HOPF BIFURCATION IN A LOW-DIMENSION SUBCRITICAL INSTABILITY MODEL

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Abstract. We discuss center manifold theory and normal form theory and their applications to bifurcation problems that arise in the study of dynamical systems. We use the developed theory to show that a particular four-dimensional shear flow model parameterized by the Reynolds number undergoes a supercritical Hopf bifurcation at a critical parameter value, giving rise to subcritical instability of the flow.

Contents

1. Introduction

In many shear flows, one can detect fully developed turbulence for sufficiently high Reynolds numbers[1] R despite the fact that the laminar state is linearly stable to small perturbations. Such observations lead to the idea that, for a given shear flow, there exists a critical value δ such that the laminar state is asymptotically stable only if the energy of the perturbation is lower than δ. Quite reasonably, δ is seen to be a function of R, but little is known about the relationship between δ and R other than the fact that for R less than some (undetermined) constant R_G, δ is infinite i.e. the laminar state is globally stable. However, with simplified,

[1] The Reynolds number is a constant (that arises in fluid dynamics) associated with different fluid flows. Roughly speaking, high Reynolds numbers indicate turbulent flows and low Reynolds numbers indicate laminar flows.

lower dimensional models, which still capture many essential features of these so-called subcritical instabilities (or subcritical transitions)[2], the task of finding δ as a function of R is much more reasonable (Cossu [3]).

In this paper, we analyze a particular four-dimensional model that arises in the study of these subcritical transitions to show that a particular "upper-branch" steady state undergoes a supercritical Hopf bifurcation at some $R = R_C$. Prior to the analysis of the model, we introduce the model in Section 2, and in Sections 3.1 and 3.2, we develop Center Manifold Theory, which will allow us to reduce the dimension of our model, so that we can study a simpler model. In Section 3.3, we develop Normal Form Theory specifically applied to Hopf bifurcations as a means to analytically derive a criterion for discerning the occurence of a Hopf bifurcation. As we will see, the criterion involves the eigenvalues of the linearization of the model at the equilibrium point, at the critical parameter value. Furthermore, it involves a parameter-dependent function (called the first Lyapunov coefficient) whose value at the critical parameter value partially determines the type of Hopf bifurcation that occurs.

We begin the analysis of the model in Section 4.1 by performing some transformations to write the model in a form conducive to the application of center manifold theory. In Sections 4.2 and 4.3, we approximate the center manifold up to cubic terms and reduce our model to two dimensions. In Section 4.4, we go through the transformations necessary to get our now two-dimensional system into a form where we can check the two-condition criteria derived via Normal Form Theory in Section 3.3. Finally, in Section 4.5, we make explicit computations to verify the two conditions and conclude that our model undergoes a supercritical Hopf bifurcation at a critical Reynolds number R_C.

2. The Model

The model to be studied is four dimensional with variables x_1, x_2, x_3, and x_4 - all functions of time. The variable x_3 corresponds to the amplitude of streamwise vortices; x_2 corresponds to the amplitude of streamwise streaks; x_4 corresponds to the amplitude of sinuous perturbations of the streaks; and x_1 corresponds to the amplitude of the mean shear induced by these perturbations. Having $x_i = 0$ $(i = 1, 2, 3, 4)$ corresponds to a laminar flow. Otherwise, we have a turbulent flow (at least instantaneously, so long as the values of x_i stay non-zero). Explicitly, the model we work with is given by

$$\dot{\mathbf{x}} = L_R\mathbf{x} + N(\mathbf{x}) \tag{2.1}$$

where $\mathbf{x} = (x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4$, and

$$L_R = \begin{bmatrix} -k_1^2/R & 0 & 0 & 0 \\ 0 & -k_2^2/R & \sigma_2 & 0 \\ 0 & 0 & -k_3^2/R & 0 \\ 0 & 0 & 0 & -k_4^2/R - \sigma_1 \end{bmatrix}$$

$$N(\mathbf{x}) = \begin{bmatrix} \sigma_1 x_4^2 - \sigma_2 x_2 x_3 \\ -\sigma_4 x_4^2 + \sigma_2 x_1 x_2 \\ \sigma_3 x_4^2 \\ (\sigma_4 x_2 - \sigma_1 x_1 - \sigma_3 x_3) x_4 \end{bmatrix}. \tag{2.2}$$

[2]Here, the term *subcritical* does not necessarily relate to the same *critical* parameter value as the term *subcritical/supercritical* (Hopf) bifurcation does. Subcritical instability is a conventional term in fluid dynamics used to refer to a particular kind of transition to turbulence from the laminar state.

Specifically, for model W97, which we are working with, we use the following parameter values[3]:

$$[k_1, k_2, k_3, k_4] = [1.57, 2.28, 2.77, 2.67]$$
$$[\sigma_1, \sigma_2, \sigma_3, \sigma_4] = [0.31, 1.29, 0.22, 0.68].$$

It is clear that for all $R > 0$, the origin (i.e. the laminar flow) is a steady state, and from the upper-triangular form of the matrix L_R, it is also clear that it is linearly stable to small perturbations (i.e. locally attracting). Furthermore, solving for zeros of (2.2) reveals that at $R \approx 106.14 =: R_S$, there is a saddle-node bifurcation such that for $R > R_S$, there are two additional branches of steady states, namely the upper and lower branches ("upper" and "lower" corresponding to the steady states of larger and smaller norm, respectively). The parameter value of interest is $R_C \approx 139.74$, where the linearization of the model centered at the upper branch steady state has a pair of purely imaginary eigenvalues and a conjugate pair of eigenvalues with negative real part.

Remark 2.3. Explicit computation of zeros using a program will reveal that for $R > R_S$ there are actually four non-zero steady state solutions to (2.1). However, one can verify that the system has a symmetry with respect to reflection over the x_4-axis, and hence the four non-zero steady states actually come in pairs, each member of a pair being identical to the other except with the opposite sign for the x_4 coordinate. There is no loss of generality in restricting ourselves to the invariant subspace defined by $x_4 > 0$ and just talking about two out of the four non-zero steady states.

For more information on and further analysis of the model, see Cossu [3].

3. Preliminaries

It has already been shown numerically that at $R = R_C$, a supercritical Hopf bifurcation takes place for the upper branch steady state (Lebovitz [5]). We aim to show analytically what has been shown numerically. To do so, we employ Center Manifold Theory and Normal Form Theory, which we discuss below.

3.1. Center Manifold Theory.

The theory we develop below will allow us to look at our model restricted to an invariant manifold (the center manifold) and allow us to study the resulting lower dimensional model. It is not obvious that studying the restricted system should tell us anything about the original model, but as we will see, center manifold theory tells us that, in fact, under certain conditions the dynamics of a system restricted to its center manifold are locally equivalent (in a sense to be described in more detail later) to the dyanamics of the original system.

To begin our treatment of the theory, suppose we have a system

$$\dot{\mathbf{w}} = f(\mathbf{w}) \tag{3.1}$$

such that $\mathbf{w} \in \mathbb{R}^n$ and f is C^r. Furthermore, suppose that $f(0) = 0$ and $Df(0)$ has n_s eigenvalues (s for stable) with negative real part and $n_c = n - n_s$ (c for center) eigenvalues with zero real part. Observe that we can, through a change of basis, rewrite (3.1) as

$$\begin{aligned} \dot{\mathbf{x}} &= A\mathbf{x} + f_c(\mathbf{x}, \mathbf{y}) \\ \dot{\mathbf{y}} &= B\mathbf{y} + f_s(\mathbf{x}, \mathbf{y}) \end{aligned} \tag{3.2}$$

[3] W97 refers to a shear flow model proposed by Fabian Waleffe in one of his papers from 1997 (Waleffe [7]). There is also a related, earlier model, W95, proposed by Waleffe in 1995 (Waleffe [6]). Both models are discussed in Cossu [3].

where $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{n_c} \times \mathbb{R}^{n_s}$, A has eigenvalues with zero real part, B has eigenvalues with negative real part, and f_c and f_s are C^r and have no linear terms. Having thus written our system in the form (3.2), we can state the following definition:

Definition 3.3. *An invariant manifold is called a center manifold for (3.2) if it can locally be represented as follows*

$$W^c(0) = \{(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^{n_c} \times \mathbb{R}^{n_s} \mid \boldsymbol{y} = h(\boldsymbol{x}), |\boldsymbol{x}| < \delta, h(0) = 0, Dh(0) = 0\} \tag{3.4}$$

for sufficiently small $\delta > 0$.

