RANDOM WALKS AND THE UNIFORM MEASURE IN GROMOV-HYPERBOLIC GROUPS

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Abstract. The study of finite Markov chains that converge to the uniform distribution is well developed. The study of random walks on infinite groups, and specifically the question of whether they asymptotically approach some analog of the uniform distribution, is an area of active research. We consider random walks on Gromov-hyperbolic groups governed by symmetric, finitely-supported probability measures. S. Gouëzel, F. Mathéus, and F. Maucourant have recently shown that such random walks do not approach the uniform distribution if the groups are not virtually free. We introduce some of the most important concepts in this result.

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1. Introduction

The mixing time of random walks, particularly those that converge to the uniform distribution, is well studied in the literature. We can consider, for example, the simple random walk on the 3-dimensional cube, which converges to the uniform distribution[1] quite quickly. This cube can be thought of as the right Cayley graph of the dihedral group $D_8 = \langle s, r, r^{-1} \mid s^2 = r^4 = (rs)^2 = e\rangle$ with respect to the generating set $\{s, r, r^{-1}\}$. In this light, a random walk on the cube can be thought of as a random walk on the group, and the fact that a random walk approaches a uniform distribution can be seen as a result about properties of D_8 itself.

We wish to generalize this study - that is, the study of the relationship between random walks and the uniform distribution - to infinite, finitely-generated groups. Since there is no uniform distribution on infinite sets, we must come up with a

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[1]There are periodicity problems that we will look past. Formally, the 3-dimensional cube is a bipartite graph and the random walk has subsequences that converge to the uniform distributions on each of the partite sets.

new benchmark against which to measure the asymptotics of a random walk. The standard solution to this is to consider the sequence $\{\rho_n\}_{n\in\mathbb{N}}$ of uniform measures on balls of radius n about the identity with respect to a word metric. Although the theory of mixing time is sufficiently developed in simple cases such as D_8 to address issues like speed of convergence, the main question in the infinite case is still the following.

Question 1.1. Given a word metric on a group and an associated sequence $\{\rho_n\}_{n\in\mathbb{N}}$ of uniform measures, does there exist a "nice" random walk which "asymptotically approximates" $\{\rho_n\}_{n\in\mathbb{N}}$?

Part of this paper will be devoted to determining a convenient formal context in which to interpret the heuristic of "asymptotic approximation." Question 1.1 has recently been answered in the negative by S. Gouëzel, F. Mathéus, and F. Maucourant for random walks on hyperbolic groups which are not virtually free [5]. This paper attempts to serve as an introduction to the most important ideas involved in their answer.

Section 2 defines hyperbolic groups by a thin-triangles property in their Cayley graphs. Random walks on general groups are defined in Section 3. Section 4 considers the Gromov boundary of a hyperbolic group as a way of obtaining limits of sequences of measures. The fundamental inequality of Guivarc'h, an extremely useful tool in this theory, is introduced in Section 5 as a way of capturing the information from the boundary. Finally, Section 6 addresses the fundamental inequality in free groups and discusses the main article [5]. Sections 3 through 5 include versions of Question 1.1, each revised from the one preceding it to incorporate new ideas and terminology. Section 6 provides a partial negative answer to this question.

2. Hyperbolic Groups

Standing Assumptions and Notations: All groups are finitely generated and carry the discrete topology. All probability measures on groups are finitely supported and symmetric in the sense of assigning the same measure to an element and to its inverse. The letter G shall always refer to a group, and the letter μ to a probability measure on that group, subject to these conditions. The letter S shall always refer to a finite generating set of G. We will work with the word metric $d = d_S$ on G with respect to S, which will be defined below. We define $|g| = d_S(e, g)$ for any g in G. Expectation with respect to a measure ν will be written $\mathbb{E}_\nu$.

Definitions 2.1. These largely follow [2].

