THE N-DIMENSIONAL STOKES' THEOREM

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ABSTRACT. This paper introduces a variety of concepts, including those of manifolds, tensors, exterior algebra, and the theory of differential forms, in order to prove the generalized form of Stokes' Theorem.

CONTENTS

1. INTRODUCTION

Stokes' Theorem, Green's Theorem, and the Divergence Theorem should all be quite familiar to the average multivariable student, but in looking to extend these theorems beyond three dimensions, the tools introduced in vector calculus quickly fall short. In order to express and prove the higher-dimensional analogues of these theorems, multilinear algebra and the theory of differential forms become necessary. A complete examination of these fields is somewhat beyond the scope of this paper, of course, but an introductory look into the basic tools and concepts of the aforementioned two fields will still prove useful, and in doing so, we will lay the groundwork for this paper's main focus: to prove the generalized version of Stokes' Theorem.

2. MANIFOLDS

In order to extend Stokes' Theorem to higher dimensions, we must introduce the concept of manifolds, which will serve as the basic setting in which the theorem can be constructed. Even before that, however, we must first define the class of linear maps that serve to describe manifolds.

Definition 2.1 ([1, Definition 2.6.2]). Let A be an open set in $\mathbb{R}^n$. A linear map f of A into $\mathbb{R}^m$ is said to be of differentiability class $\mathbf{C}^\infty$ if its partial derivatives of all orders exist and are continuous.

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This concept of a $\mathbf{C}^\infty$ map, which is perhaps more familiarly known as a smooth map, analogous to smooth functions in Euclidean space, in turn allows us to define a particular type of function, known as a diffeomorphism, which is of particular interest in developing the concept of manifolds.

Definition 2.2 ([1, Definition 2.6.4])**.** Let A and B be open sets in $\mathbb{R}^n$. A map $f : A \to B$ is a diffeomorphism if both it and its inverse map $f^{-1} : B \to A$ are of class $\mathbf{C}^\infty$ and bijective.

Diffeomorphisms play an important role in both defining and conceptualizing manifolds. Manifolds can be locally described using Euclidean space, as for any given local region of a manifold, a diffeomorphism exists that maps said local region to a region of Euclidean space. We now state this definition of a manifold more formally.

Definition 2.3 ([1, Definition 5.23.1])**.** Let $k > 0$. Suppose M is a subspace of $\mathbb{R}^n$, and that for each $p \in M$ there exists an open subset, V, of M, which contains p. Suppose there also exists a set U that is open in $\mathbb{R}^k$, and a continuous map $\alpha : U \to V$ that carries U onto V in a one-to-one fashion, such that α is a diffeomorphism. Then M is said to be a k-manifold without boundary, and the map α is said to be a coordinate patch on M about p.

In order to introduce the more general formulation of a manifold, however, we must first introduce a subspace of $\mathbb{R}^k$, the upper half space.

Definition 2.4 ([1, Definition 5.23.2])**.** The upper half space, which we denote $\mathbf{H}^k$, is the set of all $x \in \mathbb{R}^k$ for which $x_k \geq 0$.

The upper half space, essential to our discussion of manifolds, will also be especially important in discussing integrations over manifolds and in our final presentation of the generalized Stokes' Theorem.

Definition 2.5 ([1, Definition 5.23.5])**.** Let $k > 0$. Suppose M is a subspace of $\mathbb{R}^n$, and that for each $p \in M$ there exists an open subset, V, of M, which contains p. Suppose there also exists a set U that is open in $\mathbf{H}^k$, and a continuous map $\alpha : U \to V$ that carries U onto V in a one-to-one fashion, such that α is a diffeomorphism. Then M is said to be a k-manifold, and the map α is said to be a coordinate patch on M about p.

In this sense, a manifold without boundary can be considered a special case of a manifold, in which all coordinate patches are open in $\mathbb{R}^k$ as well as in $\mathbf{H}^k$. If this condition is not met, then the given manifold is considered to be a manifold with boundary. With these definitions in mind, we are able to discuss the boundary of a manifold, as well as define what it means for a point to be in the interior of a given manifold.