Observe that the conditions $h(0) = 0$ and $Dh(0) = 0$ imply that $W^c(0)$ is tangent, at the origin, to E_c, the generalized eigenspace corresponding to the eigenvalues of (3.2) with zero real part.

Now in the theorem below, we get the existence of a center manifold and the property of center manifolds that allows us to reduce the dimension of (3.1) in studying its dynamics. Note that the theorem gives us existence but not uniqueness.

Theorem 3.5. *There exists a* C^r *center manifold for (3.2). Furthermore, system (3.1) is topologically equivalent near the origin to the system*

$$\begin{aligned} \dot{\boldsymbol{x}} &= A\boldsymbol{x} + f_c(\boldsymbol{x}, h(\boldsymbol{x})) \\ \dot{\boldsymbol{y}} &= B\boldsymbol{y} \end{aligned} \tag{3.6}$$

where A, B *and* f_c *are as in (3.2) and* h *comes from Definition 3.3.*

The first part of system (3.6)

$$\dot{\mathbf{x}} = A\mathbf{x} + f_c(\mathbf{x}, h(\mathbf{x})), \ \mathbf{x} \in \mathbb{R}^{n_c} \tag{3.7}$$

is called the *restriction* of (3.2) to its center manifold.

The theorem tells us that for some neighborhoods U and V of the origin (in $\mathbb{R}^{n_c} \times \mathbb{R}^{n_s}$), there exists a homeomorphism from U to V such that orbits of (3.2) contained in U are mapped to orbits of (3.6) contained in V in such a way that preserves the direction of time. In other words, the theorem says that the orbital structures of (3.2) and (3.6) are equivalent near the origin up to some "bending and pulling." Particularly, equilibrium points and small amplitude periodic orbits of (3.2) sufficiently close to the origin exist also for system (3.6), and vice-versa. Notice, furthermore, that for system (3.6), all dynamical behaviors along the $\mathbf{y}$ coordinates decay exponentially as $t \to \infty$ so orbits of (3.6) asymptotically approach orbits of (3.7) (or, to be precise, orbits of (3.7) adjoined with $\mathbf{y} = 0$). So, Theorem 3.5 allows us, when studying a system like (3.2), to focus instead on the lower dimensional system (3.7) to learn about the local behavior near an equilibrium point.[4]

Now the problem boils down to actually computing the center manifold or, equivalently, computing the function h as given in Definition 3.3. We do so by making the following observations.

[4] Perhaps Theorem 3.5 makes it clear why, up to this point, we have assumed that (3.1) has no eigenvalues with positive real part: having eigenvalues with positive real part would still give us a theorem like Theorem 3.5 except we would have to state that the matrix B has eigenvalues of both negative and positive real part, so dynamics along the $\mathbf{y}$ coordinates would no longer be decaying and hence the center manifold would no longer be attracting. Then the center manifold loses much of its inherent value in that we cannot simply focus on (3.7) to study the local behavior of (3.2).

1) For any $(\mathbf{x}, \mathbf{y}) \in W^c(0)$, we have

$$\mathbf{y} = h(\mathbf{x}). \tag{3.8}$$

2) Differentiating (3.8) with respect to time, we find that on $W^c(0)$ we must have

$$\dot{\mathbf{y}} = Dh(\mathbf{x})\dot{\mathbf{x}}. \tag{3.9}$$

Recall from Definition 3.3 that the center manifold is invariant; therefore, we can look at the restriction of (3.2) to it by making the substitution $\mathbf{y} = h(\mathbf{x})$:

$$\begin{aligned} \dot{\mathbf{x}} &= A\mathbf{x} + f_c(\mathbf{x}, h(\mathbf{x})) \\ \dot{\mathbf{y}} &= Bh(\mathbf{x}) + f_s(\mathbf{x}, h(\mathbf{x})), \ (\mathbf{x}, \mathbf{y}) \in W^c(0). \end{aligned} \tag{3.10}$$

3) Substituting (3.10) into (3.9) we get

$$Dh(\mathbf{x})[A\mathbf{x} + f_c(\mathbf{x}, h(\mathbf{x}))] = Bh(\mathbf{x}) + f_s(\mathbf{x}, h(\mathbf{x})) \tag{3.11}$$

or equivalently,

$$\mathcal{N}(h(\mathbf{x})) := Dh(\mathbf{x})[A\mathbf{x} + f_c(\mathbf{x}, h(\mathbf{x}))] - Bh(\mathbf{x}) + f_s(\mathbf{x}, h(\mathbf{x})) = 0. \tag{3.12}$$

Therefore, solving the differential equation (3.12) would give us the desired function h; however, in most cases the equation is too complex to feasibly solve. So, we have the following theorem, which allows us to approximate h up to any desired degree of accuracy.

Theorem 3.13. *Let $\phi : \mathbb{R}^{n_c} \to \mathbb{R}^{n_s}$ be a C^1 mapping with $\phi(0) = D\phi(0) = 0$ such that $\mathcal{N}(\phi(\boldsymbol{x})) = \mathcal{O}(|\boldsymbol{x}|^q)$ as $|\boldsymbol{x}| \to 0$ for some $q > 1$. Then $|h(\boldsymbol{x}) - \phi(\boldsymbol{x})| = \mathcal{O}(|\boldsymbol{x}|^q)$ as $|\boldsymbol{x}| \to 0$.*

At this point, we make the observation that system (3.1) does not depend on a parameter, whereas system (2.1) does. Below, we discuss how we can use Theorem 3.5 and Theorem 3.13 to analyze a parameter dependent system.

3.2. **Parameter-Dependent Center Manifold.** Suppose we have a parameter-dependent system

$$\dot{\mathbf{w}} = f(\mathbf{w}, \mu), \mathbf{w} \in \mathbb{R}^n, \mu \in \mathbb{R} \tag{3.14}$$

where $f(0, 0) = 0$ and $Df_{\mathbf{w}}(0, 0)$ has n_c eigenvalues with zero real part and $n_s = n - n_c$ eigenvalues with negative real part.