A map between metric spaces is called *1-Lipschitz* if it does not increase distance. A metric space (X, d) is called a *path metric space* if the distance between any two points p and q is the infimum over L of the set of 1-Lipschitz maps $\gamma : [0, L] \to X$ that map 0 to p and L to q. If at least one such map γ achieves this infimum for each pair of points p, q in X, then (X, d) is called a *geodesic metric space*. The image of this γ, as a set of points in X, is called a *geodesic* between p and q.

Assume in the following that (X, d) is a geodesic metric space.

A *geodesic triangle* with endpoints x, y, z in X is a subset of X consisting of a geodesic between each two-element subset of $\{x, y, z\}$. A geodesic triangle is called *δ-thin* for some $\delta \geq 0$ if each side of the triangle is contained in the neighborhood of radius δ of the union of the other two sides (see Figure 1).

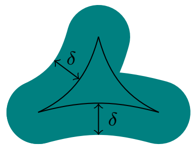

FIGURE 1. This triangle is δ-thin. This is derived from [2, Figure 2].

If each geodesic triangle in (X, d) is δ-thin for some uniform δ, then (X, d) is called *δ-hyperbolic*. A geodesic metric space (X, d) is called a *hyperbolic metric space* if it is δ-hyperbolic for some δ.

To make use of these definitions, we must first associate a metric space with a given group. The most natural choice for this space is the *(right) Cayley graph*. The right Cayley graph $C_S(G)$ of a group G with respect to a finite symmetric generating set S is an undirected graph whose vertices are the elements of G. For each g in G, and for all s in S, we form an edge between g and gs. The standard metric on $C_S(G)$ defines the distance between vertices g and h to be the length of the shortest path connecting them, where each edge is considered homeomorphic to the unit interval, and defines the distance between points on edges similarly. Since S generates G, $C_S(G)$ is connected as a graph, so this is well-defined. This metric on $C_S(G)$ lifts to a metric d on the group itself under identification of group elements with the vertices that represent them. Equivalently, if g and h are group elements, then $d(g, h)$ is the length of the shortest word in the alphabet S that evaluates to $h^{-1}g$. The metric so defined is called the *word metric* with respect to S. A metric on G is called a word metric if it is equal to the word metric with respect to some finite generating set.

Any Cayley graph is a geodesic metric space. Generally, the Cayley graphs of the same group with respect to different generating sets are not isometric (Figure 2). The notion of a quasi-isometry allows us to avoid this problem.

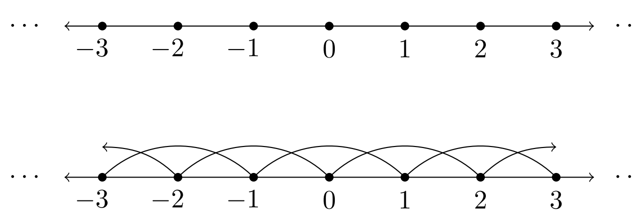

FIGURE 2. The Cayley graphs of $(\mathbb{Z}, +)$ with respect to the generating sets $\{\pm 1\}$ (top) and $\{\pm 1, \pm 2\}$. These metric spaces are not isometric, since the former is a tree and the latter is not. However, they are quasi-isometric.

Fix K and ϵ in $\mathbb{R}^+$. If (X, d_X) and (Y, d_Y) are metric spaces, we call $f : X \to Y$ a (K, ϵ)-*quasi-isometry* if the following two conditions hold:

(i) The map f does not distort distances too much. Formally, for all points x, x' in X, we have the following inequalities:

$$\frac{d_X(x, x')}{K} - \epsilon \leq d_Y(f(x), f(x')) \leq K d_X(x, x') + \epsilon.$$

(ii) Points in Y are a uniformly bounded distance away from $f(X)$. Formally, there exists a real C such that, for all y in Y, there exists x in X satisfying $d_Y(y, f(x)) \leq C$.

We call spaces *quasi-isometric* if there exists a quasi-isometry from one to the other. This is an equivalence relation on metric spaces. Condition (ii) is needed for symmetry. The following two facts are simple to check.

