LINE BUNDLES OVER FLAG VARIETIES

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Abstract. In this paper, we will prove the Borel-Weil Theorem, which relates representation theory to line bundles over flag varieties. We will cover the preliminary definitions required to understand and prove the theorem. We give some examples based on the algebraic group $\mathrm{SL}(2, \mathbb{C})$.

Contents

1. Introduction

To simplify presentation, our base field will be the field of complex numbers $\mathbb{C}$. All the results will be true if we work in any algebraically closed field of characteristic 0.

The purpose of this paper is to state and prove the Borel–Weil Theorem. Doing so requires us to know about algebraic groups and some basic representation theory, such as the theory of highest weight representations, and induced representations. This material can be found in sections 2 to 4 in this paper, but a more in-depth treatment of section 2 can be found in [3]. Section 5 is devoted to proving the Borel–Weil Theorem.

To provide a more complete picture, we discuss the Plücker embedding in section 6. The Plücker embedding provides some intuition about the flag variety, since the projective space is relatively well understood.

2. Preliminary definitions & results

This section contains some useful definitions and theorems from the theory of algebraic groups and Lie algebras. The proofs of the theorems in this section are

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omitted as they are not the main aim of the paper. Readers who are interested in the proofs are referred to [3].

2.1. **Varieties.** The idea of a variety is a geometric object that locally looks like the locus cut out by the vanishing of a collection of polynomials. A thorough treatment of varieties can be found in [1, Chapter 1].

We first define 2 classes of varieties, affine and projective varieties. Then we will define varieties in general. An **affine algebraic variety** is the set of common roots of a collection of polynomials in $\mathbb{C}^n$. We sometimes omit 'algebraic' and call it an affine variety.

Before we define projective varieties, we recall that homogeneous polynomials mean that each term has the same degree. For example, $x^2y + 5xyz + z^3 + x^3$ is a homogeneous polynomial (of degree 3), but $x^2y+5xyz+z$ is not (because there are 2 terms of degree 3 and a term of degree 1). The **projective n-space**, written $\mathbb{P}^n$, is the set of one dimensional subspaces of $\mathbb{C}^{n+1}$. A **projective algebraic variety** in $\mathbb{P}^n$ is the set of common roots of a collection of homogeneous polynomials in n+1 variables. We sometimes omit 'algebraic' and call it a projective variety.

The reader might find it strange why we only allow homogeneous polynomials in the above definition. The reason is because the collection of polynomials should have a well-defined vanishing set in projective n-space, and it makes sense only if we restrict ourselves to only homogeneous polynomials. This is because if f is a homogeneous polynomial of degree m, then $f(kx_0, kx_1, \ldots, kx_n) = k^m f(x_0, x_1, \ldots, x_n)$, so that if $(p_0, \ldots, p_n)$ is a root of f, then so is $(kp_0, \ldots, kp_n)$ for any $k \in \mathbb{C}$.

We wish to define a topology on affine and projective varieties. A way to do this is to give them the Zariski topology, which we now define. This captures the relationship of affine and projective varieties with vanishing polynomials.

Let X be an affine variety in $\mathbb{C}^n$ (resp. a projective variety in $\mathbb{P}^n$). Let S be a collection of polynomials in n variables (resp. collection of homogeneous polynomials in $n+1$ variables) over $\mathbb{C}$. Define $V(S)$ to be the vanishing set of S, explicitly $\{x \in X \mid f(x) = 0$ for any $f \in S$. The **Zariski topology** of X is the topology on X where $V(S)$ for any collection S of polynomials in n variables (resp. collection S of homogeneous polynomials in $n+1$ variables) are all the closed sets of X.

An example of a closed set in affine n-space is a singleton set. This is because $p = (p_1, \ldots, p_n)$ is the vanishing set of $\{x_1 - p_1, \ldots, x_n - p_n\}$. By the definition of a topology, this implies that all finite sets in affine n-space are closed under the Zariski topology.

We proceed to define varieties in general. A **quasi-affine algebraic variety** is an open subset (under the Zariski topology) of an affine variety. A **quasi-projective algebraic variety** is an open subset (under the Zariski topology) of a projective variety. An **algebraic variety** (called a **variety** for short) is any affine, quasi-affine, projective, or quasi-projective variety. This definition (which agrees with [1]) is sufficient for our purposes.

In fact, varieties contain a bit more structure. Let X be a variety. A map $f\colon X \to \mathbb{C}^1$ is a **regular function** if f is locally a rational function. Explicitly, for any $x \in X$, there is an open U containing x such that f, when restricted to U, can be written as f_1/f_2, where f_1, f_2 are polynomials and f_2 is never 0 on U.

Now that we have varieties, we should define the maps that preserve the structure of the varieties. In other words, we want to define morphisms of varieties. Let X be a variety. A **morphism of varieties** is a continuous map $\varphi\colon X \to Y$ such that for any regular function $f\colon U \to \mathbb{C}$ on an open set $U \subset Y$, the function $f \circ \varphi$ is regular on the open set $\varphi^{-1}(U) \subset X$.

$$\begin{tikzcd} \varphi^{-1}(U) \arrow[r, "\varphi"] \arrow[dr, dashed, "f\circ\varphi"'] & U \arrow[d, "f"] \\ & \mathbb{C} \end{tikzcd}$$

In short, a morphism of varieties is a map such that the pullback of open sets are open, and the pullback of regular functions are regular. This is a good way to remember the definition.

An **algebraic group** is a variety G with group structure such that multiplication and inversion are morphisms of varieties. A **morphism** of algebraic groups is a morphism of varieties that is also a group homomorphism.

A **linear algebraic group** is an algebraic group where the underlying variety is an affine variety. From now on, when we say 'algebraic group', we mean 'linear algebraic group', unless stated otherwise.

2.2. **Lie algebras.** Lie algebras are a useful tool to study representation theory. A standard book to learn about Lie algebras is [2]. The aim of this subsection is to give some basic definitions and results in the theory of Lie algebras, and use it to say something about the representation theory of algebraic groups. But first we need to show how to 'linearize' algebraic groups to get Lie algebras. This is the content of the next definition.