For $\mu = 0$, the center manifold theory from above gives us a center manifold (after rewriting (3.14) in its eigenbasis)

$$W_0^c(0) = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{n_c} \times \mathbb{R}^{n_s} \mid \mathbf{y} = h(\mathbf{x}), |\mathbf{x}| < \delta, h(0) = 0, Dh(0) = 0\} \tag{3.15}$$

for $\delta > 0$ sufficiently small. However, we would like to study the system as we vary μ near 0, so we treat μ as a variable (that has no dependence on time) and hence study the $(n+1)$-dimensional system

$$\begin{aligned} \dot{\mathbf{w}} &= f(\mathbf{w}, \mu) \\ \dot{\mu} &= 0, \ \mathbf{w} \in \mathbb{R}^n, \ \mu \in \mathbb{R}. \end{aligned} \tag{3.16}$$

It is easy to see that system (3.16) has a $(n_c + 1)$-dimensional center manifold. Through a change of basis, we can change system (3.16) into the form[5]

$$\begin{aligned} \dot{\mathbf{x}} &= A\mathbf{x} + f_c(\mathbf{x}, \mathbf{y}, \mu) \\ \dot{\mu} &= 0 \\ \dot{\mathbf{y}} &= B\mathbf{y} + f_s(\mathbf{x}, \mathbf{y}, \mu), \;\; \mathbf{x} \in \mathbb{R}^{n_c}, \;\; \mathbf{y} \in \mathbb{R}^{n_s}, \;\; \mu \in \mathbb{R} \end{aligned} \tag{3.17}$$

where A has eigenvalues with zero real part and B has eigenvalues with negative real part, and f_c and f_s are nonlinear with respect to $\mathbf{x}, \mathbf{y}, \mu$. Note that since μ is now considered a variable, A and B can be made independent of μ.

Having gone through these coordinate changes, the parameter-dependent center manifold of (3.17) is given by

$$W^c_\mu(0) = \{(\mathbf{x}, \mathbf{y}, \mu) \in \mathbb{R}^{n_c} \times \mathbb{R}^{n_s} \times \mathbb{R} \mid \mathbf{y} = h(\mathbf{x}, \mu), |\mathbf{x}| < \delta, |\mu| < \delta^*, h(0,0) = 0, Dh(0,0) = 0\}$$

for $\delta, \delta^* > 0$ sufficiently small.

Now, all the theorems from the parameter-free center manifold theory can easily be extended for parameter-dependent center manifolds by viewing μ as an extra variable: system (3.17) is topologically equivalent, near the origin, to

$$\begin{aligned} \dot{\mathbf{x}} &= A\mathbf{x} + f_c(\mathbf{x}, h(\mathbf{x}, \mu), \mu) \\ \dot{\mu} &= 0 \\ \dot{\mathbf{y}} &= B\mathbf{y}. \end{aligned} \tag{3.18}$$

And, for later use, we state below the differential equation whose solution gives us our parameter-dependent center manifold (cf. 3.12):

$$\mathcal{N}(h(\mathbf{x}, \mu)) := D_{\mathbf{x}} h(\mathbf{x}, \mu)[A\mathbf{x} + f_c(\mathbf{x}, h(\mathbf{x}, \mu), \mu)] - Bh(\mathbf{x}, \mu) + f_s(\mathbf{x}, h(\mathbf{x}, \mu), \mu) = 0. \tag{3.19}$$

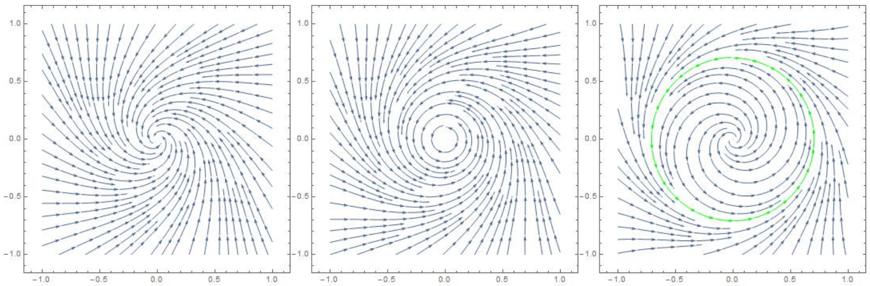
3.3. **Normal Form Theory Applied to the Hopf Bifurcation.** In this section, we develop the analytic machinery that we need to show that a two dimensional system satisyfing certain properties undergoes a Hopf bifurcation. The general method is to use a series of smooth variable changes to change the form of our given system into the *normal* form of the Hopf bifurcation - a "simple" system which we know beforehand undergoes a Hopf bifurcation. The normal forms of the supercritical and subcritical Hopf bifurcations, respectively, are given below:

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} \alpha & -1 \\ 1 & \alpha \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} x_1(x_1^2 + x_2^2) \\ x_2(x_1^2 + x_2^2) \end{bmatrix} \tag{3.20}$$

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} \alpha & -1 \\ 1 & \alpha \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} x_1(x_1^2 + x_2^2) \\ x_2(x_1^2 + x_2^2) \end{bmatrix} \tag{3.21}$$

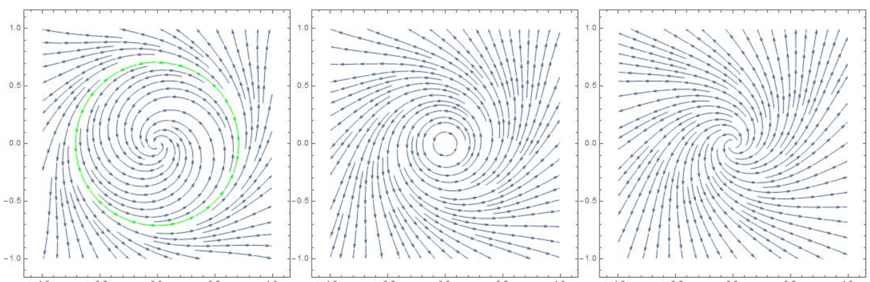
Solutions to (3.20) and (3.21) are shown in Figures 1 and 2. In both of these examples, $\alpha = 0$ is the critical parameter value at which the stability of the origin changes. A supercritical Hopf bifurcation occurs when a limit cycle forms as α goes from negative to positive. A subcritical Hopf bifurcation occurs when a limit cycle forms as α goes from positive to negative. The stability of the limit cycle depends on the stability of the equilibrium point it surrounds. One

[5]There may be a concern that going through the change of basis from (3.16) to (3.17) may change the variable μ from (3.16) in such a way that in our new system (3.17), varying μ would vary more than just μ in our original system, in which case we are no longer studying the dynamics of (3.16) as just μ varies. In general, one would expect this to be the case, but in our case, because in (3.16), $\dot{\mu}$ is precisely equal to zero (i.e. it is independent of time), one can verify that the variable μ does not actually go through any change through the change of variables. In particular μ from (3.17) is not dependent on $\mathbf{w}$ from (3.16).



(A) $\alpha < 0$ (B) $\alpha = 0$ (C) $\alpha > 0$

FIGURE 1. Supercritical Hopf Bifurcation



(A) $\alpha < 0$ (B) $\alpha = 0$ (C) $\alpha > 0$

FIGURE 2. Subcritical Hopf Bifurcation

can feasibly see that by changing α to $-\alpha$ above, there are a total of four different kinds of Hopf bifurcations each of which is fully characterized by 1) whether a limit appears for α positive or negative and 2) whether the equilbrium point goes from stable to unstable or unstable to stable as α goes from negative to positive.

Now, given a two-dimensional system

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mu), \ \mathbf{x} \in \mathbb{R}^2, \ \mu \in \mathbb{R} \tag{3.22}$$

with $f(0,0) = 0$, we can show that if our system satisfies certain properties, then we can, through a smooth change of coordinates, transform system (3.22) into the form (3.20) or (3.21), and in doing so we will have shown that the system (3.22) undergoes a Hopf bifurcation at $\mu = 0$.

For the remainder of Section 3, we determine what properties (3.22) must satisfy and what kinds of changes of coordinates we must use to reduce our system into normal form.

Our first lemma below tells us that if we have a system *almost* in normal form - *almost* in the sense of (3.24) below - then there is a locally defined homeomorphism that maps orbits of (3.20) (or (3.21)) to orbits of (3.24) for orbits that start sufficiently near the origin. This means that stable (unstable) equilibrium points are mapped to stable (unstable) equilibrium points and small amplitude periodic orbits are mapped to periodic orbits. So, our task has been reduced to that of performing the necessary change of coordinates to transform system (3.22) into the form of (3.24).