Definition 2.6 ([1, Definition 5.24.1])**.** Let M be a k-manifold in $\mathbb{R}^n$, and let $p \in M$. If there exists a coordinate patch $\alpha : U \to V$ in M about p such that U is open in $\mathbb{R}^k$, then we call p an interior point of M. Otherwise, we call p a boundary point of M. The set of boundary points in M is denoted as ∂M, which we call the boundary of M. We denote the set of all interior points of M, also known as the interior of M, as $\operatorname{Int} M$.

In this sense, the concepts of manifolds with or without boundary are very similar to the concepts of open and closed sets in basic topology. The generalized Stokes' Theorem places great importance on relating the integral of a manifold to the integral of its boundary, and it is therefore important to note that a manifold's boundary is itself a manifold.

Definition 2.7 ([1, Theorem 5.24.3])**.** Let M be a k-manifold in $\mathbb{R}^n$. If ∂M is non-empty, then ∂M is a $k-1$ manifold without boundary in $\mathbb{R}^n$.

To end our discussion of manifolds, we introduce a few definitions that will be of use throughout the rest of the paper.

Definition 2.8 ([1, Definition 6.29.3])**.** Let x be a point on a k-manifold M in $\mathbb{R}^n$, with $U \subset \mathbb{R}^k$. Let $\beta : U \to M$ be a local diffeomorphism around x, with $d\beta : \mathbb{R}^k \to \mathbb{R}^n$ its derivative map. We call the image of $d\beta$ the tangent space of M at x, which we denote as $T_x(M)$.

A manifold's tangent space is particularly important when it comes to defining a manifold's orientation. First, however, we look at what it means for a vector space to be oriented.

Definition 2.9 ([2, Proposition 1.9.4])**.** Let V be a finite-dimensional, real vector space. An orientation is the arbitrary assignment of either a positive or negative value to the equivalence classes of a linear transformation, $A : V \to V$. This assignment is based on the sign of the determinant of A.

In a similar fashion, a diffeomorphism of open sets can be said to preserve or reverse the orientation of an open set. Whether or not such a diffeomorphism is orientation-preserving or reversing is dependent on the sign of the Jacobian determinant, much like in the previous definition.

Definition 2.10 ([1, Definition 7.34.2])**.** Let $f : A \to B$ be a diffeomorphism of open sets in $\mathbb{R}^k$. We say that f is orientation-preserving if $\det Dg > 0$ on A, and that f is orientation-reversing if $\det Dg < 0$ on A.

It is using these definitions of orientation-preserving and orientation-reversing diffeomorphisms that we are able to define what it means for a manifold to be orientable, a definition that is reliant upon the existence of coordinate patches which can both cover M and have consistently positive Jacobian determinants on their intersections.

Definition 2.11 ([1, Definition 7.34.3])**.** Let M be a k-manifold in $\mathbb{R}^n$. Given coordinate patches $\alpha_i : U_i \to V_i$ on M for $i = 0, 1$, we say the patches overlap if $V_0 \cap V_1$ is non-empty. We say the patches overlap positively if the transition function $\alpha_1^{-1} \circ \alpha_0$ is orientation-preserving. If M can be covered by a collection of coordinate patches, each pair of which overlaps positively if at all, M is said to be orientable. Otherwise, M is said to be non-orientable.

With this definition in mind, we are now able to define what it means for a manifold to be oriented.

Definition 2.12 ([1, Definition 7.34.4])**.** Let M be a k-manifold in $\mathbb{R}^n$. Suppose M is orientable. Given a collection of coordinate patches covering M that overlap positively, let us adjoin to this collection all other coordinate patches on M that overlap these patches positively. The patches in this expanded collection obviously overlap each other positively, and we call this collection an orientation on M. A manifold M, together with an orientation on M, is called an oriented manifold.

We end our discussion of orientable manifolds with two useful facts about orientable manifolds, which are both presented without proof.

