

TOPOLOGICAL K -THEORY

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ABSTRACT. The goal of this paper is to introduce some of the basic ideas surrounding the theory of vector bundles and topological K -theory. To motivate this, we will use K -theoretic methods to prove Adams' theorem about the non-existence of maps of Hopf invariant one in dimensions other than $n = 1, 2, 4, 8$. We will begin by developing some of the basics of the theory of vector bundles in order to properly explain the Hopf invariant one problem and its implications. We will then introduce the theory of characteristic classes and see how Stiefel-Whitney classes can be used as a sort of partial solution to the problem. We will then develop the methods of topological K -theory in order to provide a full solution by constructing the Adams operations.

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1. INTRODUCTION AND BACKGROUND

1.1. **Preliminaries.** This paper will presume a fair understanding of the basic ideas of algebraic topology, including homotopy theory and (co)homology. As with most writings on algebraic topology, all maps are assumed to be continuous unless otherwise stated.

1.2. Vector bundles. The basic building block of K -theory is the *vector bundle*. Intuitively, we can think of a vector bundle as a way of assigning a vector space to each point of a topological space in a way that varies continuously. We give the full definition here:

Definition 1.1. A real *vector bundle* over a base space B is a topological space E called the *total space* along with a map $p : E \rightarrow B$ satisfying the following:

- (1) For any element $b \in B$, the fiber $p^{-1}(b)$ is homeomorphic to \mathbb{R}^k for some positive integer k .
- (2) (*Local Triviality*) There exists a collection of open subsets $\{U_\alpha\}$ of B such that every point $x \in B$ is contained in one of the U_α and there exists a homeomorphism $\varphi_\alpha : p^{-1}(U_\alpha) \rightarrow U_\alpha \times \mathbb{R}^k$ for some integer k .
- (3) (*Transition Maps*) For each pair (U_α, U_β) such that $U_\alpha \cap U_\beta \neq \emptyset$, the map $\varphi_\alpha^{-1} \circ \varphi_\beta : p^{-1}(U_\alpha \cap U_\beta) \rightarrow p^{-1}(U_\alpha \cap U_\beta)$ is a linear isomorphism.

A vector bundle is said to have *dimension* n if the fiber over every point is homeomorphic to \mathbb{R}^k . Note in particular that the dimension of a vector bundle need not necessarily be defined if the base space is not connected.

Notation 1.2. We will denote a vector bundle $p : E \rightarrow B$ as one of the following, which will be clear based on context:

- (1) The triple (p, E, B)
- (2) The total space E
- (3) The map p

Remark 1.3. We can also define complex vector bundles in exactly the same way, except we require that the fibers be \mathbb{C}^n instead of \mathbb{R}^n .

For the purpose of this paper, we will be assuming that the base space B is compact and Hausdorff. Though some results may only require strictly weaker assumptions, such as paracompactness, we will nevertheless follow the convention of [1] and use this stronger assumption.

Example 1.4. We can define the *trivial bundle* over a space X to be simply the product $X \times \mathbb{R}^n$.

Example 1.5. Consider the *tangent bundle* τM of an n -manifold M . This example is of fundamental importance in differential topology as well as algebraic topology. It assigns to each point the n -dimensional vector space lying tangent to the manifold. For low-dimensional manifolds, we can easily visualize the tangent bundle.

Consider, for example, τS^1 . Points in the tangent bundle are pairs (x, v) , with x representing a point on the circle and v representing a point on the tangent line at the point x . It is possible to exhibit a diffeomorphism from τS^1 to the cylinder $S^1 \times \mathbb{R}$, so we see in particular that this tangent bundle is not only locally trivial but globally trivial.

Example 1.6. Consider the line bundle over S^1 obtained by attaching lines to it with a “twist”. More concretely, we can visualize S^1 embedded in \mathbb{R}^3 and assign to each point on the circle a line in a way such that the orientation of the lines changes when one goes around the circle once by starting with a vertical line at $\theta = 0$ and rotating it by $\frac{\theta}{2}$ at the angle θ . This is called the *Möbius bundle*, as one can visualize it like the Möbius band. One important property of this vector bundle

that was not the case with either of the previous example is that it is nontrivial - that is, it is not homeomorphic to $S^1 \times \mathbb{R}$.

Because the fibers are finite-dimensional vector spaces, vector bundles inherit a few important operations from vector spaces by applying the operations fiber-wise. Examples include:

- (1) The *direct sum* or *Whitney sum* operation takes vector bundles E_1 and E_2 of dimension m and n over the same base space B and returns a vector bundle $E_1 \oplus E_2$ of dimension $m+n$. This is obtained by applying the direct sum operation fiber-wise.
- (2) The *tensor product* operation takes vector bundles E_1 and E_2 of dimension m and n over the same base space B and returns a vector bundle $E_1 \otimes E_2$ of dimension mn by applying the tensor product operation fiberwise.
- (3) The *exterior power* of a vector bundle. Given a vector bundle E of dimension n , the k -th exterior power $\Lambda^k(E)$ returns a vector bundle of dimension $\binom{n}{k}$ whose fibers are homeomorphic to $\Lambda^k(\mathbb{R}^n)$.
- (4) The *Hom-bundle* of two vector bundles E and F of dimension m and n over the same base space B gives a vector bundle $\text{Hom}(E, F)$ of dimension mn . It is defined such that the fiber over a given point $x \in B$ is the set of linear maps from the fiber E_x to the fiber F_x .