Given an affine variety X, define $P(X)$ to be the set of polynomials in n variables vanishing on X. Note that $P(X)$ is an ideal in the polynomial ring in n variables. We define **the affine algebra** of an affine variety $X \subset \mathbb{C}^n$ to be $\mathbb{C}[X] = \mathbb{C}[x_1, \ldots, x_n]/P(X)$.

Definition 2.1. For any $x \in G$, let λ_x denote the map defined by

$$(\lambda_x f)(g) = f(x^{-1}g)$$

for all $f \in \mathbb{C}[G]$, the affine algebra of G. The **Lie algebra** $\mathfrak{g}$ of an algebraic group G is $\mathscr{L}(G) = \{\delta \in \mathrm{Der}(\mathbb{C}[G]) \mid \delta\lambda_x = \lambda_x\delta \text{ for any } x \in G\}$ where $\mathrm{Der}(\mathbb{C}[G])$ denotes the set of derivations of the affine algebra of G.

Definition 2.2. A **Lie algebra** is a vector space $\mathfrak{g}$ over $\mathbb{C}$ with a bilinear form $[,]\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$, called the Lie bracket, satisfying

(1) $[x, y] = -[y, x]$

(2) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$

for any $x, y, z \in \mathfrak{g}$.

An ideal of a Lie algebra $\mathfrak{g}$ is a vector subspace I such that $[x, y] \in I$ for any $x \in \mathfrak{g}, y \in I$.

In particular, the first property implies that $[x, x] = 0$. An example of a Lie algebra is $\mathfrak{gl}_n$, the vector space of all $n \times n$ complex matrices with Lie bracket $[x, y] = xy - yx$. Another example is $\mathfrak{sl}_n$, the vector space of all $n \times n$ complex matrices of trace 0.

The Lie algebra of an algebraic group is an example of a Lie algebra, defined abstractly in Definition 2.2. Geometrically, the Lie algebra of an algebraic group can be visualised as the 'tangent space at 1'. This suggests that any morphism between algebraic groups $\varphi\colon G \to G'$ can be 'linearised' to get a Lie algebra homomorphism which we will define below. This is, in fact, true and we denote the associated Lie algebra homomorphism by $\mathrm{d}\varphi\colon \mathfrak{g} \to \mathfrak{g}'$ where $\mathfrak{g}$ and $\mathfrak{g}'$ are the Lie algebras of G and G' respectively.

Before moving on to representations of Lie algebras, we will need some structure theory of Lie algebras. Some notation: $[\mathfrak{g}, \mathfrak{g}]$ denotes the vector subspace of $\mathfrak{g}$ spanned by elements of the form $[x, y]$, where $x, y \in \mathfrak{g}$.

A Lie algebra $\mathfrak{g}$ is **simple** if $[\mathfrak{g}, \mathfrak{g}] \neq 0$ and the only ideals of $\mathfrak{g}$ are 0 or $\mathfrak{g}$. A Lie algebra $\mathfrak{g}$ is **semisimple** if it is a direct sum of simple Lie algebras. In particular, simple Lie algebras are semisimple.

A Lie algebra $\mathfrak{g}$ is **nilpotent** if some finite succession of taking Lie bracket with $\mathfrak{g}$ gives the 0 Lie algebra. In other words,

$$[\mathfrak{g}, [\mathfrak{g}, [\ldots[\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]]\ldots]]] = 0.$$

A Lie algebra $\mathfrak{g}$ is **solvable** if some finite succession of taking Lie bracket with itself gives the 0 Lie algebra. That is, if the sequence

$$\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}], [[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]], \ldots$$

eventually reaches the 0 Lie algebra. Any nilpotent Lie algebra is solvable, but the converse is not true in general.

In this paper, we will only need the representation theory of semisimple Lie algebras, and hence we will only discuss representations of semisimple Lie algebras.

A **Cartan subalgebra** of a semisimple Lie algebra $\mathfrak{g}$ is defined to be a nilpotent subalgebra $\mathfrak{h}$ satisfying $\mathfrak{h} = \{x \in \mathfrak{g} \mid [x\mathfrak{h}] \subset \mathfrak{h}\}$. A **Borel subalgebra** $\mathfrak{b}$ of a semisimple Lie algebra $\mathfrak{g}$ is defined to be a maximal solvable subalgebra. We remark that if $\mathfrak{g}$ is semisimple, then Cartan subalgebras and Borel subalgebras exist, but may not be unique. Furthermore, any Borel subalgebra contains a Cartan subalgebra.

Suppose we have a semisimple Lie algebra $\mathfrak{g}$, and we fix a Borel subalgebra $\mathfrak{b}$ containing a fixed Cartan subalgebra $\mathfrak{h}$. For any non-zero $\alpha\colon \mathfrak{h} \to \mathbb{C}$, consider $\mathfrak{g}_\alpha := \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x$ for all $h \in \mathfrak{h}$. If $\mathfrak{g}_\alpha$ is non-zero, we call α a **root** and $\mathfrak{g}_\alpha$ a **root space**. It can then be shown that $\mathfrak{g}$ is the direct sum of $\mathfrak{h}$ and all its root spaces. In other words, denote the set of roots by Φ. Then $\mathfrak{g} = \oplus_{\alpha\in\Phi}\mathfrak{g}_\alpha$. The theory of Lie algebras tells us that Φ is a finite set.

Next we define the positive root space $\mathfrak{g}^+$ to be the vector subspace of $\mathfrak{b}$ such that $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{g}^+$ as vector spaces. In particular, $\mathfrak{g}^+$ can be decomposed as a direct sum of some root spaces. In other words, there exists a subset $\Phi^+ \subset \Phi$ such that $\mathfrak{g}^+ = \oplus_{\alpha\in\Phi^+}\mathfrak{g}_\alpha$. We call the elements of Φ^+ to be **positive roots**. Lastly, given a positive root α, it is a non-trivial fact from the structure theory of Lie algebras that there exists a unique element $h_\alpha \in [g_\alpha, g_{-\alpha}] \subset \mathfrak{h}$ such that $\alpha(h_\alpha) = 2$. We call each h_α a **positive coroot**.