Remark 2.13 ([1, Definition 7.34.7]). A connected, orientable manifold with boundary admits exactly two orientations.

Remark 2.14 ([1, Definition 6.29.5]). Every orientation of a manifold induces an orientation on its boundary, which will either be directed inwards or outwards. These orientations are known as inward and outward normals, respectively.

3. Multilinear Algebra

In order to more fully develop the machinery necessary to prove Stokes' Theorem, we must develop the theory of differential forms, which itself must be preceded by a discussion of the algebra of multilinear functions. We will first define what it means for a function to be multilinear.

Definition 3.1 ([1, Definition 6.26.1]). Let V be a vector space. We let

$$V^k = V \times ... \times V$$

denote the set of all k-tuples $(v_1, ..., v_k)$ of vectors of V. We define a function $f : V^k \to \mathbb{R}^n$ to be multilinear if it is linear in all k variables, and we call such a function a k-tensor on V.

It is important to note that, in many cases, tensors are simply more generalized versions of functionals, which take vectors as inputs and output scalars. A 1-tensor, for instance, is a real-valued functional, and the elementary dot product is a 2-tensor. Our focus on tensors, however, will mainly be concentrated on a specific method of combining them, known as the tensor product. Additionally, our use of tensors will be restricted to only those tensors which are themselves functionals.

Definition 3.2 ([1, Definition 6.26.4]). Let f be a k-tensor and g be an l-tensor, both on V. We define the $k+l$-tensor $f \otimes g$, also on V, by the formula

$$f \otimes g(v_1, ..., v_k, v_{k+1}, ..., v_{k+l}) = f(v_1, ..., v_k) \cdot g(v_{k+1}, ..., v_{k+l})$$

We call $f \otimes g$ the tensor product of f with g.

While the tensor product itself is certainly of interest, we will only be using a relative of the tensor product, the wedge product, which will be necessary to develop the theory of differential forms. In order to produce the wedge product, however, we must first introduce a specific type of tensor, the alternating tensor.

Definition 3.3 ([1, Definition 6.27.3]). A tensor f is called alternating if any transposition of two variables reverses the sign of f:

$$f(v_1, ..., v_i, ..., v_j, ..., v_k) = -f(v_1, ..., v_j, ..., v_i, ..., v_k)$$

We define the set of all alternating k-tensors as $\Lambda^k(V)$, which is itself a subset of the set of all k-tensors, $L^k(V)$.

In order to fully define the wedge product, however, we must find some method of relating any tensor to an associated alternating tensor. To that end, we introduce a method of constructing such a tensor.

We first note that given a permutation $\sigma \in S_k$, where S_k is the symmetric group of k elements, σ is said to be odd an odd permutation if, when decomposed into a product of transpositions, σ is the product of an odd number of transpositions. Similarly, σ is said to be an even permutation if it can be expressed as the product of an even number of transpositions. The sign of a permutation, denoted $\operatorname{sgn}\sigma$, assigns a value of $+1$ to σ if σ is an even permutation and a value of -1 if σ is an odd permutation.

Using these properties of permutations, we are immediately able to define a useful property of alternating tensors that will allow us to readily construct an alternating tensor from any arbitrary tensor. Given f, an alternating tensor, and taking $f^\sigma(v_1, ..., v_k) = f(v_{\sigma(1)}, ..., v_{\sigma(k)})$, it immediately follows that $f^\sigma = (\operatorname{sgn}\sigma)f$.