- If S and T are two finite generating sets of G, then $C_S(G)$ and $C_T(G)$ are quasi-isometric.
- If (X, d_X) and (Y, d_Y) are quasi-isometric and (X, d_X) is hyperbolic, then (Y, d_Y) is hyperbolic. Note that (X, d_X) and (Y, d_Y) need not be δ-hyperbolic for the same δ.

We may thus make the following definition.

Definition 2.2. A group G is called *(Gromov-)hyperbolic* if its Cayley graph with respect to some finite symmetric generating set is hyperbolic. This is true if and only if its Cayley graph is hyperbolic with respect to every finite symmetric generating set.

The following are standard examples of hyperbolic groups.

- So-called *virtually free groups*, or groups with a finite-index free subgroup.
- Finite groups.
- Fundamental groups of surfaces of genus g for $g \geq 2$.

On the other hand, for example, $\mathbb{Z}^d$ is non-hyperbolic for $d \geq 2$: If we denote the standard generating set for $\mathbb{Z}^d$ by $\{e_1^{\pm 1}, \ldots, e_d^{\pm 1}\}$, then there is no uniform bound on thinness for the family of geodesic triangles on the points $\{0, ne_1, ne_2\}$ as $n \to \infty$.

Remark 2.3. The class of hyperbolic groups is quite large. A. Y. Ol'shanskii proved M. Gromov's claim that almost every finitely presented group with large relations is hyperbolic, in the following sense [8].

Consider presentations of a group G using a generating set $S = \{g_1^{\pm 1}, \ldots, g_k^{\pm 1}\}$, for $k \geq 2$, subject to relations $r_1, \ldots, r_m$, where each r_i is a reduced word of n_i letters in S that is equal to the identity:

$$G = \langle g_1, \ldots, g_k \mid r_1, \ldots, r_m \rangle.$$

Fix k and m. If N is the number of such presentations (dependent on $k, n_1, \ldots, n_m$) and N_{hyp} is the number of such presentations for which the resulting group is hyperbolic, then

$$\lim_{n \to \infty} \frac{N_{\text{hyp}}}{N} = 1$$

where $n = \min(n_1, \ldots, n_m)$.

This remark emphasizes the wide applicability of results on hyperbolic groups.

3. Random Walks on Groups

Let G be a group and μ be a probability measure on G. Choose a sequence $\{X_i\}_{i \geq 1}$ of i.i.d. G-valued random variables according to μ. Given these, a *right random walk on G according to μ* is a sequence $\{Z_n\}_{n \geq 1}$ of G-valued random variables defined by the following equation, where $\cdot$ is group multiplication:

$$Z_n = X_1 \cdot \ldots \cdot X_n.$$

If μ and ν are measures on the same σ-algebra of measurable subsets of G, we define their *convolution*, written $\mu * \nu$, by the following formula, defined on any measurable subset $H \subset G$:

$$\mu * \nu(H) = \int_G \int_G \mathbf{1}_H(y \cdot z) d\mu(y) d\nu(z).$$

Here, $\mathbf{1}_H$ is the characteristic function on H. Convolution is an associative binary operation on probability measures. If μ and ν are probability measures that govern independent G-valued random variables X and Y, respectively, then $\mu * \nu$ is a probability measure that governs the G-valued random variable $X \cdot Y$. This tells us that the distribution of Z_n is the nth convolution of μ with itself, written μ^{*n}.

Let $\mathrm{supp}(\mu)$ denote the support of μ and H the subgroup of G that it generates. A right random walk on G according to μ can be viewed as a random walk on the right Cayley graph $C_{\mathrm{supp}(\mu)}(H)$ starting at the identity. This identification with the well-developed theory of random walks on graphs may aid the reader with visualization.

Example 3.1. If $G = \mathbb{Z}^d$ and μ is the uniform measure on the standard generating set, then the process here defined is just the simple random walk on the d-dimensional integer lattice.