A **representation** of a Lie algebra $\mathfrak{g}$ is a vector space V over $\mathbb{C}$with a $\mathfrak{g}$ -action, such that $[x, y].v = x.(y.v) - y.(x.v)$ for any $x, y \in \mathfrak{g}, v \in V$. For example, the

Lie algebra $\mathfrak{gl}_n$ acts on $\mathbb{C}^n$ by left multiplication: For any $X \in \mathfrak{gl}_n, v \in \mathbb{C}^n$, define $X.v = Xv$ by matrix multiplication.

There is a very special type of representation with good properties. For this, we let $\mathfrak{g}$ be a semisimple Lie algebra, and fix a Borel subalgebra $\mathfrak{b}$ containing the Cartan subalgebra $\mathfrak{h}$. Let $\lambda\colon \mathfrak{h} \to \mathbb{C}$ be a functional on $\mathfrak{h}$. The **irreducible highest weight representation of weight** λ of $\mathfrak{g}$ is a representation V such that V is generated by some $v_\lambda \in V$ satisfying $h.v_\lambda = \lambda(h)v_\lambda$ for any $h \in \mathfrak{h}$ and $x.v_\lambda = 0$ for any $x \in \mathfrak{g}^+$, where $\mathfrak{g}^+$ is the positive root space. We call v_λ a **highest weight vector**. A highest weight vector is unique up to scalar multiplication.

A functional $\lambda\colon \mathfrak{h} \to \mathbb{C}$ is said to be a **dominant weight** if $\lambda(h_\alpha) \geq 0$ for any positive coroot h_α. It is said to be an **integral weight** if $\lambda(h_\alpha) \in \mathbb{Z}$ for any positive coroot h_α. A functional is a **dominant integral weight** if it is both dominant and integral. It can be proved that the irreducible highest weight representation of weight λ is finite dimensional if and only if λ is dominant integral.

2.3. **Algebraic groups.** Now that we have these results from the theory of Lie algebras, we wish to lift these results to the level of algebraic groups.

Let $D(n, \mathbb{C})$ be the group of diagonal n by n matrices. A **torus** T of an algebraic group G is a subgroup that is isomorphic to $D(n, \mathbb{C})$ for some n. The tori are partially ordered by inclusion, and a maximal element is called a **maximal torus**. A **Borel subgroup** B of an algebraic group G is a maximal solvable connected subgroup of G. The term 'solvable' is in the group theoretic sense. We remark that every Borel subgroup contains a maximal torus. In fact, Cartan subalgebras and Borel subalgebras in Lie algebras correspond to maximal tori and Borel subgroups in algebraic groups.

The reader might ask what the analogue of semisimple Lie algebras are in the level of algebraic groups. They are called semisimple algebraic groups: A connected algebraic group is **semisimple** if it has no non-trivial closed connected abelian normal subgroup. With this definition, it turns out that the Lie algebra of a semisimple algebraic group is a semisimple Lie algebra, as we wanted.

What is the analogue in algebraic groups for positive root space? Suppose G is an algebraic group, with a Borel subgroup B containing a maximal torus T. Let U be the **unipotent radical** of B, which is defined to be the subgroup of B containing the unipotent elements in B. It can be found in [3] that $B = TU$, so that U is the analogue of positive root space.

The analogue of representations of Lie algebras are representations of algebraic groups. A **representation** of the algebraic group G is a morphism of algebraic groups $\rho\colon G \to \mathrm{GL}(V)$ for a vector space V. The representation ρ is said to be finite-dimensional if V is a finite dimensional vector space.

A representation is essentially a vector space V being acted upon by the algebraic group G. We sometimes write $g.v$ to mean $\rho(g)v$. We also sometimes abuse terminology and call V a representation of G.

To define a highest-weight representation, we will need to define characters.

Definition 2.3. A character of an algebraic group G is a morphism of algebraic groups $\chi\colon G \to \mathbb{C}^\times$, where $\mathbb{C}^\times$ is the multiplicative group of non-zero complex numbers.

Notice that if χ_1, χ_2 are characters of an algebraic group G, then

$$\chi_1 + \chi_2 \colon g \mapsto \chi_1(g)\chi_2(g)$$

for any $g \in G$ is also a character. The set $\mathrm{X}(G)$ of all characters of the algebraic group G is actually a commutative group, called the character group of G. It is an unfortunate fact that the group operation in $\mathrm{X}(G)$ is conventionally written as addition, which might lead to confusion sometimes.

Now let λ be a character of the chosen maximal torus T. The **irreducible highest weight representation of G with highest weight** λ is the representation of G generated by a vector v_λ satisfying $t.v_\lambda = \lambda(t)v_\lambda$ for $t \in T$ and $u.v_\lambda = 0$ for $u \in U$. We call v_λ a **highest weight vector**. Note that for fixed λ, the irreducible highest weight representation of G with highest weight λ is unique up to isomorphism, and our use of 'the' is justified.

There is one more concept for weights we need. Let G be a semisimple algebraic group with a chosen maximal torus T. Let λ be a character of T. Then λ is said to be a **dominant integral weight** if $d\lambda \colon \mathfrak{h} \to \mathbb{C}$ is a dominant integral weight, which was defined in the weight theory of Lie algebras above.

Lastly, we now discuss flag varieties, which is a main object of study in this paper. Let G be an algebraic group and B be a Borel subgroup. We state the fact that the quotient group G/B can be given the structure of a quasi-projective variety, so that it becomes an algebraic group. We call G/B a flag variety. Proving the above fact is rather tedious, and can be found in [3, Sections 11-12]. The main point the reader should get from this paragraph is that G/B is an algebraic group.