Theorem 3.4 ([3, Proposition 1.4.8])**.** *Let f be a k-tensor. We then construct a k-tensor, $Alt(f) = \frac{1}{k!}\sum_{\sigma \in S_k}(\operatorname{sgn}\sigma)f^\sigma$, where S_k is the symmetric group of k elements. Then $Alt(f)$ is an alternating tensor.*

Proof. Take f to be a k-tensor, and let $\pi \in S_k$. Then

$$Alt(f)^\pi = (\frac{1}{k!}\sum_{\sigma \in S_k}(\operatorname{sgn}\sigma)f^\sigma)^\pi$$

$$= \frac{1}{k!}\sum_{\sigma \in S_k}(\operatorname{sgn}(\sigma \circ \pi))(f^\sigma)^\pi = \frac{1}{k!}(\operatorname{sgn}\pi)\sum_{\sigma \in S_k}(\operatorname{sgn}(\sigma \circ \pi))f^{\sigma \circ \pi}$$

Let $\rho = \sigma \circ \pi$. Then

$$Alt(f)^\pi = (\operatorname{sgn}\pi)(\frac{1}{k!}\sum_{\rho \in S_k}(\operatorname{sgn}\rho)f^\rho) = (\operatorname{sgn}\pi)(Alt(f)),$$

and $Alt(f)$ must therefore be an alternating tensor. □

We are now able to define the wedge product, which allows us to take the tensor product of any two arbitrary tensors and create a new tensor, which will be an alternating tensor.

Definition 3.5 ([1, Definition 6.28.1])**.** Given f, a k-tensor, and g, an l-tensor, we define the wedge product of f and g, $f \wedge g$, as $Alt(f \otimes g)$.

We now state a few properties of the wedge product.

Proposition 3.6 ([1, Theorem 6.28.1])**.** *The wedge product is associative and anti-commutative. That is to say, given f, a k-tensor, g, an l-tensor, and h, an m-tensor,*

$$(f \wedge g) \wedge h = f \wedge (g \wedge h)$$

and

$$f \wedge g = (-1)^{kl} g \wedge f$$

To end our discussion of the wedge product and tensors, we note a particularly powerful property of the wedge product, which relates a linear map of the wedge product of tensors to the determinant of said linear map. To do so, we first define the pullback, or dual transformation, of a linear map.

Definition 3.7 ([1, Definition 6.29.2]). Given a linear map, $f : V \to W$, the pullback, $f^* : \Lambda^k(W) \to \Lambda^k(V)$ is defined such that if some $T \in \Lambda^k(W)$, then $f^*T(v_1, ..., v_k) = T(fv_1, ..., fv_k)$.

The pullback, much like its name suggests, is a means of "pulling back" a linear transformation from one vector space to another. It is using this terminology that we can present the aforementioned property of the wedge product: the determinant theorem, which we will state without proof.

Theorem 3.8 ([1, Definition 6.29.5]). *If $f : V \to V$ is a linear map, and $\theta_1, ..., \theta_k \in \Lambda^1(V)$, then $f^*\theta_1 \wedge ... \wedge f^*\theta_k = \det(f)\theta_1 \wedge ... \wedge \theta_k$.*

This theorem, which we will extend further in our discussion of differential forms, is reminiscent of the change of variables formula from multivariable calculus. The determinant of the linear map, $\det(f)$, is of the same form as the Jacobian used to change from Cartesian coordinates to other coordinate systems when integrating, a similarity that we note and will reference during our discussion of changes of variables in integration over manifolds.

4. Differential Forms

Having defined the wedge product, we are now able to begin examining the basics of the theory of differential forms: namely, tangent fields, tensor fields, and differential forms.

Definition 4.1 ([1, Definition 6.29.5]). For a manifold M, the union of the tangent spaces $T_x(M)$ for $x \in M$ is called the tangent bundle of M, which is itself a manifold and denoted $T(M)$. A tangent vector field to M is a continuous function $F : M \to T(M)$ such that $F(x) \in T_x(M)$ for each $x \in M$.

Definition 4.2 ([1, Definition 6.29.6]). Let A be an open set in $\mathbb{R}^n$. A k-tensor field in A is a function ω assigning to each $x \in A$ a k-tensor defined on the space $T_x(\mathbb{R}^n)$. That is,

$$\omega(x) \in \Lambda^k(T_x(\mathbb{R}^n))$$

for each x. For a given k-tuple of tangent vectors at an x, we require $\omega(x)$ to be continuous as a function of $(x, v_1, ..., v_k)$.