Since the sequence $\{\mu^{*n}\}_{n \in \mathbb{N}}$ captures all the information of the random walk, we can now reframe Question 1.1 in terms of sequences of measures.

Question 3.2. Given a sequence $\{\rho_n\}_{n \in \mathbb{N}}$ of uniform measures, does there exist a symmetric, finitely supported probability measure μ such that the sequence $\{\mu^{*n}\}_{n \in \mathbb{N}}$ is a good "asymptotic approximation" for $\{\rho_n\}_{n \in \mathbb{N}}$?

There are several perfectly good ways to interpret "asymptotic approximation." We could, for example, choose some metric d on the space of probability measures on G and consider the limit $\lim_{n \to \infty} d(\mu^{*n}, \rho_n)$. However, several nice properties of hyperbolic groups allow us to situate them in larger spaces and then look at the limit points of each sequence of measures. This is the interpretation we wish to explore.

4. The Gromov Boundary

A random walk is called *recurrent* if the probability of ever returning to the starting point is one and *transient* otherwise. In our context, a random walk on a group is transient if $\mathbb{P}(Z_n = e \text{ for some } n)$ is strictly less than one.

Fact 4.1. If G is a nonamenable hyperbolic group, then a random walk on G is transient [7, p. 5].

Fact 4.2. If G is an amenable hyperbolic group, then G is finite or virtually $\mathbb{Z}$ [4, p. 244]. Such groups are called *elementary hyperbolic groups.*

Facts 4.1 and 4.2 suggest that we will not lose too much generality by assuming that our random walks escape and by asking therefore how they behave "at infinity." Boundary theory allows us to formalize this intuition.

Definition 4.3. Let (X, d_X) be a proper[2] path-connected hyperbolic metric space with fixed basepoint x_0. The *(Gromov) boundary* of X, written $\partial_\infty X$, is a compactification of X made by adding a point to X for each equivalence class of geodesic rays originating from x_0, where two rays are equivalent if they are a bounded Hausdorff distance apart. The topology on $\partial_\infty X$ is that of convergence on compact subsets of equivalence classes. Different basepoints give rise to homeomorphic spaces, so we can define a basepoint-invariant boundary for any such space.

In fact, two quasi-isometric hyperbolic spaces have homeomorphic Gromov boundaries. Since Cayley graphs are proper and path-connected, this allows us to define the boundary of a hyperbolic group G to be the boundary of any of its Cayley graphs [2].

Definition 4.4. Let G be a hyperbolic group and μ a probability measure on $G \cup \partial_\infty G$ that is supported on G and that governs a transient random walk. The associated *exit measure* μ_∞ on $G \cup \partial_\infty G$, supported on $\partial_\infty G$, is the limit point of the sequence of measures $\{\mu^{*n}\}_{n\in\mathbb{N}}$. Existence and uniqueness of such a limit point are nontrivial. Intuitively, the exit measure captures, for any measurable subset of the boundary, the probability that a random walk will escape along a geodesic ray whose equivalence class belongs to that subset.

The results of [1] allow us to consider measures up to the following equivalence relation: If ν_1 and ν_2 are measures on the same σ-algebra of measurable subsets of a space X, then we say that ν_1 and ν_2 are equivalent if they agree on whether each set has positive measure or zero measure.

Definition 4.5. Let G be a hyperbolic group. There exists a measure ρ_∞ on the boundary, called the *Patterson-Sullivan measure*, that is equivalent to the limit points of the sequence $\{\rho_n\}_{n\in\mathbb{N}}$ of uniform measures [5, p. 2]. The construction of a Patterson-Sullivan measure in our spaces is due to M. Coornaert [3].

We can now interpret the "asymptotic approximation" of Question 3.2 by comparing just two measures.

Question 4.6. Let G be a non-elementary hyperbolic group. Does there exist a symmetric, finitely supported probability measure μ on G such that the exit measure μ_∞ is equivalent to the Patterson-Sullivan measure ρ_∞?