Lemma 3.23. *The system*

$$\begin{bmatrix}\dot{x_1}\\ \dot{x_2}\end{bmatrix} = \begin{bmatrix}\alpha & -1\\ 1 & \alpha\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} \pm \begin{bmatrix}x_1(x_1^2+x_2^2)\\ x_2(x_1^2+x_2^2)\end{bmatrix} + \mathcal{O}((|x_1|+|x_2|)^4) \tag{3.24}$$

is topogically equivalent near the origin to system (3.20) or (3.21) (depending on the sign of the cubic terms).

Proof. See Kuznetsov [2]. □

Returning to the task at hand, suppose that in system (3.22), $Df(0,0)$ has eigenvalues $\lambda_{1,2} = \pm i\omega_0$, $\omega_0 > 0$. Then, by the Implicit Function Theorem, for $|\mu|$ small enough, we have a unique equilibrium point $\mathbf{x}_0(\mu) \in \mathbb{R}^2$ i.e. $f(\mathbf{x}_0(\mu), \mu) = 0$. So by making the change $\mathbf{x} \to \mathbf{x} + \mathbf{x}_0(\mu)$, we can, for small $|\mu|$, place the equilibrium point at the origin. Having done so, (3.22) can be written

$$\dot{\mathbf{x}} = A(\mu)\mathbf{x} + F(\mathbf{x}, \mu) \tag{3.25}$$

where $F = \mathcal{O}(|\mathbf{x}|^2)$. Furthermore, we may assume that we have performed a μ-dependent change of variables such that for small $|\mu|$, $A(\mu)$ is of the form

$$A(\mu) = \begin{bmatrix}Re(\lambda(\mu)) & -Im(\lambda(\mu))\\ Im(\lambda(\mu)) & Re(\lambda(\mu))\end{bmatrix} \tag{3.26}$$

where $\lambda(\mu), \overline{\lambda(\mu)}$ are the eigenvalues of $D_{\mathbf{x}}f(\mathbf{x}, \mu)|_{\mathbf{x}=0}$ for small $|\mu|$.

Now we can identity the complex variable z with $x_1 + ix_2$, and in doing so can write (3.25) as a one-dimensional complex system

$$\dot{z} = \lambda(\mu)z + g(z, \overline{z}, \mu) \tag{3.27}$$

where $g = \mathcal{O}(|z|^2)$. Now observe that system (3.24) can similarly be written in complex form:

$$\dot{z} = (\alpha + i)z \pm z|z|^2 + \mathcal{O}(|z|^4). \tag{3.28}$$

Hence, our goal will be to reduce (3.27) into the form (3.28) through a series of smooth changes of variables. In doing so, we will have to impose two conditions on (3.27), and having done so, we will have shown that the system undergoes a Hopf bifurcation at $\mu = 0$ given that the two conditions (to be discussed) hold.

We begin the process with the following lemma, which allows us to eliminate all quadratic terms from our system:

Lemma 3.29. *The equation*

$$\dot{z} = \lambda(\mu)z + \frac{g_{20}(0,0,\mu)}{2}z^2 + g_{11}(0,0,\mu)z\overline{z} + \frac{g_{02}(0,0,\mu)}{2}\overline{z}^2 + \mathcal{O}(|z|^3) \tag{3.30}$$

can be transformed by an invertible μ-dependent change of complex coordinates

$$z = w + \frac{h_{20}(\mu)}{2}w^2 + h_{11}(\mu)w\overline{w} + \frac{h_{02}(\mu)}{2}\overline{w}^2 \tag{3.31}$$

for all sufficiently small $|\mu|$, into an equation without quadratic terms:

$$\dot{w} = \lambda(\mu)w + \mathcal{O}(|w|^3). \tag{3.32}$$

Remark 3.33. In the above lemma, we assume $g_{ij}(0,0,\mu) := \frac{\partial^{i+j}g}{\partial z^i \partial \overline{z}^j}|_{z=\overline{z}=0}$ and $\lambda(\mu) = R(\mu) + iI(\mu)$, $R(0) = 0$, $I(0) = \omega_0 > 0$. Furthermore note that the subscripts for h do not correspond to partial derivatives; h_{ij} are smooth functions of μ to be determined. We further remark that for the remainder of the paper we will use $g_{ij}(\mu) := g_{ij}(0,0,\mu)$.

Proof. (of Lemma 3.29) The inverse change of variables of (3.31) is given by

$$w = z - \frac{h_{20}(\mu)}{2} z^2 - h_{11}(\mu) z\overline{z} - \frac{h_{02}(\mu)}{2}\overline{z}^2 + \mathcal{O}(|z|^3) \tag{3.34}$$

as can be obtained by setting

$$w = a_0 + a_1 z + a_2\overline{z} + a_3 z^2 + a_4 z\overline{z} + a_5\overline{z}^2 + \mathcal{O}(|z|^3), \tag{3.35}$$

then substituting (3.35) into (3.31) and solving for the coefficients a_0, a_1, a_2, a_3, a_4 and a_5.

Now from (3.34), we get

$$\dot{w} = \dot{z} - h_{20}(\mu) z\dot{z} - h_{11}(\mu)(z\dot{\overline{z}} + \dot{z}\overline{z}) - h_{02}(\mu)\overline{z}\dot{\overline{z}} + \cdots \tag{3.36}$$

Then substituting (3.30) and (3.31) into (3.36) we find (after algebraic manipulations) that the constant term on the RHS is zero and the coefficients of $w, \overline{w}, w^2, w\overline{w}$, and $\overline{w}^2$ are $\lambda(\mu), 0, \frac{1}{2}(g_{20}(\mu) - \lambda(\mu)h_{20}(\mu)), g_{11}(\mu) - \overline{\lambda(\mu)}h_{11}(\mu)$, and $\frac{1}{2}(g_{02}(\mu) - (2\overline{\lambda(\mu)} - \lambda(\mu))h_{02}(\mu))$, respectively. Thus by setting

$$h_{20}(\mu) = \frac{g_{20}(\mu)}{\lambda(\mu)} \tag{3.37}$$

$$h_{11}(\mu) = \frac{g_{11}(\mu)}{\overline{\lambda(\mu)}} \tag{3.38}$$

$$h_{02}(\mu) = \frac{g_{02}(\mu)}{2\overline{\lambda(\mu)} - \lambda(\mu)} \tag{3.39}$$

we get our desired form (3.32). Note that (3.37-39) are well-defined for small $|\mu|$ since $\lambda(0) \neq 0$ by assumption. □

Having thus eliminated the quadratic terms, we attempt to eliminate cubic terms, too:

Lemma 3.40. *The equation*

$$\dot{z} = \lambda(\mu) z + \frac{g_{30}(\mu)}{6} z^3 + \frac{g_{21}(\mu)}{2} z^2\overline{z} + \frac{g_{12}(\mu)}{2}\overline{z}^2 z + \frac{g_{03}(\mu)}{6}\overline{z}^3 + \mathcal{O}(|z|^4), \tag{3.41}$$

with the same assumptions as in the previous lemma, can be transformed by an invertible μ-dependent change of complex coordinates

$$z = w + \frac{h_{30}(\mu)}{6} w^3 + \frac{h_{21}(\mu)}{2} w^2\overline{w} + \frac{h_{12}(\mu)}{2} w\overline{w}^2 + \frac{h_{03}(\mu)}{6}\overline{w}^3 \tag{3.42}$$

for all sufficiently small $|\mu|$ into an equation with only one cubic term:

$$\dot{w} = \lambda(\mu) w + c_1(\mu) w^2\overline{w} + \mathcal{O}(|w|^4). \tag{3.43}$$