In essence, the tensor field $\omega(x)$ is a function that maps k-tuples of tangent vectors to $\mathbb{R}^n$ at an x into the space of $\mathbb{R}$. If $\omega(x)$ is an alternating k-tensor for each value of x, then ω is said to be a differential form of order k on A. We now state a more general definition of differential forms.

Definition 4.3 ([1, Definition 6.29.7]). Let M be an m-manifold in $\mathbb{R}^n$. A differential form of order k, or a k-form, on M is a function ω that assigns to each $p \in M$ an alternating k-tensor on the tangent space of M at p.

Perhaps the easiest differential forms to understand are 0-forms, which are simply real-valued functions over a manifold, also known as scalar fields, which should be familiar from multivariable calculus and its examinations of gradients, curl, and divergence. Differential forms of order greater than 0 are less familiar, but the introduction of a mechanism known as the differential operator both allows a greater understanding of such forms and allows us to manufacture any k-form.

Definition 4.4 ([1, Definition 2.5.5]). Let $A \subset \mathbb{R}^m$, and $f : A \to \mathbb{R}^n$. Suppose A contains a neighborhood of a. We say that f is differentiable at a if there exists an $n \times m$ matrix B such that

$$\frac{f(a+h) - f(a) - B \cdot h}{|h|} \to 0 \qquad \text{as} \qquad h \to 0.$$

B is called the derivative of f at a, and is denoted $Df(a)$.

Definition 4.5 ([1, Theorem 6.30.1]). Let A be open in $\mathbb{R}^n$, and $f : A \to \mathbb{R}$. A 1-form df on A is defined by the formula

$$df(x)(x; v) = f'(x; v) = Df(x) \cdot v$$

where $f'(x; v)$ is the directional derivative of f with respect to the vector v. We call d the differential operator, which is linear for 0-forms.

The differential operator can, therefore, be conceptualized as a more general form of the directional derivative discussed in multivariable calculus. We now seek to define an arbitrary k-form, and in doing so, explain some of the basic notation of differential forms, so as to avoid confusion.

Definition 4.6 ([1, Convention 6.30.3]). If $I = (i_1, ..., i_k)$ is an ascending k-tuple from the set $1, ..., n$, then an elementary k-form θ_I in $\mathbb{R}^n$ is denoted as

$$dx_I = dx_{i_1} \wedge ... \wedge dx_{i_k}.$$

The general k-form can therefore be expressed as

$$\omega = \sum_I b_I dx_I$$

for some scalar functions b_I, and the exterior derivative of ω, $d\omega$ is the $(k+1)$-form

$$d\omega = \sum_I db_I \wedge dx_I.$$

It is extremely important to note that while dx_i denotes a 0-form, dx_I denotes a k-form that is the wedge product of k 1-forms. To conclude our discussion of the differential operator, we introduce and prove a few of its useful properties.

Theorem 4.7 ([1, Theorem 6.30.4]). *The differential operator d, defined for forms of arbitrary order on manifolds with boundary, has the following properties:*

1. *$d(df) = 0$, if f is a 0-form.*
2. *If ω and η are arbitrary forms of orders k and l, respectively, then*

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.$$

3. *$d(d\omega) = 0$.*

Proof. 1. We have

$$d(df) = d\sum_{j=1}^{n} D_j f dx_j = \sum_{j=1}^{n} d(D_j f) \wedge dx_j = \sum_{j=1}^{n}\sum_{i=1}^{n} D_i D_j f dx_i \wedge dx_j.$$

We can then omit all terms for which $i = j$, which results in the following:

$$d(df) = \sum_{i<j} (D_i D_j f - D_j D_i f) dx_i \wedge dx_j.$$

Since $D_i D_j = D_j D_i$ by the equality of mixed partial derivatives, $d(df) = 0$.