3. Vector Bundles

Definition 3.1. Let X be a variety. A vector bundle of rank k over X is a variety E with a surjective morphism of varieties $\pi \colon E \to X$ such that

(1) For any $x \in X$, $\pi^{-1}(x)$ is a vector space (over $\mathbb{C}$) of dimension k.

(2) For any $x \in X$, there exists an open $U \subset X$ containing x and a morphism $\Phi \colon \pi^{-1}(U) \to U \times \mathbb{C}^k$, such that for any $p \in U$, $\Phi|_{\pi^{-1}(p)} \colon \pi^{-1}(p) \to p \times \mathbb{C}^k$ is a vector space isomorphism, and the following diagram commutes.

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[rr, "\Phi"] \arrow[dr, "\pi"'] & & U \times \mathbb{C}^k \arrow[dl, "\pi_1"] \\
& U &
\end{tikzcd}$$

We call Φ a local trivialization over U.

Definition 3.2. A line bundle is a vector bundle of rank 1.

Example 3.3. The trivial line bundle over a variety X is $X \times C$, the surjection being $\pi(x, z) = x$ for any $(x, z) \in X \times \mathbb{C}$.

Example 3.4. The tautological line bundle over the projective variety $\mathbb{P}^n$ is $\mathscr{L} = \{(x, l) \in \mathbb{C}^{n+1} \times \mathbb{P}^n \mid x \in l\}$, with the natural projection map $\pi \colon \mathscr{L} \to \mathbb{P}^n$ defined as $\pi(x, l) = l$.

For the name to make sense, we have to justify that the tautological line bundle is a line bundle. Observe that for any $l \in \mathbb{P}^n$, $\pi^{-1}(l) \cong C$ is a vector space. As for local trivialization, for any $l = [z_0\colon z_1\colon \cdots : z_n] \in \mathbb{P}^n$, there is $z_i \neq 0$. Choose

$$U = U_i = \{[y_0\colon y_1\colon \cdots : y_n] \mid y_i = 1, y_0, \ldots, y_n \in \mathbb{C}\}$$

which is clearly open in $\mathbb{P}^n$. Then an explicit description of a local trivialization $\Phi\colon \pi^{-1}(U) \to U \times \mathbb{C}$ is

$$\Phi((x_0, x_1, \ldots, x_n), [z_0\colon z_1\colon \cdots : z_n]) = ([z_0\colon z_1\colon \cdots : z_n], x_i).$$

Clearly the diagram in Definition 3.1 commutes, and also the overlapping $U_i \cap U_j$ is open with transition map x_i/x_j between local trivializations, which is a morphism of varieties.

Next we introduce another example of a vector bundle, called a homogeneous vector bundle.

Definition 3.5. A map $\pi\colon G \to G'$ between algebraic groups is said to be locally trivial if for any $h \in G'$, there exists open $U \subset G'$ with $h \in U$ and a morphism $\sigma\colon U \to G$ such that $\pi \circ \sigma = id_U$.

Theorem 3.6. *Let G be an algebraic group, and H is a closed subgroup of G. Suppose $\pi\colon G \to G/H$ is locally trivial. Let V be a finite dimensional H-module. Define an equivalence relation $\sim$ on $G \times V$ by $(g, v) \sim (gh, h^{-1}.v)$. Then $G \times_H V$ defined as $(G \times V)/\sim$ is a vector bundle over G/H of rank* $\dim V$.

The vector bundle defined in Theorem 3.6 is called a homogeneous vector bundle. We shall denote the elements of $G \times_H V$ by $[x, v]$, the equivalence class of $(x, v) \in G \times V$.

Proof. Define the surjection $\pi\colon G \times_H V \to G/H$ by $\pi[x, v] = xH$. To show that it is well-defined, notice that $\pi[xh, h^{-1}.v] = xhH = xH = \pi[x, v]$.

Condition 1 in Definition 3.1 is satisfied: For any $xH \in G/H$,

$$\pi^{-1}(xH) = \{[x, v] \mid v \in V\} \cong V$$

is a vector space.

Condition 2 in Definition 3.1 is satisfied: For any $xH \in G/H$, there is an open set U containing xH and a morphism $\sigma\colon U \to G$ such that $\pi \circ \sigma = id_U$, by locally trivial property. For the local trivialization, we take

$$\Phi\colon \pi^{-1}(U) \to U \times V$$

to be $\Phi([g, v]) = (gH, \sigma(gH)g.v)$. To show well-definedness, observe that

$$\Phi([gh, h^{-1}.v]) = (gH, \sigma(gH)gh.h^{-1}v) = (gH, \sigma(gH)g.v) = \Phi([g, v]).$$

Lastly, the diagram in Definition 3.1 clearly commutes, with $k = \dim V$. Hence $G \times_H V$ is a vector bundle of rank $\dim V$. □

Proposition 3.7. *Let B be a Borel subgroup of a semisimple algebraic group G. Then the projection $\pi\colon G \to G/B$ is locally trivial.*

Before we prove the above proposition, we need a lemma (Bruhat decomposition). Pick any maximal torus T, and define the Weyl group W to be $N_G(T)/T$, where

$$N_G(T) = \{g \in G \mid gTg^{-1} = T\}$$

Remark 3.8. The Weyl group is a finite group. It is well-defined, in that the choice of maximal torus T does not matter. This is because of the non-trivial fact that all maximal tori are conjugate in G.

Lemma 3.9. *(Bruhat decomposition) Let G be a semisimple algebraic group with a chosen Borel subgroup B containing a maximal torus T. Then $G = \bigsqcup_{\sigma \in W} B\sigma B$, where W is the Weyl group.*

The proof of the Bruhat decomposition can be found in [3].

Proof of Proposition 3.7: G has Bruhat decomposition $G = \bigsqcup_{w \in W} B\sigma B$. For any $xB \in G/B$, there is only one $\delta \in W$ such that $xB \in B\delta B$. Let $U = B\delta B \subset G/B$. Since B is closed and W is finite, so $G - U$ is a finite union of closed sets, hence closed. This implies that U is open.

