(\bigvee_{\kappa} S^1)$ are equivalent using the thinning and thickening functors appropriate for these categories would be a very long and involved process. However, these functors are useful purely for the lovely geometric connection they formalize between the two categories.

Acknowledgments. It is a pleasure to thank my mentors, Henry Chan and Jonathan Rubin, for all their help in my studies. Their guidance, patience, encouragement, and constructive suggestions have all been very much appreciated. Special thanks are also due to Peter May for organizing the REU program at the University of Chicago.

References

[1] Allen Hatcher. Algebraic Topology. Cambridge University Press. 2002.
[2] J. P. May. Finite Spaces and Larger Contexts
[3] J. P. May. A Concise Course in Algebraic Topology University of Chicago Press. 1999.
[4] James R. Munkres. Topology. New Jersey: Pearson Education, 2000.