5. The Fundamental Inequality

In many cases, the information contained in the exit and Patterson-Sullivan measures can be captured in a single inequality. Comparison of measures then reduces to determining whether equality holds. This inequality relates three real numbers associated with our stochastic process: the exponential growth rate, the drift, and the entropy.

[2]A metric space is called proper if closed balls are compact. Since every hyperbolic group in this paper is finitely generated, closed balls in our Cayley graphs are finite, so our hyperbolic groups are proper.

5.1. **Exponential Growth Rate.** Let B_n denote the open ball of radius n about the identity according to the word metric and let $\overline{B_n}$ denote its closure. For any n and m in $\mathbb{N}$, observe that every reduced word of $n+m$ letters over the alphabet S can be written as the group product of a reduced word of n letters and a reduced word of m letters. This yields the inequality $|B_{n+m}| \leq |B_n| \cdot |B_m|$, which is equivalent to $\log |B_{n+m}| \leq \log |B_n| + \log |B_m|$. The sequence $\{\log |B_n|\}_{n\in\mathbb{N}}$ is therefore subadditive, so Fekete's lemma tells us that the following limit exists:

$$v = \lim_{n\to\infty} \frac{\log |B_n|}{n}.$$

This limit is called the *exponential growth rate* of G. In fact, there are several equivalent definitions:

$$v = \lim_{n\to\infty} \frac{\log \left|\overline{B_n}\right|}{n} = \lim_{n\to\infty} \frac{\log \left|\overline{B_n} \setminus B_n\right|}{n}. \tag{5.1}$$

5.2. **Drift.** This existence proof follows [9].

We will denote the σ-algebra of μ-measurable subsets of G by $\mathscr{B}$. Let G^∞ be the countably infinite Cartesian product of G with itself. We equip G^∞ with a σ-algebra $\mathscr{B}^\infty$ generated by so-called *cylinders*, or sets of the form $\prod_{i=1}^\infty U_i$ where $U_i \in \mathscr{B}$ for all integers i and $U_i = G$ for all but finitely many integers i. We define a probability measure μ^∞ on cylinders by the following equation:

$$\mu^\infty \left(\prod_{i=1}^\infty U_i \right) = \prod_{i=1}^\infty \mu(U_i).$$

Since all but finitely many U_i have measure one, there is no concern about convergence here. The extension of μ^∞ from a measure on the generating set of $\mathscr{B}^\infty$ to a measure on all of $\mathscr{B}^\infty$ is standard.

Given an integer n and an infinite sequence $\omega = (s_1, s_2, \ldots)$ of letters of S, we define a family $\{L_n(\omega)\}_{n\in\mathbb{N}}$ of integer-valued random variables by right-multiplying the first n letters of ω in G and taking absolute values:

$$L_n(\omega) = |s_1 \cdot \ldots \cdot s_n|.$$

We will apply Kingman's subadditive ergodic theorem to this sequence, using the left-shift operator $T : G^\infty \to G^\infty$ defined by the following equation:

$$T(\omega) = T((s_1, s_2, s_3, \ldots)) = (s_2, s_3, \ldots).$$

Since T is measure-preserving on cylinders, which generate $\mathscr{B}^\infty$, it is measure-preserving on G^∞. Since G acts by isometries on itself, the triangle inequality gives us the following inequality for all n, m in $\mathbb{N}$:

$$L_{n+m}(\omega) \leq L_n(\omega) + L_m(T^n(\omega)).$$

Hence the Kingman subadditive ergodic theorem lets us conclude that the limit

$$\ell(\omega) = \lim_{n\to\infty} \frac{L_n(\omega)}{n}$$

exists almost surely. Since $\ell(\omega)$ is invariant under any change in finitely many elements of ω, Kolmogorov's zero-one law tells us that T is ergodic, so that $\ell(\omega)$ is constant in ω. This limit, which we shall simply denote ℓ, is called the *drift* of the random walk and, intuitively, captures the linear rate of escape of a random walk from the origin.