Remark 3.44. Observe that we are actually not able to eliminate all cubic terms. The term $w^2\overline{w}$ above is called a *resonant term.*

Proof. (of Lemma 3.40) The proof of this lemma is essentially identical to that of Lemma 3.29. We find by the same method that the inverse change of variables is given by

$$w = z - \frac{h_{30}(\mu)}{6} z^3 - \frac{h_{21}(\mu)}{2} z^2\overline{z} - \frac{h_{12}(\mu)}{2} z\overline{z}^2 - \frac{h_{03}(\mu)}{6}\overline{z}^3 + \mathcal{O}(|z|^4). \tag{3.45}$$

Then, by differentiating (3.45) and making the necessary substitutions, we find that by setting

$$h_{30}(\mu) = \frac{g_{30}(\mu)}{2\lambda(\mu)} \tag{3.46}$$

$$h_{12}(\mu) = \frac{g_{12}(\mu)}{2\overline{\lambda(\mu)}} \tag{3.47}$$

$$h_{03}(\mu) = \frac{g_{03}(\mu)}{3\overline{\lambda(\mu)} - \lambda(\mu)} \tag{3.48}$$

the coefficients of $w^3, w\overline{w}^2, \overline{w}^3$ all become zero. If we attempt to make the coefficient of $w^2\overline{w}$ zero, then we must set

$$h_{21}(\mu) = \frac{g_{21}(\mu)}{\lambda(\mu) + \overline{\lambda(\mu)}}, \tag{3.49}$$

but we observe that $h_{21}(\mu)$ then becomes undefined at $\mu = 0$. Hence we set $h_{21}(\mu) = 0$, which results in

$$c_1(\mu) = \frac{g_{21}(\mu)}{2}. \tag{3.50}$$

□

Now observe that applying Lemma 3.29 and Lemma 3.40 in succession, we can reduce the equation

$$\dot{z} = \lambda(\mu)z + \sum_{2\le k+l\le 3} \frac{1}{k!l!} g_{kl}(\mu) z^k \overline{z}^l + \mathcal{O}(|z|^4) \tag{3.51}$$

into the form (3.43):

$$\dot{w} = \lambda(\mu)w + c_1(\mu)w^2\overline{w} + \mathcal{O}(|w|^4).$$

However, note that in first eliminating the quadratic terms, the coefficients of the cubic terms change, too, so we cannot simply say that $c_1(\mu) = \frac{g_{21}(\mu)}{2}$. We get around this problem by doing the following: we can express $\dot{z}$ in terms of $w, \overline{w}$ in two ways. First, we can substitute (3.31) into (3.51). Second, since we know that after all the necessary reductions we end up with an equation of the form (3.43), $\dot{z}$ can be computed by differentiating (3.31) and then substituting in $\dot{w}$ and $\dot{\overline{w}}$ using (3.43). Then, equating the coefficients of $w|w|^2$ (i.e. $w^2\overline{w}$), we get

$$c_1(\mu) = \frac{g_{20}(\mu)g_{11}(\mu)(2\lambda(\mu) + \overline{\lambda(\mu)})}{2|\lambda(\mu)|^2} + \frac{|g_{11}(\mu)|^2}{\lambda(\mu)} + \frac{|g_{02}(\mu)|^2}{4\lambda(\mu) - 2\overline{\lambda(\mu)}} + \frac{g_{21}(\mu)}{2}. \tag{3.52}$$

At the critical bifurcation value $\mu = 0$, we have

$$c_1(0) = \frac{i}{2\omega_0}(g_{20}(0)g_{11}(0) - 2|g_{11}(0)|^2 - \frac{1}{3}|g_{02}(0)|^2) + \frac{g_{21}(0)}{2}. \tag{3.53}$$

So far, we can reduce an equation of the form (3.51) into the form (3.43):

$$\dot{w} = \lambda(\mu)w + c_1(\mu)w^2\overline{w} + \mathcal{O}(|w|^4).$$

Now we would like to transform (3.43) into the normal form discussed previously:

$$\dot{z} = (\alpha + i)z \pm z|z|^2 + \mathcal{O}(|z|^4). \tag{3.54}$$

Lemma 3.55. *Given two conditions to be determined, (3.43) can be transformed into the Hopf bifurcation normal form (3.54).*

Proof. Rewrite (3.43) in the following way by setting $\lambda(\mu) = R(\mu) + iI(\mu)$:

$$\frac{dw}{dt} = (R(\mu) + iI(\mu))w + c_1(\mu)w|w|^2 + \mathcal{O}(|w|^4). \tag{3.56}$$

By assumption $R(0) = 0$ and $I(0) = \omega_0 > 0$. Now, introduce the μ-dependent time variable $\tau = I(\mu)t$. The time direction is preserved since $I(\mu)$ is positive for sufficiently small $|\mu|$. Also, define $\beta(\mu) = \frac{R(\mu)}{I(\mu)}$. Then, if we impose the condition that $R'(0) \neq 0$, we have

$$\beta(0) = 0,\ \beta'(0) = \frac{I(0)R'(0) - R(0)I'(0)}{I^2(0)} = \frac{R'(0)}{I(0)} \neq 0.$$

Hence we can consider β as our new parameter and write μ as a function of β for small $|\beta|$, $\mu = \mu(\beta)$. So now with the new time variable and new parameter, (3.56) turns into

$$\frac{dw}{d\tau} = (\beta + i)w + d_1(\beta)w|w|^2 + \mathcal{O}(|w|^4) \tag{3.57}$$

where $d_1(\beta) = \frac{c_1(\mu(\beta))}{I(\mu(\beta))}$.

We now introduce a new time parameterization along orbits of (3.57), $\theta = \theta(\tau, \beta)$, where

$$d\theta = (1 + Im(d_1(\beta))|w|^2)d\tau. \tag{3.58}$$

With this new definition of time, (3.57) becomes

$$\frac{dw}{d\theta} = (\beta + i)w + l_1(\beta)w|w|^2 + \mathcal{O}(|w|^4) \tag{3.59}$$

where $l_1(\beta) = Re(d_1(\beta)) - \beta \cdot Im(d_1(\beta))$ and

$$l_1(0) = \frac{Re(c_1(0))}{\omega_0} \tag{3.60}$$

It is not a particularly illuminating process going from (3.57) to (3.59), but one can easily verify the result by multiplying the RHS of (3.59) by $1 + Im(d_1(\beta))|w|^2$ to check it results in the RHS of (3.57).

Finally, if we impose the condition that $Re(c_1(0)) \neq 0$, then $l_0(0) \neq 0$, so we can introduce a new complex variable u:

$$w = \frac{u}{\sqrt{|l_1(\beta)|}}. \tag{3.61}$$

With this new complex variable, (3.59) becomes

$$\begin{aligned}\frac{du}{d\theta} &= (\beta + i)u + \frac{l_1(\beta)}{|l_1(\beta)|}u|u|^2 + \mathcal{O}(|u|^4) \\ &= (\beta + i)u + su|u|^2 + \mathcal{O}(|u|^4)\end{aligned} \tag{3.62}$$

where $s = \frac{l_1(0)}{|l_1(0)|} = \text{sign } l_1(0) = \text{sign } Re(c_1(0))$.