As with many types of mathematical objects, vector bundles form a category which we will denote **Vect**. The objects will be vector bundles and the morphisms from a vector bundle (p_1, E_1, B_1) to another vector bundle (p_2, E_2, B_2) will be pairs (f, g) with $f : B_1 \rightarrow B_2$ and $g : E_1 \rightarrow E_2$ such that the following diagram commutes:

$$\begin{array}{ccc} E_1 & \xrightarrow{g} & E_2 \\ \downarrow p_1 & & \downarrow p_2 \\ B_1 & \xrightarrow{f} & B_2 \end{array}$$

and, for a given fiber $p_1^{-1}(b)$, the map g restricted to that fiber is a linear map between vector spaces.

As a category, vector bundles naturally inherit an equivalence relation of isomorphism. A morphism (f, g) of vector bundles is an *isomorphism* if and only if f is a homeomorphism and g restricts to an isomorphism on each fiber. Two vector bundles over the same base space B are said to be *equivalent* if they are isomorphic by a morphism (id_B, g) . Let $\mathcal{E}_n(B)$ denote the set of equivalence classes of n -dimensional vector bundles over B .

Remark 1.7. Let $f : X \rightarrow Y$ be a continuous map and let $p : E \rightarrow Y$ be a vector bundle. Then we can define a vector bundle f^*E over X as the pullback of the following diagram:

$$\begin{array}{ccc} f^*E & \dashrightarrow & E \\ \downarrow & & \downarrow p \\ X & \xrightarrow{f} & Y \end{array}$$

Thus, one can think of $\mathcal{E}_n(-)$ as a contravariant set-valued functor from the category **Top** of topological spaces to the category **Vect**. The vector bundle f^*E is appropriately called the *pullback bundle*. We could similarly define a functor $\mathcal{E}(-)$ which gives all vector bundles over a given topological space.

This categorical perspective is a useful tool for understanding a lot of the basic results surrounding vector bundles. For example, we will show in the next section that in fact this functor is representable. Before that, though, there is one important result involving pullback bundles that we will refer to multiple times in this paper:

Lemma 1.8. (*The Splitting Principle*) *Let $\xi : E \rightarrow X$ be a vector bundle of dimension n . Then there exists a space $F(E)$ (called the flag bundle of E) and a map $p : Y \rightarrow X$ such that:*

- (1) *The induced homomorphism of cohomology rings $p^* : H^*(X) \rightarrow H^*(Y)$ is injective*
- (2) *The pullback bundle p^*E can be written as the direct sum of line bundles over $F(E)$*

This useful lemma allows us to prove statements and give constructions involving arbitrary vector bundles by reducing to the case of line bundles.

1.3. Classifying Spaces. One of the basic problems in the theory of vector bundles is to classify all vector bundles over a space X . The main result that we will discuss is the following theorem:

Theorem 1.9. *There exists a space $BO(n)$ such that there is a bijection between $\mathcal{E}_n(X)$ and homotopy classes of maps $[X, BO(n)]$. Furthermore, there exists a vector bundle $p : EO(n) \rightarrow BO(n)$ such that any vector bundle can be realized as the pullback bundle of a homotopy class of maps into $BO(n)$ along this bundle. Furthermore, there is a space BO and a vector bundle $p : EO \rightarrow BO$ such that there is a bijection between $\mathcal{E}(X)$ and $[X, BO]$.*

The goal of this section is to explicitly state what the space $BO(n)$ is and to understand the motivation behind the theorem. The first important result for understanding classifying spaces is the following:

Lemma 1.10. *Let $f, g : X \rightarrow Y$ be two homotopic maps and let $p : E \rightarrow Y$ be a vector bundle. Then the pullback bundles f^*E and g^*E are equivalent.*

The idea of this is a fairly straightforward verification: because a homotopy is a map $H : X \times I \rightarrow Y$, we want to look at vector bundles over the cylinder $X \times I$, which are locally trivial, and then use compactness to show that we can deform one vector bundle to the other in finitely many steps through isomorphisms. Then the pullback bundles can be seen as the restrictions of the bundle H^*E to the opposite ends of the cylinder, which have to be equivalent by this argument.

Equipped with this fact, we can now turn to the problem of determining what our space $BO(n)$ is. One way to think about what this space might be is to think of vector bundles as assigning a vector space to each point. Thus, we want our space to be the “space of all n -dimensional vector spaces”. As it turns out, we can define such an object, which is called a *Grassmannian*, and it turns out to be a manifold and a CW-complex.