We record the following proposition, an immediate consequence of the above, for later use [9, Lemma 1].

Proposition 5.2. *For any $\epsilon > 0$, there exists an integer N such that, for all $n \geq N$, we have the following bound for μ^∞-almost-every ω:*

$$\left| \frac{L_n(\omega)}{n} - \ell \right| < \epsilon.$$

We include an equivalent definition of drift [9, Section 2.3]:

$$\ell = \lim_{n\to\infty} \frac{\mathbb{E}_{\mu^{*n}}|g|}{n}. \tag{5.3}$$

5.3. **Entropy.** For n in $\mathbb{N}$, we consider the expectation

$$H(\mu^{*n}) = \sum_{g\in G} \mu^{*n}(g)(-\log(\mu^{*n}(g)))$$

with the convention $0 \cdot \log(0) = 0$. The quantity $H(\mu)$ is called the *time-one entropy* of μ. This gives us a real subadditive sequence $\{H(\mu^{*n})\}_{n\in\mathbb{N}}$. Fekete's lemma then tells us that the following limit exists and is equal to the infimum:

$$h = \lim_{n\to\infty} \frac{H(\mu^{*n})}{n} = \inf_n \frac{H(\mu^{*n})}{n}.$$

This is called the *(asymptotic) entropy* of the measure μ. Intuitively, the random walk is supported by roughly e^{hn} points for large time n.

The following lemma will be useful for proving the fundamental inequality.

Lemma 5.4. *The uniform probability measure on any finite subset $F \subset G$ has time-one entropy $\log|F|$ and maximizes time-one entropy among measures supported on F.*

Proof. Let ν be the uniform probability measure on F. The first statement follows from the following chain of equalities:

$$H(\nu) = \sum_{g\in F} \nu(g)(-\log(\nu(g))) = \sum_{g\in F} \frac{1}{|F|}\left(-\log\left(\frac{1}{|F|}\right)\right) = -\log\left(\frac{1}{|F|}\right) = \log|F|.$$

Consider the function $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = x(-\log(x)) + (1-x)(-\log(1-x)).$$

The function f is the time-one entropy of a measure that assigns measure x to one point and measure $1-x$ to another point, and this function takes its maximum at $x = \frac{1}{2}$. This proves the second statement if $|F| = 2$; the general case follows from induction on the cardinality of F. □

5.4. **Fundamental Inequality.** The following relationship between these three quantities, first established in [6], is so central that many authors call it the "fundamental inequality" [5, 7, 9].

Theorem 5.5.

$$h \leq \ell \cdot v. \tag{5.6}$$

Proof. This follows [9].

Fix a small $\epsilon > 0$, and choose n in $\mathbb{N}$ according to Proposition 5.2 such that the set

$$V = \{g \in G : |g| \in [(1-\epsilon)\ell \cdot n, (1+\epsilon)\ell \cdot n]\}$$

satisfies $\mu^{*n}(V) > 1-\epsilon$. We restrict μ^{*n} to V and normalize to obtain a probability measure μ_1 on V. Similarly, we restrict μ^{*n} to $V^c = \overline{B_n} \setminus V$ and normalize to obtain a probability measure μ_2 on V^c. That is, we can write

$$\mu^{*n} = (1-k)\mu_1 + k\mu_2$$

for a real constant $0 \leq k \leq \epsilon$. Time-one entropy is additive on mutually singular measures; since μ_1 and μ_2 are supported on disjoint sets, they are certainly mutually singular. Furthermore, if α is real and μ is a probability measure, then $H(\alpha\mu) = \alpha H(\mu) - \alpha \log \alpha$. This gives us the following:

$$\begin{aligned} H(\mu^{*n}) &= (1-k)H(\mu_1) - (1-k)\log(1-k) + kH(\mu_2) - k\log(k) \\ &\leq (1-k)\log(|V|) - (1-k)\log(1-k) + k\log(|\overline{B_n}|) - k\log(k). \end{aligned}$$