Define $\sigma \colon U = B\delta B \to G$ by $\sigma(p\delta B) = p\delta$ (δ acts on the right). Then we have $\pi \circ \sigma(p\delta B) = p\delta B$ for any $p \in B$, so that the projection $\pi \colon G \to G/B$ is locally trivial. $\square$

We now know that $G \times_B V$ is a vector bundle, by Proposition 3.7 and Theorem 3.6.

Example 3.10. For the algebraic group $G = \mathrm{SL}(n, \mathbb{C})$, choose the Borel subgroup B to be the upper triangular matrices of determinant 1. There is an obvious action of G on the set of flags of the form

$$0 \subset V_1 \subset V_2 \subset \ldots \subset V_n = \mathbb{C}^n,$$

where each V_i is a vector subspace of $\mathbb{C}^n$ of dimension i. This is clearly a transitive action, by acting on a flag by a change-of-basis matrix, to get the standard flag

$$0 \subset <e_1> \subset <e_1, e_2> \subset \ldots \subset <e_1, \ldots, e_n> = \mathbb{C}^n,$$

where $\{e_i\}$ is the standard basis for $\mathbb{C}^n$. The stabilizer of the standard flag is the Borel subgroup B, so that G/B is the set of flags. Recall that G/B can be given the structure of a quasi-projective variety. This implies that the set of flags can be given the structure of a quasi-projective variety too. This is also the reason why we call G/B a flag variety.

In particular, for $n = 2$, each flag corresponds to a one-dimensional subspace V_1 of $\mathbb{C}^2$, so that the set of flags is the projective space $\mathbb{P}^1$. In particular, it has the structure of a projective variety. We hence get some line bundles over G/B such as the tautological line bundle over $\mathbb{P}^1$. We will see more examples of line bundles after we prove the Borel–Weil Theorem.

4. Review of Basic Representation Theory

Given a closed subgroup H of an algebraic group G, we would like to get a representation of G from a representation of H, and vice versa. This is done via the induced and restricted representations.

Definition 4.1. (Global section) Given a vector bundle over X, $\pi \colon E \to X$, a **global section** of π is a morphism $\sigma \colon X \to E$ satisfying $\pi \circ \sigma = id_X$.

Definition 4.2. (Induced representation) Suppose we have an algebraic group G and a closed subgroup H. Let V be a representation of H. Consider the homogeneous vector bundle $\pi \colon G \times_H V \to G/H$. The **induced representation**, denoted by $\mathrm{Ind}_H^G V$, is the vector space of all global sections of this vector bundle, with G-action $(g.\sigma)(xH) = g\sigma(g^{-1}x)$ for any $\sigma \in \mathrm{Ind}_H^G$ and $xH \in G/H$.

Definition 4.3. (Restricted representation) Let G be an algebraic group, and H be a closed subgroup. Let V be a representation of G. Then the restricted representation of H is V, with G-action restricted to H.

Remark 4.4. The restricted representation is sometimes written as $\mathrm{Res}_H^G V$, to emphasize that we consider it as a representation of H instead of G.

No review of representation theory is complete without Schur's lemma, which we will now state and prove.

Lemma 4.5 (Schur)**.** *If V and W are finite dimensional irreducible representations of a group G, then* $\dim(\mathrm{Hom}_G(V, W))$ *is 1 if V and W are isomorphic, and 0 otherwise.*

Proof. For any $f \in \mathrm{Hom}_G(V, W)$, it is easy to check that $\mathrm{Ker}(f)$ and $\mathrm{Image}(f)$ are G-invariant subspaces of V and W respectively. By irreducibility of V, $\mathrm{Ker}(f)$ is 0 or V. The former case implies f is injective, and by irreducibility of W, we conclude $\mathrm{Image}(f) = W$, so that f is bijective. The latter case implies $f = 0$. Hence $\dim(\mathrm{Hom}_G(V, W)) = 0$ if V is not isomorphic to W.

Suppose V is isomorphic to W. We compute $\dim(\mathrm{Hom}_G(V, W))$, which is equivalent to computing $\dim(\mathrm{Hom}_G(V, V))$. We claim that any G-invariant map from V to itself is scalar multiplication. To prove this, let f be any such map. Since $\mathbb{C}$ is algebraically closed, it has an eigenvalue $\alpha \in \mathbb{C}$. Consider $f - \alpha\mathrm{Id}$, which is a G-invariant linear transformation from V to itself. Furthermore, since α is an eigenvalue, so $f - \alpha\mathrm{Id}$ has non-zero kernel. By the same argument as the first paragraph in this proof, this means $\mathrm{Ker}(f - \alpha\mathrm{Id}) = V$, so that $f = \alpha\mathrm{Id}$ for some $\alpha \in \mathbb{C}$. Hence $\mathrm{Hom}_G(V, W) = \mathbb{C}$ and in particular, has dimension 1. $\square$

5. Borel–Weil Theorem

We can now state and prove the Borel–Weil Theorem, which relates induced representations with the dominance of the character. Roughly, it says that there is a one-one correspondence between dominant integral weights and irreducible induced representations.

Let G be an algebraic group, and B be a Borel subgroup containing a maximal torus T. We first classify irreducible finite dimensional representations of B. To do this, we need a preliminary result, the Lie–Kolchin Theorem, which can be found as Theorem 17.6 in [3].

Theorem 5.1 (Lie–Kolchin Theorem)**.** *Let G be a connected solvable subgroup of* $\mathrm{GL}(V)$ *for some finite dimensional non-zero vector space V. Then G has a common eigenvector in V.*

Readers interested in the proof can refer to [3].