2. First, we take $\omega = f$ and $\eta = g$ to both be 0-forms. Then

$$\begin{aligned}
d(\omega \wedge \eta) &= \sum_I d(f \cdot g) dx_I \\
&= \sum_W (df \cdot g + f \cdot dg) dx_I = \sum_I df \cdot g dx_I + \sum_I f \cdot dg dx_I \\
&= df \wedge f + f \wedge dg \\
&= d\omega \wedge \eta + \omega \wedge \eta.
\end{aligned}$$

We then let ω be a k-form and η be an l-form. Using the previously proven case and the anti-commutativity of the wedge product, we have

$$\begin{aligned}
d(\omega \wedge \eta) &= d(f \cdot g \ dx_I \wedge dx_J) \\
&= d((f \cdot g) \wedge d_I \wedge d_J) \\
&= ((df \wedge g + f \wedge dg) \wedge dx_I \wedge dx_J) \\
&= (df \wedge g \wedge dx_I \wedge dx_J) + (f \wedge dg \wedge dx_I \wedge dx_J) \\
&= (d \wedge dx_I) \wedge (g \wedge dx_J) + (-1)^k (f \wedge dx_I) \wedge (dg \wedge dx_J) \\
&= d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.
\end{aligned}$$

3. Since d is linear, it suffices to consider only the case $\omega = f dx_I$. Then

$$d(d\omega) = d(df \wedge dx_I) = d(df) \wedge dx_I - df \wedge d(dx_I)$$

Since $d(df) = 0$, we have

$$d(dx_I) = d(1) \wedge dx_I = 0$$

and hence, $d(d\omega) = 0$. □

5. Integration and Stokes' Theorem

We will end our overview of manifolds and differential forms with an examination of the operation most critical to expressing Stokes' Theorem: integration. It is here that the process of changing variables learned in multivariable calculus become particularly useful, as it helps to define such the same process over manifolds.

Theorem 5.1 ([1, Theorem 7.33.2]). *Let A be open in $\mathbb{R}^k$, and let $\alpha : A \to \mathbb{R}^n$ be an orientation-preserving diffeomorphism onto its image. The set of $Y = \alpha(A)$, together with the map α, constitute the parametrized manifold Y_α. If ω is a k-form defined in an open set of $\mathbb{R}^n$ containing Y, we define the integral of ω over Y_α by the equation*

$$\int_{Y_\alpha} \omega = \int_A \alpha^* \omega.$$

Proof. Let $\omega = a dx_1 ... dx_k$. Using the Determinant Theorem (Theorem 3.8), we know that $\alpha^* \omega = (a \circ f) \det(df) dy_1 \wedge .. \wedge dy_k$. Since α is orientation preserving, $\det(df) > 0$, so $|\det(df)| = \det(df)$. We recall the standard change of variables formula from multivariable calculus as

$$\int_{Y_\alpha} a dx_1 ... dx_k = \int_A (a \circ f) |\det(df)| dy_1 ... dy_k.$$

We substitute $\alpha^* \omega = (a \circ f) |\det(df)| dy_1 \wedge .. \wedge dy_k$ and $\omega = a dx_1 ... dx_k$ into the above equation, yielding our desired equation. □

We notice that this change of variables formula is very similar to the formula from multivariable calculus, with one notable exception: the Jacobian has seemingly vanished. It is this "disappearance," with the Jacobian forming a part of the pullback's calculation, that make differential forms especially useful as integrands. We now add the last definition necessary for our examination of Stokes' Theorem: the support of a differential form.

Definition 5.2 ([1, Defintion 7.33.5])**.** Let ω be a k-form on M, a k-dimensional manifold with boundary. The support of ω is the closure of the set of points where $\omega(x) \neq 0$.

The support of a differential form is therefore all points on the manifold M such that the form is non-zero when evaluated. Having developed the necessary tools to address the generalized version of Stokes' Theorem, we now prove this paper's primary focus. Before tackling Stokes' Theorem itself, however, we must first begin with a lemma, which is something of a special case of the theorem.