The inequality follows from Lemma 5.4. We divide through by n and pass to the limit to obtain the following bound:

$$h \leq (1-k)\left(\lim_{n\to\infty} \frac{\log|V|}{n}\right) + k\left(\lim_{n\to\infty} \frac{\log|\overline{B_n}|}{n}\right).$$

Equation (5.1) tells us that the first limit is $\ell \cdot v$ (as $\epsilon \to 0$) and that the second is v. Since $k \to 0$ as $\epsilon \to 0$, we obtain the desired result. □

We required that our hyperbolic groups be non-elementary in order to pass to boundary theory. If, in addition, we add some mild conditions to the measure μ, then the following proposition, which is a strictly weaker reformulation of [5, Theorem 1.2], tells us that we can pass to the fundamental inequality instead.

Proposition 5.7. *If G is a nonamenable hyperbolic group and μ is a probability measure on G whose finite support generates G, then the following are equivalent:*

- *The fundamental inequality is an equality: $h = \ell \cdot v$.*
- *The exit measure μ_∞ and the Patterson-Sullivan measure ρ_∞ are equivalent.*

Hence we can rephrase Question 4.6 as follows.

Question 5.8. Does there exist a symmetric, finitely supported probability measure μ such that the fundamental inequality (5.6) is an equality?

6. Strictness in the Fundamental Inequality

We begin by discussing the fundamental inequality (5.6) for free groups on multiple generators.

Proposition 6.1. *If $G = \mathbb{F}_d = \langle g_1^{\pm 1}, \ldots, g_d^{\pm 1} \mid \rangle$ is the free group on $d \geq 2$ letters and μ is the uniform measure on $S = \{g_1^{\pm 1}, \ldots, g_d^{\pm d}\}$, then*

$$\begin{aligned} v &= \log(2d-1), \\ \ell &= \frac{d-1}{d}, \\ h &= \frac{(d-1)\log(2d-1)}{d}, \end{aligned}$$

and equality holds in Equation 5.6.

Proof. Since $\mathbb{F}_d$ is nonamenable when $d \geq 2$, Fact 4.1 tells us that a random walk on G according to μ is transient.

- **Exponential Growth Rate:** For $n > 1$, there are $(2d)(2d-1)^{n-1}$ reduced words of length n over the alphabet S. Since G is free, this is precisely $|B_n|$. Thus we have
$$v = \lim_{n\to\infty} \frac{\log |B_n|}{n} = \lim_{n\to\infty} \frac{\log((2d)(2d-1)^{n-1})}{n} = \log(2d-1).$$
- **Drift:** We are interested in the expected length of a random word at time n, or the expected distance of the random walk from the identity at time n. If $Z_n = e$, then $|Z_{n+1}| = 1$ with probability one, in the notation of Section 3. If $Z_n \neq e$, then the random walk moves $\frac{2d-1}{2d} - \frac{1}{2d} = \frac{d-1}{d}$ steps away from the origin at time $n+1$, on average. This gives us the following equation for all n in $\mathbb{N}$:
$$\mathbb{E}_{\mu^{*(n+1)}}|g| = \mathbb{E}_{\mu^{*n}}|g| + \left[\mathbb{P}(Z_n = e) + \mathbb{P}(Z_n \neq e)\left(\frac{d-1}{d}\right)\right].$$
Since the random walk on G is transient, $\mathbb{P}(Z_n = e) \to 0$ as $n \to \infty$. Since $\mathbb{E}_\mu(g) = 1$, we can compute the drift according to Equation 5.3:
$$\ell = \lim_{n\to\infty} \frac{\mathbb{E}_{\mu^{*n}}(g)}{n} = \frac{d-1}{d}.$$
- **Entropy:** Since this calculation involves several theories not otherwise needed in this paper, we will only present a short proof sketch. We follow the rigorous computation at [7, pp. 13-14]. Our notion of a random walk on a group can be thought of as a random walk starting at the identity. A random walk starting at an arbitrary group element g can simply be defined by the sequence $\{gZ_n\}_{n\in\mathbb{N}}$ of G-valued random variables. Since the random walk on G is transient, the function $\mathscr{G} : G \times G \to \mathbb{R}$ given by the following equation is well-defined:
$$\mathscr{G}(x, y) = \sum_{n\in\mathbb{N}} \mathbb{P}(\text{random walk starting at } x \text{ hits } y \text{ at time } n).$$
This is called the *Green function*,[3] and represents the expected number of visits to a group element y of a random walk according to μ starting at x. The function $d_{\mathscr{G}} : G \times G \to \mathbb{R}$ defined by
$$d_{\mathscr{G}}(x, y) = \log(\mathscr{G}(e, e)) - \log(\mathscr{G}(x, y))$$
is a metric on G, called the *Green metric*, that satisfies the following equation in this free case:
$$d_{\mathscr{G}}(x, y) = d(x, y) \log(2d - 1).$$