Theorem 5.2. *Let G be a connected and solvable algebraic group with a chosen maximal torus T. Then there is only one Borel subgroup $B = G$, and hence $G = B = TU$ where U is the unipotent radical of G. Any character of T, say $\chi\colon T \to \mathbb{C}^\times$, determines a one-dimensional irreducible representation V of G, with G-action given by $g.v = \tilde{\chi}(g)v$ for $g \in G$ and $v \in V$, where $\tilde{\chi}$ is χ on T and 1 on U. Furthermore, all irreducible finite dimensional representations of G are one-dimensional and of this form.*

Proof. Firstly, G is a maximal solvable connected subgroup of itself, and hence is the only Borel subgroup.

Given a character χ of T, it is a straightforward verification from the definition that the vector space V described in the statement of the theorem is indeed a representation.

Suppose V is an irreducible finite dimensional representation of G. This means there is a morphism of algebraic groups, $\phi\colon G \to \mathrm{GL}(V)$. Hence $\phi(G)$ is a connected solvable subgroup of $\mathrm{GL}(V)$, and we can apply Lie–Kolchin Theorem. There exists $v \in V$ such that for any $g \in G$, $g.v = k_g v$ for some $k_g \in \mathbb{C}$ which depends on g. Notice that this implies that the one-dimensional vector subspace spanned by v is a subrepresentation, and by irreducibility of V, means that V is one-dimensional spanned by v.

Observe that a one-dimensional representation V is determined completely by specifying what the G-action on some non-zero $v \in V$ is. This is equivalent to specifying a morphism $\tilde{\chi}\colon B \to \mathbb{C}^\times$, by saying the G-action is just $g.v = \tilde{\chi}(g)v$. However, we can further simplify this, by seeing that $\tilde{\chi}(g)$ has to be 0 for $g \in U$, because g is unipotent. Hence, it is sufficient to just specify a character χ of T in order to determine what the one-dimensional representation of G is. Therefore, all irreducible finite dimensional representations of G come from a character χ of T. $\square$

In particular, we can take G to be a Borel subgroup of any linear algebraic group, and hence we have classified all irreducible finite dimensional representations of a Borel subgroup.

Theorem 5.3 (Borel–Weil Theorem)**.** *Let G be a semisimple algebraic group. Let B be a Borel subgroup of G, and W be a one-dimensional representation of B corresponding to a character χ of T. If $-\chi$ is a dominant integral weight, then $\mathrm{Ind}_B^G W$ is the dual of the irreducible representation of G of highest weight $-\chi$. If $-\chi$ is not a dominant integral weight, then $\mathrm{Ind}_B^G W$ is 0.*

Before we prove the Borel–Weil Theorem, we have the following lemma. This is a different version of Frobenius reciprocity.

Lemma 5.4. *Let G be an algebraic group and H be a closed subgroup. For any finite dimensional representations V of G and W of H, we have an isomorphism of vector spaces $\mathrm{Hom}_G(V, \mathrm{Ind}_H^G W) \cong \mathrm{Hom}_H(\mathrm{Res}_H^G V, W)$.*

Proof. Define the map $\phi\colon \mathrm{Hom}_G(V, \mathrm{Ind}_H^G W) \xrightarrow{\sim} \mathrm{Hom}_H(\mathrm{Res}_H^G V, W)$ by $\phi(F)(v) = F(v)(1)$ for $F \in \mathrm{Hom}_G(V, \mathrm{Ind}_H^G W), v \in V$. We will show that it has an inverse map ψ given by $\psi(f)(v)(g) = f(g^{-1}.v)$ for $f \in \mathrm{Hom}_H(\mathrm{Res}_H^G V, W)$, $v \in V, g \in G$. It is a straightforward computation to check that they are inverse maps and conclude that the Hom sets are isomorphic. $\square$

We now use this lemma to prove the Borel–Weil Theorem.

Proof of Theorem 5.3. Given a one-dimensional representation W of B corresponding to the character χ, we wish to calculate $\mathrm{Ind}_B^G W$. For any dominant integral weight λ, let us define V_λ to be the irreducible representation of G with highest weight vector v_λ, of weight λ. We will compute $\dim(\mathrm{Hom}_B(\mathrm{Res}_B^G V_\lambda, W))$. By Lemma 5.4 (we can apply it because V_λ is finite dimensional if λ is dominant integral),

$$\mathrm{Hom}_B(V_\lambda, W) \cong \mathrm{Hom}_G(V_\lambda, W)^B \cong (V_\lambda^* \otimes W)^B$$

where the superscript B means the subspace of B-invariant elements (elements that are fixed under the B-action).

Let U be the unipotent radical of B, the subgroup containing unipotent elements in B. Note that $B = TU$ [3], so

$$(V_\lambda^* \otimes W)^B = ((V_\lambda^* \otimes W)^U)^T.$$

We remark that the U-action on $(V_\lambda^* \otimes W)$ is $u.(f \otimes w) = (f \circ u^{-1}) \otimes w$ for $u \in U, f \in V_\lambda^*, w \in W$.

Thus $(V_\lambda^* \otimes W)^U$ is naturally identified with $(V_\lambda^*)^U \otimes W$, because U acts trivially on W. But by definition, the highest weight vectors are the vectors fixed by U, so $(V_\lambda^*)^U$ is a highest weight space and in particular, is one-dimensional.

The next step is to use this to compute $((V_\lambda^* \otimes W)^U)^T$. Fix the highest weight vectors of V_λ^* and W, which we call f and w_χ respectively. Since $(V_\lambda^* \otimes W)^U$ is one-dimensional, $f \otimes w_\chi$ spans it. The T-action on the basis vector is

$$t.(f \otimes w_\chi) = (\lambda(t^{-1})f \otimes \chi(t)w_\chi.$$

The condition for $f \otimes w_\chi$ to be T-invariant is that λ should satisfy $-\lambda(t^{-1}) = \chi(t)$ for any $t \in T$. Another way to say this is that we take $\lambda = -\chi$ (treating the characters as elements of the character group; refer to the paragraph after Definition 2.3 for notations) at the start of the proof, so that V is the dual of the irreducible representation of highest weight $-\chi$. But we can find such a λ only if $-\chi$ is dominant integral, as we required λ to be dominant integral at the start of the proof. If $-\chi$ is not dominant integral, then there is no such λ, and $((V_\lambda^* \otimes W)^U)^T$ is 0 for any dominant integral weight λ. Since $\operatorname{Ind}_B^G W$ is finite-dimensional, it can be decomposed as a direct sum of irreducible representations of G. Suppose there were some non-zero irreducible representation of G with highest weight λ in the decomposition. Then by mapping V_λ into itself (in the decomposition of $\operatorname{Ind}_B^G W$), Lemma 4.5 (Schur's lemma) says that $\operatorname{Hom}_G(V_\lambda^*, \operatorname{Ind}_B^G W)$ is non-trivial, which is a contradiction.