This completes the proof. □

Remark 3.63. The real function $l_1(\beta)$ is called the *first Lyapunov coefficient*. Observe that from (3.53) and (3.60), it follows that the first Lyapunov coefficient evaluated at $\beta = 0$ is given by

$$l_1(0) = \frac{1}{2\omega_0^2} Re(ig_{20}(0)g_{11}(0) + \omega_0 g_{21}(0)). \tag{3.64}$$

We summarize Lemma 3.23, Lemma 3.29, Lemma 3.40 and Lemma 3.55 in one theorem below, making sure to include the two conditions that we imposed in the process:

Theorem 3.65. *Suppose we have a complex system*

$$\dot{z} = \lambda(\mu)z + g(z, \overline{z}, \mu) \tag{3.66}$$

where $g = \mathcal{O}(|z|^2)$ is a smooth function of $(z, \overline{z}, \mu)$ such that $g(0,0,0) = 0$, $\lambda(\mu)$ is a smooth function of μ and $\lambda(0) = i\omega_0$ ($\omega_0 > 0$). Then (3.66) can be reduced to the form (3.62) if the following two conditions hold:

1. *$\frac{d}{d\mu} Re(\lambda(\mu))|_{\mu=0} \neq 0$*
2. *The first Lyapunov coefficient evaluated at 0 (given by the formula (3.64)) is nonzero.*

Finally from an application standpoint, we have (due to Lemma 3.23):

Corollary 3.67. *Given a two-dimensional, parameter-depedent system*

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}, \mu) \tag{3.68}$$

with smooth f, having at $\mu = 0$ the equilibrium $\boldsymbol{x} = 0$ with $Df(0,0)$ having eigenvalues $\lambda_{1,2} = \pm i\omega_0$, $\omega_0 > 0$, the system undergoes a Hopf bifurcation around the origin as μ crosses zero provided the two conditions from Theorem 3.65 hold.

Remark 3.69. Note, however, that to check the conditions, one must first get (3.68) into the form (3.66) (see text following Lemma 3.23). Furthermore, observe that whether the Hopf bifurcation is supercritical or subcritical depends on the sign of the first Lyapunov first coefficient evaluated at zero. Whether the periodic solutions are stable or unstable depends on the sign of $\frac{d}{d\mu} Re(\lambda(\mu))|_{\mu=0}$.

4. Application to Model

Now we are ready to analyze our shear flow model. We begin by adjoining the four dimensional model with $\dot{R} = 0$, making it into a five dimensional system. This allows us to investigate the dynamics of our model as our parameter R varies. Then we make the necessary transformations to set up for the application of center manifold theory. Then, applying the theory, we reduce our system to two dimensions and, after applying further transformations to turn our system into the form (3.25), we introduce a complex variable, resulting in a one-dimensional complex system. At this point, we can check our two conditions from Theorem 3.65 to verify the occurence of a supercritical Hopf bifurcation.

4.1. Preliminary Transformations.

Recall that our model is given by

$$\dot{\mathbf{x}} = L_R\mathbf{x} + N(\mathbf{x}) \tag{4.1}$$

where

$$L_R = \begin{bmatrix} -k_1^2/R & 0 & 0 & 0 \\ 0 & -k_2^2/R & \sigma_2 & 0 \\ 0 & 0 & -k_3^2/R & 0 \\ 0 & 0 & 0 & -k_4^2/R - \sigma_1 \end{bmatrix}$$

$$N(\mathbf{x}) = \begin{bmatrix} \sigma_1 x_4^2 - \sigma_2 x_2 x_3 \\ -\sigma_4 x_4^2 + \sigma_2 x_1 x_2 \\ \sigma_3 x_4^2 \\ (\sigma_4 x_2 - \sigma_1 x_1 - \sigma_3 x_3)x_4 \end{bmatrix} \tag{4.2}$$

with constants

$$[k_1, k_2, k_3, k_4] = [1.57, 2.28, 2.77, 2.67]$$
$$[\sigma_1, \sigma_2, \sigma_3, \sigma_4] = [0.31, 1.29, 0.22, 0.68].$$

Since we are studying the dynamics of (4.1) as R varies near the critical value, we adjoin $\dot{R} = 0$ (cf. Section 3.2). Now, we recall that our critical bifurcation value is given by $R_C \approx 139.74$, where the linearization of (4.1) at the equilibrium point ($x_1^* \approx -0.8015, x_2^* \approx 0.1923, x_3^* \approx 0.0828, x_4^* \approx 0.1438$) has two eigenvalues with negative real part ($-a \pm bi$, $a, b > 0$) and two eigenvalues with zero real part ($\pm i\omega$, $\omega > 0$).

For convenience, we make the linear shifts to move the equilbrium point in question to the origin. Furthermore, we make the change of variables $r' = 1/R$, and then $r = r' - 1/R_C$, so that now, with r as the parameter, we can view our system as being made up of *polynomials* in x_1, x_2, x_3, x_4 and r, where $r = 0$ is the critical bifurcation value. Also, note the inverse relation of r and R: this will be necessary for the interpretation of results later on.

So, starting with the five-dimensional system

$$\begin{aligned} \dot{\mathbf{x}} &= L_R \mathbf{x} + N(\mathbf{x}) \\ \dot{R} &= 0. \end{aligned} \tag{4.3}$$

we perform the changes of variables mentioned in the previous paragraph and then rewrite the resulting system in real canonical form, resulting in the system

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \dot{x_4} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} -a & -b & 0 & 0 & 0 \\ b & -a & 0 & 0 & 0 \\ 0 & 0 & 0 & -\omega & 0 \\ 0 & 0 & \omega & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ r \end{bmatrix} + \begin{bmatrix} N_1^*(x_1, x_2, x_3, x_4, r) \\ N_2^*(x_1, x_2, x_3, x_4, r) \\ N_3^*(x_1, x_2, x_3, x_4, r) \\ N_4^*(x_1, x_2, x_3, x_4, r) \\ 0 \end{bmatrix} \tag{4.4}$$

where N_i^* are nonlinear with respect to x_1, x_2, x_3, x_4, r.

4.2. **Cubic Approximation of Center Manifold.** Now we can apply the center manifold theorem to find the restriction of our system to its center manifold. We begin by approximating the center manifold of (4.4) at the equilibrium point (0,0,0,0,0). Notice that the center manifold in our case is three dimensional, and the theorem gives us two smooth functions $x_1 = h_1(x_3, x_4, r)$ and $x_2 = h_2(x_3, x_4, r)$ whose graph near the origin of (x_1, x_2, x_3, x_4, r)-space gives us our center manifold. At this point, we use Theorem 3.13 to approximate our two functions. For our purposes, it is sufficient to approximate $h_1(x_3, x_4, r)$ and $h_2(x_3, x_4, r)$ up to cubic terms.[6] From center manifold theory, we also know $h_i(0, 0, 0) = 0$ and $Dh_i(0, 0, 0) = 0$, hence we can write our functions as

$$\begin{aligned} h_i(x_3, x_4, r) = &a_{i1}x_3^2 + a_{i2}x_4^2 + a_{i3}r^2 + a_{i4}x_3x_4 + a_{i5}x_3r + a_{i6}x_4r \\ &a_{i7}x_3^3 + a_{i8}x_4^3 + a_{i9}r^3 + a_{i10}x_3^2x_4 + a_{i11}x_3x_4^2 \\ &+ a_{i12}x_3^2r + a_{i13}x_3r^2 + a_{i14}x_4^2r + a_{i15}x_4r^2 \\ &+ a_{i16}x_3x_4r + \mathcal{O}((|x_3| + |x_4| + |r|)^4) \end{aligned} \tag{4.5}$$

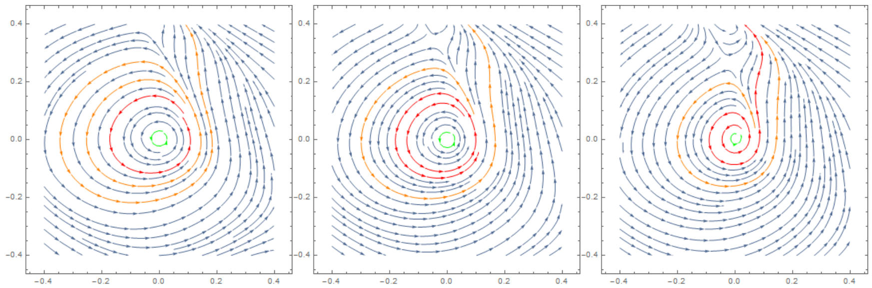
for $i = 1, 2$ where the coefficients are currently unknown. We solve for the coefficients a_{ij} using the fact that $h_i(x_3, x_4, r)$ must satisfy (cf. 3.19)

$$\begin{aligned} D\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} [\begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} N_3^*(h_1, h_2, x_3, x_4, r) \\ N_4^*(h_1, h_2, x_3, x_4, r) \end{bmatrix}] \\ - \begin{bmatrix} -a & -b \\ b & -a \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} + \begin{bmatrix} N_1^*(h_1, h_2, x_3, x_4, r) \\ N_2^*(h_1, h_2, x_3, x_4, r) \end{bmatrix} = 0 \end{aligned} \tag{4.6}$$

where the derivative D is with respect to x_3 and x_4, and h_1 and h_2 are functions of x_3, x_4, r.