[3] Given x, y in G, some authors define the Green function as a complex power series:
$$\mathscr{G}_{x,y}(z) = \sum_{n\in\mathbb{N}} \mathbb{P}(\text{random walk starting at } x \text{ hits } y \text{ at time } n) \cdot z^n.$$
Our interpretation corresponds to $\mathscr{G}_{x,y}(1)$ in the notation of these authors.

The argument above tells us that $\ell_{\mathscr{G}}$, the drift with respect to the Green metric, satisfies

$$\ell_{\mathscr{G}} = \frac{(d-1)\log(2d-1)}{d}.$$

After checking that G has finite entropy, we may apply a theorem of Blachère, Haïssinsky, and Mathieu to conclude that $h = \ell_{\mathscr{G}}$.

□

A remarkable recent paper says that, among hyperbolic groups, this is virtually the only case where equality holds.

Theorem 6.2. [5, Theorem 1.3].
If G is a hyperbolic group which is not virtually free, and $F \subset G$ is finite, then there exists $c < 1$ such that

$$h \leq c \cdot \ell \cdot v$$

for any symmetric probability measure μ supported in F.

The uniform bound makes this a fairly strong negative answer to our question. We finish with three remarks on the conditions of the theorem, following the authors of the article in question, who spend some time thinking about the extendability of their approach.

Remark 6.3. Unlike Proposition 5.7, this theorem does not require that the support of μ generate G. This means that the inequality is strict even for random walks that may not witness the full geometry of G.

Remark 6.4. On the other hand, our assumption that μ be finitely supported appears to be necessary. Theorem 1.4 of [5] states that

$$\lim_{n\to\infty} \frac{h(\rho_n)}{\ell(\rho_n)} = v.$$

That is, the fundamental inequality for the uniform measure on balls approaches equality as the balls grow.

Remark 6.5. The authors of [5] have not determined whether the assumption that μ is symmetric is necessary. To simplify this paper, we have assumed it throughout, even if some of our statements hold true in the non-symmetric setting. Certainly, there must be some finer delineation; they prove that any group with infinitely many ends and no element of finite order admits a non-symmetric probability measure μ with respect to which $h = \ell \cdot v$, but the measure has exponential moment and its support does not generate the group.

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Annotations

The main focus of this paper, the recent article of **Gouëzel, Mathéus, and Maucourant**, provides clear definitions and helpful intuition but is probably best for readers already familiar with the main ideas [5]. The monograph of **Calegari** is an excellent, careful introduction to hyperbolic groups and includes a short treatment of random walks [2]. The best English-language introduction to random walks on hyperbolic groups that this author knows of is **Vershik** [9]. **Haïssinsky** provides a terrific French-language introduction to the same [7].

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