If $-\chi$ is dominant integral, we have computed above that $\operatorname{Hom}_G(V_\lambda^*, \operatorname{Ind}_B^G W)$ has dimension 1 if $\lambda = -\chi$ and 0 if $\lambda \neq -\chi$. Since $\operatorname{Ind}_B^G W$ is finite-dimensional, it can be decomposed as a direct sum of irreducible representations of G. We now apply Lemma 4.5 (Schur's lemma) to conclude that $\operatorname{Ind}_B^G W$ includes $V_{-\chi}^*$ in the decomposition into irreducible representations. Suppose this decomposition contains another irreducible representation of weight $\lambda \neq -\chi$. Then by mapping V_λ into itself (in the decomposition of $\operatorname{Ind}_B^G W$, Schur's lemma says that $\operatorname{Hom}_G(V_\lambda^*, \operatorname{Ind}_B^G W)$ is non-trivial, which is a contradiction. We conclude that $\operatorname{Ind}_B^G W$ is $V_{-\chi}^*$, the dual of the irreducible representation of G of highest weight $-\chi$, if $-\chi$ is a dominant integral weight. □

Example 5.5. Let G be $\mathrm{SL}(2, \mathbb{C})$. Our choice of maximal torus is the set T of 2×2 diagonal matrices of determinant 1, and our choice of Borel subalgebra B is the set of 2×2 upper triangular matrices of determinant 1. Then G/B is the flag variety $\mathbb{P}^1$, by Example 3.10. We compute all the line bundles over G/B.

First we compute what the representations of B are. Recall that T is the set of matrices of the form

$$\begin{bmatrix} z & 0 \\ 0 & z^{-1} \end{bmatrix}$$

where z is a non-zero complex number. By the mapping

$$\begin{bmatrix} z & 0 \\ 0 & z^{-1} \end{bmatrix} \in T \mapsto z \in \mathbb{C},$$

we see that T is isomorphic to $\mathbb{C}^\times$, the algebraic group of non-zero complex numbers. By Theorem 5.2, an irreducible representation is the same as specifying a morphism of algebraic groups $T \to \mathbb{C}^\times$, and hence the same as specifying a morphism of algebraic groups $\mathbb{C}^\times \to \mathbb{C}^\times$. But the only morphisms of algebraic groups $\mathbb{C}^\times \to \mathbb{C}^\times$ are the 'power to the n' maps. Explicitly, $z \mapsto z^n$, where n is a non-negative integer. Tracing back the mappings, this means that the irreducible representations of B are in one-one correspondence with non-negative integers. A non-negative integer n corresponds to the one-dimensional representation V_n with B-action

$$\begin{bmatrix} z & w \\ 0 & z^{-1} \end{bmatrix} .v = z^n v$$

for $v \in V_n$. Hence we have all the line bundles over G/B, namely $G \times_B V_n$ for each non-negative integer n.

Remark 5.6. The Borel–Weil Theorem says that there is a one-one correspondence between dominant integral weights and irreducible induced representations. By computing the global section for each line bundle $G \times_B V_n$, the induced representations are homogeneous polynomials of degree n in 2 variables. Hence these are all the irreducible representations of $\mathrm{SL}(2, \mathbb{C})$, which is indeed well-known to be true.

6. Plücker Embedding

The purpose of this section is to give us an idea of what the flag variety feels like, by embedding it in projective space. In this section we assume the following:

Let G be a semisimple algebraic group, with Lie algebra $\mathfrak{g}$. Pick a maximal torus T of G, with corresponding Lie algebra $\mathfrak{h}$. Pick a Borel subgroup B of G containing T, with corresponding Lie algebra $\mathfrak{b}$. Let λ be a dominant regular weight of $\mathfrak{g}$. Let V_λ be the irreducible finite dimensional representation of $\mathfrak{g}$ with highest weight λ and highest weight vector v_λ.

Theorem 6.1. *The map $i \colon G/B \to \mathbb{P}(V_\lambda)$ given by $i(gB) = [g.v_\lambda]$ is well-defined and injective.*

We call the map in Theorem 3.1 the Plücker embedding. The idea of the proof is to first work in Lie algebras, then use the normalizer theorem to show that the statement is true for algebraic groups.

Proof. I claim that $\mathfrak{b} = \mathrm{Stab}_{\mathfrak{g}}(\mathbb{C}v_\lambda)$. By the definition of highest weight representation, all elements of $\mathfrak{b}$ have eigenvector v_λ. Hence $\mathfrak{b} \subset \mathrm{Stab}_{\mathfrak{g}}(\mathbb{C}v_\lambda)$. Conversely, for any $x \notin \mathfrak{b}$, there exists $y \in \mathfrak{g}^+$ such that $[x, y] \in \mathfrak{h}$, where $\mathfrak{h}$ is the Cartan subalgebra. Suppose $x \in \mathrm{Stab}_{\mathfrak{g}}(\mathbb{C}v_\lambda)$. Then $[x, y].v_\lambda = x.(y.v_\lambda) - y.(x.v_\lambda) = 0$, so that $\lambda([x, y])v_\lambda = 0$. This means that $\lambda([x, y]) = 0$, contradicting the regularity of λ. Hence $\mathfrak{b} \supset \mathrm{Stab}_{\mathfrak{g}}(\mathbb{C}v_\lambda)$.