Substituting (4.5) into (4.6) and setting the coefficients of all the quadratic and cubic

[6]This statement is justified by Lemma 3.23.



(A) $r < 0$ (B) $r = 0$ (C) $r > 0$

FIGURE 3. System (4.7)

terms equal to zero, we get a unique real solution set for a_{ij}, where $i = 1, 2$, $j = 1, ..., 16$. Thus, by Theorem 3.13, we have obtained approximations whose difference with the actual functions h_1 and h_2 is $\mathcal{O}((|x_3| + |x_4| + |r|)^4)$. For the remainder of the paper, we will call the approximating functions h_1 and h_2.

4.3. **Center Manifold Reduction.** Now as a direct application of center manifold theory, we can simply study our system (4.4) *restricted* to its center manifold (cf. 3.7) to study the dynamics of (4.4) near the origin as r is varied near 0.

The restriction to the center manifold is given by

$$\begin{bmatrix} \dot{x_3} \\ \dot{x_4} \end{bmatrix} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} N_3^*(h_1, h_2, x_3, x_4, r) \\ N_4^*(h_1, h_2, x_3, x_4, r) \end{bmatrix}$$
$$\dot{r} = 0.$$

Notice, however, that there are no dynamics along the r-axis, and hence we need only focus on the system

$$\begin{bmatrix} \dot{x_3} \\ \dot{x_4} \end{bmatrix} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} N_3^*(h_1, h_2, x_3, x_4, r) \\ N_4^*(h_1, h_2, x_3, x_4, r) \end{bmatrix} =: \begin{bmatrix} F_3(x_3, x_4, r) \\ F_4(x_3, x_4, r) \end{bmatrix}. \tag{4.7}$$

Before proceeding, we can use a program to graph the above system (Figure 3).

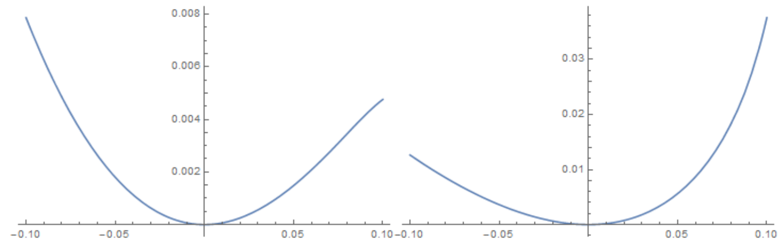
Just based on the plots from Figure 3, one can see that as r goes from negative to positive, the equilibrium goes from stable to unstable, and comparing the red and orange orbits in (A), one can see that a periodic orbit should exist somewhere "in between" those two orbits. However, for $r > 0$, all orbits are leaving the equilibrium and there is no periodic solution. Thus, we have at least graphically verified the occurence of a (subcritical) Hopf bifurcation at $r = 0$.

4.4. **Normal Form Reduction.** Now, to actually apply the theory from Section 3.3, we must get system (4.7) in the form (3.66):

$$\dot{z} = \lambda(\mu)z + g(z, \overline{z}, \mu).$$

We recall, however, that in order to make the transformation to the form (3.66), it is necessary to make a r-dependent coordinate change such that for all small $|r|$, we have $F_3(0, 0, r) = F_4(0, 0, r) = 0$. This is done by observing that

$$\begin{bmatrix} F_3(0, 0, 0) \\ F_4(0, 0, 0) \end{bmatrix} = 0$$



(A) $x_3^*(r)$ (B) $x_4^*(r)$

FIGURE 4

and the Jacobian of $\begin{bmatrix} F_3(x_3, x_4, r) \\ F_4(x_3, x_4, r) \end{bmatrix}$ with respect to x_3, x_4 evaluated at $(0, 0, 0)$ is invertible. Hence, it is theoretically possible to write x_3 and x_4 as functions of r near the origin. That is, we have functions

$$\begin{aligned} x_3 &= x_3^*(r) \\ x_4 &= x_4^*(r) \end{aligned} \tag{4.8}$$

such that $F_3(x_3^*(r), x_4^*(r), r) = F_4(x_3^*(r), x_4^*(r), r) = 0$ for all small $|r|$. In practice, we can use a program to find $x_i^*(r)$ at sufficiently many points r and then interpolate to acquire a function that is "correct" up to high orders. For this model, we restricted ourselves to the domain $r \in (-0.1, 0.1)$ and interpolation resulted in functions $x_3^*(r)$ and $x_4^*(r)$ as shown in Figure 4.

Now making the variable changes $x_3 \mapsto x_3 - x_3^*(r)$ and $x_4 \mapsto x_4 - x_4^*(r)$, (4.7) turns into

$$\begin{bmatrix} \dot{x_3} \\ \dot{x_4} \end{bmatrix} = \begin{bmatrix} G_3(x_3, x_4, r) \\ G_4(x_3, x_4, r) \end{bmatrix} \tag{4.9}$$

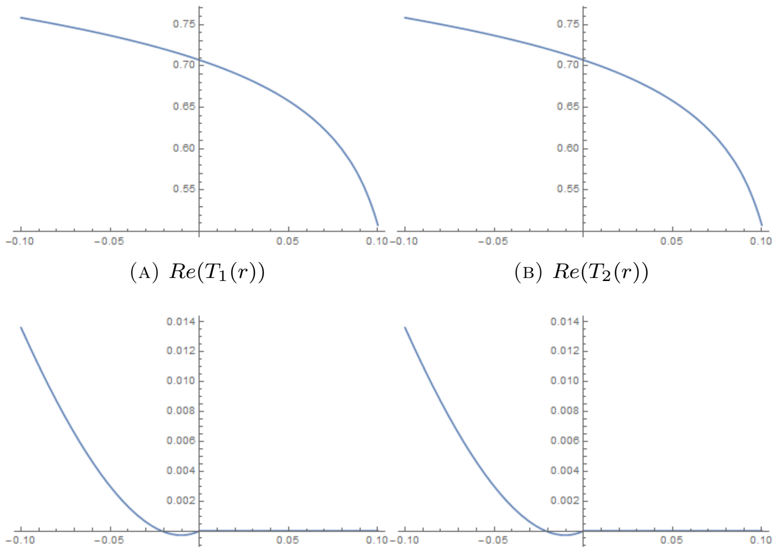
where $(0, 0, r)$ is an equilibrium point for all small $|r|$. We can try graphing (4.9) at this intermediate stage, but because $x_{3,4}^*(r)$ is very small for small $|r|$, one will see that the resulting plots are visually almost indistinguishable from Figure 3. In particular, all the observations we made for the graphs of system (4.7) still hold for system (4.9).