Lemma 5.3 ([1, Lemma 7.37.1])**.** *Let $k > 1$. Let η be a $k-1$ form defined in an open set U of $\mathbb{R}^k$ containing the unit k-cube I^k. Assume that η vanishes at all points of ∂I^k except possibly at points of the subset $(\text{Int}\, I^{k-1}) \times 0$. Then*

$$\int_{\text{Int}\, I^k} d\eta = (-1)^k \int_{\text{Int}\, I^{k-1}} b^*\eta,$$

where $b : I^{k-1} \to I^k$ is the map

$$b(u_1, ..., u_{k-1}) = (u_1, ..., u_{k-1}, 0)$$

Proof. We denote the general point of $\mathbb{R}^k$ as x and the general point of $\mathbb{R}^{k-1}$ as u. Given $1 \leq j \leq k$, we let I_j denote the $k-1$ tuple

$$I_j = (1, ..., \widehat{j}, ..., k)$$

Here, $\widehat{j}$ means that the term j is omitted. The typical elementary $k-1$ form in $\mathbb{R}^k$ is

$$dx_{I_j} = dx_1 \wedge ... \wedge \widehat{dx_j} \wedge ... \wedge dx_k$$

We notice that the differential operator d is linear, as is the pullback b^* of the previously defined map b. Since both integrals are also linear, it therefore suffices to prove only the special case where

$$\eta = f dx_{I_j}.$$

We therefore assume this value of η throughout the proof, and compute the integral

$$\int_{\text{Int}\, I^k} d\eta.$$

We note

$$\begin{aligned} d\eta &= df \wedge dx_{I_j} \\ &= (\sum_{i=1}^{k} D_i f dx_i) \wedge dx_{I_j} \\ &= (-1)^(j-1)(D_j f) dx_1 \wedge ... \wedge dx_k. \end{aligned}$$

We then compute

$$\int_{\text{Int}\, I^k} d\eta = (-1)^{j-1} \int_{\text{Int}\, I^k} D_j f$$

$$= (-1)^{j-1} \int_{I^k} D_j f$$

$$= (-1)^{j-1} \int_{v \in I^{k-1}} \int_{x_j \in I} D_j f(x_1, ..., x_k)$$

as the Fubini Theorem allows us to take the iterated integrals in any order. We take $v = (x_1, ..., \widehat{x_j}, ..., x_k)$. The Fundamental Theorem of Calculus allows us to compute the inner integral as

$$\int_{x_j \in I} D_j f(x_1, ..., x_k) = f(x_1, ..., 1, ..., x_k) - f(x_1, ..., 0, ..., x_k)$$

where the 0 and 1 both appear in the jth place. The form η, and therefore f vanish at all points of ∂I^k except possibly at points of the open bottom face $(\text{Int } I^{k-1}) \times 0$. If $j < k$, the right side of the equation vanishes, while if $j = k$, it equals

$$-f(x_1, ..., x_{k-1}, 0).$$

We then conclude that

$$\int_{\text{Int } I^k} d\eta = 0$$

if $j < k$, and

$$\int_{\text{Int } I^k} d\eta = (-1)^k \int_{I^{k-1}} (f \circ b)$$

if $j = k$. We then compute the other integral in the equation, $\int_{\text{Int } I^{k-1}} b * \eta$. The map b has derivative

$$Db = \begin{bmatrix} I^{k-1} \\ 0 \end{bmatrix}$$

We therefore have

$$b^*(dx_{I_j}) = (\det Db(1, ..., \widehat{j}, ..., k)) du_1 \wedge ... \wedge du_{k-1}$$

which is equal to 0 if $j < k$ and equal to $du_1 \wedge ... \wedge du_{k-1}$ if $j = k$. We then conclude that

$$\int_{\text{Int } I^{k-1}} b * \eta = 0$$

if $j < k$, and

$$\int_{\text{Int } I^{k-1}} b * \eta = \int_{I^{k-1}} (f \circ b)$$

if $j = k$. Comparing our results from the two computed integrals then yields our desired equation. □

It is with this lemma that we can approach the generalized form of Stokes' Theorem.