Now by the correspondence with algebraic groups, $B = \mathrm{Stab}_G(\mathbb{C}v_\lambda)$. We conclude that G acts on $\mathbb{P}(V_\lambda)$ with $\mathrm{Stab}_G([v_\lambda]) = B$. By the Orbit-Stabilizer Theorem, we get a bijection of sets $G/B \cong \mathrm{Orb}_G([v_\lambda])$. Hence this map is well-defined and injective.

□

The Plücker embedding can be generalized to the following theorem:

Theorem 6.2. *Suppose $w_1, w_2, ...w_n$ are the fundamental weights of $\mathfrak{g}$. Then the map*
$$i\colon G/B \to \mathbb{P}(V_{w_1}) \times \mathbb{P}(V_{w_2}) \times \cdots \times \mathbb{P}(V_{w_n})$$
given by $i(gB) = ([g.v_{w_1}], [g.v_{w_2}], \ldots, [g.v_{w_n}])$ is well-defined and injective.

Proof. Consider the map $f\colon G \to \mathbb{P}(V_{w_1}) \times \mathbb{P}(V_{w_2}) \times ... \times \mathbb{P}(V_{w_n})$ given by
$$f(g) = ([g.v_{w_1}], [g.v_{w_2}], ..., [g.v_{w_n}]).$$
We compute the stabilizer of the right hand side. This is equal to $\bigcap_{i=1}^n \mathrm{Stab}_G([v_{w_i}])$. We compute it similarly to the proof of the Plücker embedding, by first computing on the level of Lie algebras.

I claim that $\mathfrak{b} = \bigcap_{i=1}^n \mathrm{Stab}_{\mathfrak{g}}([v_{w_i}])$. Clearly $\mathfrak{b} \subset \bigcap_{i=1}^n \mathrm{Stab}_{\mathfrak{g}}([v_{w_i}])$. Conversely, for any $x \notin \mathfrak{b}$, there exists $y \in \mathfrak{g}^+$ such that $[x, y] \in \mathfrak{h}$, where $\mathfrak{h}$ is the Cartan subalgebra. Suppose $x \in \mathrm{Stab}_{\mathfrak{g}}(\mathbb{C}v_\lambda)$, then $[x, y].v_\lambda = x.(y.v_\lambda) - y.(x.v_\lambda) = 0$, so that $\lambda([x, y])v_\lambda = 0$. This means that $\lambda([x, y]) = 0$. Since this is true for any $i = 1, 2, ..., n$, we see that it contradicts the definition of fundamental weights, and hence the claim is true.

Therefore, f induces a well-defined injection i. $\square$

We end the paper with one last theorem. The Plücker embedding gives us an embedding of flag varieties in projective space. What does this say about vector bundles over flag varieties? Is there an interpretation of vector bundles over flag varieties in terms of vector bundles over projective space? Our last theorem does this for homogeneous line bundles. We let $\mathbb{C}_\lambda$ be the one-dimensional irreducible representation of B with a dominant regular highest weight λ. It's underlying vector space is just $\mathbb{C}$, and the action is described by $b.z = \lambda(b)z$ for any $b \in B, z \in \mathbb{C}$.

Theorem 6.3. *Let $i\colon G/B \to \mathbb{P}(\mathbb{C}_\lambda) \cong \mathbb{P}^1$ be the Plücker embedding, and $\mathscr{L}$ the tautological line bundle over $\mathbb{P}^1$. Then $G \times_B \mathbb{C}_\lambda = i^*\mathscr{L}$, the pullback of the tautological line bundle along the Plücker embedding.*

The tautological line bundle is defined in Example 3.4, in case the reader has forgotten the definition. We warn the reader that for $g \in G$, the expression $g.1$ means g acting on $1 \in \mathbb{C}_\lambda$. On the other hand, if $z \in \mathbb{C}$ is a complex number, then $z1$ means the scalar multiplication of z on $1 \in \mathbb{C}_\lambda$. The notation differs by a dot.

Proof. First observe that 1 is a highest weight vector of $\mathbb{C}_\lambda$. This follows directly from the definition. The pullback of the tautological line bundle is defined to be
$$i^*\mathscr{L} = \{(gB, (x, l)) \in G/B \times \mathscr{L} \mid [g.1] = l\}.$$
We show that this is just $G \times_B \mathbb{C}_\lambda$. This is because we have a correspondence:
$$(gB, (x, l)) \in i^*\mathscr{L} \longleftrightarrow [g, \frac{x}{g.1}] \in G \times_B \mathbb{C}_\lambda$$
where the expression
$$\frac{x}{g.1}$$
is the complex number z such that $x = z(g.1)$ in $\mathbb{C}_\lambda$. We treat this complex number z as an element of $\mathbb{C}_\lambda$. We make a remark on its existence. This complex number z must exist because both x and $g.1$ are elements of a one-dimensional subspace of $\mathbb{C}^2$,

by definition of the tautological line bundle. We now show that this correspondence is well-defined.

Since $(gbB, (x, l))$ corresponds to

$$[gb, \frac{x}{gb.1}] = [g, b.\frac{x}{gb.1}] \tag{6.4}$$
$$= [g, \lambda(b)\frac{x}{gb.1}]$$
$$= [g, \lambda(b)\frac{x}{\lambda(b)g.1}]$$
$$= [g, \frac{x}{g.1}],$$

hence the correspondence between $i^*\mathscr{L}$ and $G \times_B \mathbb{C}_\lambda$ is well-defined. Note that $b.\frac{x}{gb.1}$ in (6.4) is the B action on an element of $\mathbb{C}_\lambda$.

Therefore we have $i^*\mathscr{L} = G \times_B \mathbb{C}_\lambda$. □

Our main result is the Borel–Weil Theorem, which in particular can be applied to $\mathrm{SL}(2, \mathbb{C})$. The Plücker embedding also gives us a nice interpretation of flag varieties as well as line bundles over them. This gives us a very useful description of representations and relates line bundles over flag varieties with irreducible representations.

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