Now we are ready to perform a change of variables on (4.9) so that the resulting system is of the (intermediary) form

$$\begin{bmatrix} \dot{x_3} \\ \dot{x_4} \end{bmatrix} = \begin{bmatrix} Re(\lambda(r)) & -Im(\lambda(r)) \\ Im(\lambda(r)) & Re(\lambda(r)) \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} G_3^*(x_3, x_4, r) \\ G_4^*(x_3, x_4, r) \end{bmatrix} \tag{4.10}$$

where $\lambda(r), \overline{\lambda(r)}$ are the eigenvalues of the Jacobian of (4.9) evaluated at the equilibrium $(0, 0, r)$, and $G_{3,4}^*$ are $\mathcal{O}((|x_3| + |x_4|)^2)$. To perform the necessary change of variables, note that for a fixed r, one can compute the change of basis matrix that transforms the Jacobian of (4.9) evaluated at $(0, 0, r)$ into the linear part of (4.10). Now, for varying r, one can see that the change of basis matrix will be a smooth function of r. Thus, since an explicit, closed form of this matrix is practically unfeasible to compute, we again resort to to computing the change of basis matrix for sufficiently many points of r in the domain $(-0.1, 0.1)$ and then interpolating. Suppose, the r-dependent change of basis matrix is of the form

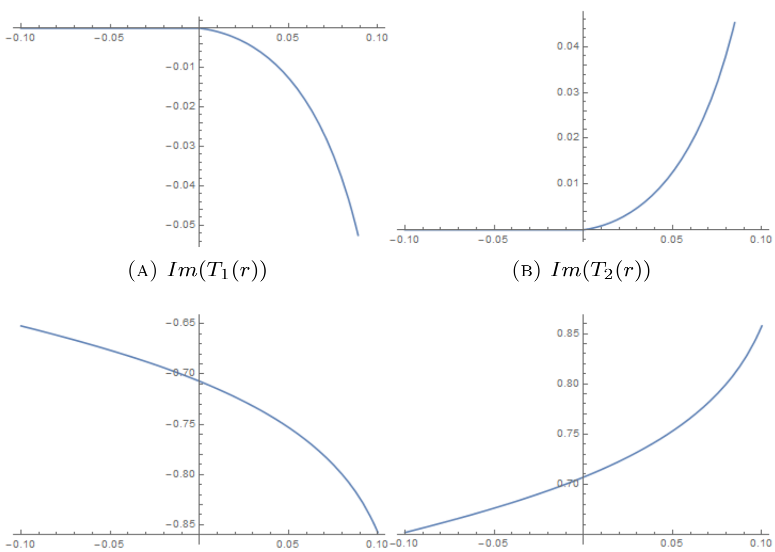
$$\mathbf{T}(r) = \begin{bmatrix} T_1(r) & T_2(r) \\ T_3(r) & T_4(r) \end{bmatrix} X. \tag{4.11}$$



(A) $Re(T_1(r))$ (B) $Re(T_2(r))$

(C) $Re(T_3(r))$ (D) $Re(T_4(r))$

FIGURE 5. Real Parts of T_i



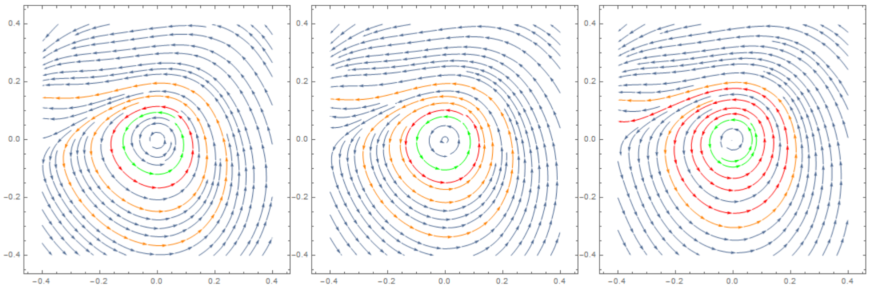
(A) $Im(T_1(r))$ (B) $Im(T_2(r))$

(C) $Im(T_3(r))$ (D) $Im(T_4(r))$

FIGURE 6. Imaginary Parts of T_i

where X is the constant 2×2 change of basis matrix that transforms a matrix in the Jordan canonical form into the real canonical form. Then, by the process described above, we obtain functions $T_i(r)$, whose real and imaginary parts we plot separately in Figures 5 and 6.

Now using the transformation given by $\mathbf{T}(r)$, we can transform (4.9) into the form



(A) $r < 0$ (B) $r = 0$ (C) $r > 0$

FIGURE 7. System (4.10)

(4.10) (Figure 7 shows solutions to (4.10)). Having done so, we finally introduce the complex variable $z = x_3 + ix_4$ and write (4.10) in the form

$$\dot{z} = \lambda(r)z + g(z, \overline{z}, r) \tag{4.12}$$

where g is $\mathcal{O}(|z|^2)$.

4.5. **Conclusion.** Finally, we can check the two conditions from Theorem 3.65. To check the first condition, we must evaluate $\frac{d}{dr}Re(\lambda(r))$ at $r = 0$. Observe that $Re(\lambda(r))$ is given in the $(1, 1)$ coordinate of the linear part of (4.10). Since we have an explicit (albeit approximate) function for $\mathbf{T}(r)$, we can express $Re(\lambda(r))$ for small $|r|$ as

$$Re(\lambda(r)) = \frac{\partial}{\partial x_3}\left[\begin{bmatrix}1 & 0\end{bmatrix}\left(\mathbf{T}^{-1}(r)\begin{bmatrix}G_3(X_3(x_3,x_4,r), X_4(x_3,x_4,r), r)\\ G_4(X_3(x_3,x_4,r), X_4(x_3,x_4,r), r)\end{bmatrix}\right)\right]_{x_3,x_4=0}. \tag{4.13}$$

where

$$X_3(x_3, x_4, r) = (\mathbf{T}_{11}(r), \mathbf{T}_{12}(r)) \cdot (x_3, x_4)$$
$$X_4(x_3, x_4, r) = (\mathbf{T}_{21}(r), \mathbf{T}_{22}(r)) \cdot (x_3, x_4)$$

with the subscripts ij under $\mathbf{T}$ indicating the (i, j) component of the matrix $\mathbf{T}(r)$, and $\cdot$ indicating the standard dot product.

Rather than thinking of (4.13) as a convenient formula, one should see it as an algorithm to isolate the (1,1) component of the linear part of (4.10). In practice, with most programs, isolating the required $Re(\lambda(r))$ should be a relatively simple task. Now computing the derivative of $Re(\lambda(r))$ at $r = 0$, we find $\frac{d}{dr}Re(\lambda(r))|_{r=0} \approx 0.0513906 > 0$.

To check condition (2), since we have transformed our equation into the required form (4.12), we can simply apply the formula given by (3.64):

$$l_1(0) = \frac{1}{2\omega_0^2}Re(ig_{20}(0)g_{11}(0) + \omega_0 g_{21}(0)).$$

In our case, we find $l_1(0) \approx 0.696056 > 0$.

Interpreting the signs of $\frac{d}{dr}Re(\lambda(r))|_{r=0}$ and $l_1(0)$, we conclude that our system (4.7) undergoes a subcritical Hopf bifurcation at $r = 0$ and the periodic solutions that exist for $r < 0$ are unstable (the equilibrium is stable). Now returning to our original model (4.1), we conclude that it undergoes a *supercritical* Hopf bifurcation at $R = R_C$ since R and r have an inverse relation. For the same reason, the equilibrium point in question goes from unstable to stable as R goes from $R < R_C$ to $R > R_C$.

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