Theorem 5.4 ([1, Theorem 7.37.2]). *Let $k > 1$. Let M be a compact oriented k-manifold in $\mathbb{R}^n$. We give ∂M the induced orientation if it is not empty. Let ω be a $k-1$ form defined in an open set of $\mathbb{R}^n$ containing M. Then*

$$\int_M d\omega = \int_{\partial M} \omega$$

if ∂M is not empty, and $\int_M = 0$ if ∂M is empty.

Proof. We first cover M with carefully-chosen coordinate patches. For a first case, we assume that $p \in M - \partial M$. We choose a coordinate patch $\alpha : U \to V$ belonging to the orientation of M, such that U is open in $\mathbb{R}^k$ and contains the unit cube I^k. Such an α carries a point of $\operatorname{Int} I^k$ to the point p. We then let $W = \operatorname{Int} I^k$ and $Y = \alpha(W)$. The map $\alpha : W \to Y$ is therefore still a coordinate patch belonging to the orientation of M about p, with W open in $\mathbb{R}^k$.

As a second case, we assume that $p \in \partial M$. We then choose a coordinate patch $\alpha : U \to V$ belonging to the orientation of M such that U is open in the upper half space and contains I^k. We also let α carry a point of $(\operatorname{Int} I^{k-1}) \times 0$ to the point p. We then let

$$W = (\operatorname{Int} I^k) \cup ((\operatorname{Int} I^{k-1}) \times 0),$$

and let $Y = \alpha(W)$. The map $\alpha : W \to Y$ is then still a coordinate patch belonging to the orientation of M about p, with W open in the upper half space $\mathbf{H}^k$.

The differential operator and the integrals involved in the equation are all linear, so it suffices to prove the theorem only when ω is a $(k-1)$-form such that the set

$$C = M \cap (\operatorname{Support} \omega)$$

can be covered by a single coordinate patch. Since the support of $d\omega$ is contained in the support of ω, $M \cap (\operatorname{Support} d\omega)$ is contained in C and covered by the coordinate patch. We let η denote the form $\alpha^* \omega$. η vanishes at all points of ∂I^k except possibly the points of the bottom face, thus satisfying the requirements of the previously proven lemma.

We now seek to prove the theorem when C is covered by a coordinate patch of the first case we constructed. Since $\alpha^* d\omega = d\alpha^* \omega = d\eta$, we have, from the lemma,

$$\int_M d\omega = \int_{\operatorname{Int} I^k} \alpha^* d\omega = \int_{\operatorname{Int} I^k} d\eta = (-1)^k \int_{\operatorname{Int} I^{k-1}} b^* \eta.$$

We note that η vanishes outside $\operatorname{Int} I^k$, so $b^* \eta = 0$. Then $\int_M d\omega = 0$. If ∂M is empty, we have found the desired equation. If ∂M is non-empty, then the theorem holds trivially, since the support of ω is disjoint from ∂M, and the integral of ω over ∂M must be 0.

We then consider our second case. We compute, as before, that

$$\int_M d\omega = (-1)^k \int_{\operatorname{Int} I^{k-1}} b^* \eta.$$

We then seek to compute $\int_{\partial M} \omega$. The set $\partial M \cap (\operatorname{Support} \omega)$ is covered by the coordinate patch

$$\beta = \alpha \circ b : \operatorname{Int} I^{k-1} \to Y \cap \partial M.$$

on ∂M, which is found by restricting α. β belongs to the induced orientation if k is even, and the opposite orientation if k is odd. If we are to use β to compute our integral, we must reverse the sign of the integral when k is odd. We then have

$$\int_{\partial M} \omega = (-1)^k \int_{\operatorname{Int} I^{k-1}} \beta^* \omega.$$

$\beta^* \omega = b^*(\alpha^* \omega) = b^* \eta$, and substituting into the previous equation, we obtain

$$\int_{\partial M} \omega = (-1)^k \int_{\operatorname{Int} I^{k-1}} b^* \eta = \int_M d\omega.$$

□

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