Definition 1.11. Let X be a vector space and define the *Stiefel manifold* $V_n(X)$ to be the set of all ordered sets of n linearly independent vectors in X . To topologize this space, we can think of it as a subspace of $X \oplus X \oplus \dots \oplus X$. We then define an equivalence relation defined by allowing two elements of $V_n(X)$ to be equivalent if they span the same subspace of X . We then define the *Grassmannian* $Gr_n(X)$ as the quotient of $V_n(X)$ by this relation.

Note that, in the case of where $X = \mathbb{R}^{n+k}$, this is equivalent to quotienting out by an action of $O(n)$ on the Stiefel manifold. To see this, we simply note that the group of transformations taking a basis for an n -dimensional vector space to another basis is precisely $O(n)$ and there is a well-defined action precisely because we defined the Stiefel manifold to consist of ordered n -tuples.

To see how this connects to the problem of classifying vector bundles, we can think of vector bundles as a way of assigning a vector space to each point in the base space. In the case of an n -dimensional vector bundle, this is precisely a map $B \rightarrow Gr_n(\mathbb{R}^{n+k})$ for some integer $k \geq 0$. However, we cannot necessarily bound the value of k . The solution to this is to take the direct limit of this process. Specifically, the natural inclusions $\mathbb{R}^{n+k} \hookrightarrow \mathbb{R}^{n+k+1}$ give inclusions $Gr_n(\mathbb{R}^{n+k}) \hookrightarrow Gr_n(\mathbb{R}^{n+k+1})$. Passing to the direct limit of this gives a space $Gr_n(\mathbb{R}^\infty)$, which is a union of the $Gr_n(\mathbb{R}^{n+k})$ over non-negative integers k . This space is precisely the classifying space $BO(n)$ from the theorem above. We will refer to this space as the *infinite Grassmannian*.

Remark 1.12. There is an entirely analogous construction for the space $BU(n)$, the classifying space for complex vector bundles using the complex Grassmannian for \mathbb{C}^∞ .

Of note is the fact that, in our theorem, we claimed that there was a canonical bundle $p : EO(n) \rightarrow BO(n)$ that any vector bundle could be realized as the pullback of. We will give the construction of this bundle, which we will refer to as the *universal* or *canonical bundle*:

Construction 1.13. We will define our total space $EO(n)$ as the infinite Stiefel manifold $V_n(\mathbb{R}^\infty)$, which can be defined as the direct limit of the $V_n(\mathbb{R}^{n+k})$ in the same sense as the infinite Grassmannian. Then, as before, there is an action of $O(n)$ on this space and $BO(n)$ is the orbit space $EO(n)/O(n)$. Then the projection map into the orbit space will specify a vector bundle. To see this, note that the pre-image of a point is the set of all orthogonal transformations under which an n -dimensional subspace is stable. This is the same as the set of all linear endomorphisms of this subspace, which is itself an n -dimensional vector space, so this does specify a vector bundle.

Another important question is to see if there is a classifying space for all vector bundles, not just vector bundles of a fixed dimension. As noted before, a vector bundle over a space B that is not connected does not necessarily have a dimension defined for the whole space. In categorical terms, if $\mathcal{E}_n(-)$ is representable as $[-, BO(n)]$, the question is to see if the functor $\mathcal{E}(-)$, sending a space to the set of all real vector bundles over it, is similarly representable. It turns out that we can define a classifying space as the disjoint union of all of the $BO(n)$, because over any given component of the base space we will have a vector bundle of some fixed dimension. Thus, we define our classifying space for $\mathcal{E}(-)$ as the union $BO = \cup BO(n)$.

2. THE HOPF INVARIANT

Now that we have some preliminary information about vector bundles, we can define the Hopf invariant one problem and understand some of its consequences in both algebra and the theory of vector bundles. The Hopf invariant takes maps $f : S^{2n-1} \rightarrow S^n$ and associates to them an integer $h(f)$. The idea is as follows: We can think of S^n as a CW complex with one 0-cell and one n -cell. Then we can think of f as an attaching map of a $2n$ -cell. Call the resulting CW complex $C_f = S^n \cup_f D^{2n}$. Then, if we look at the cohomology ring of C_f , the groups $H^n(C_f)$ and $H^{2n}(C_f)$ will be free groups on one generator (that is, they are isomorphic to \mathbb{Z}). Let $\alpha \in H^n(C_f)$ and $\beta \in H^{2n}(C_f)$ be generators. Then we can use the cup product to get that $\alpha \smile \alpha = h\beta$ for some integer h which depends only on the homotopy class of f .

Definition 2.1. The integer h is called the *Hopf invariant* of the map f . Note in particular that it is only defined up to sign depending on the choice of generator.

The fundamental result that this paper will prove is the following, which we will refer to as Adams' Theorem:

Theorem 2.2. (Adams) *Let $f : S^{2n-1} \rightarrow S^n$ be a map such that $h(f) = \pm 1$. Then $n = 1, 2, 4, 8$.*

Before we go in to the proof, we will discuss some consequences of this result.

2.1. H -spaces, division algebras, and tangent bundles of spheres. An H -space is a topological space equipped with a specific type of multiplicative structure.

Definition 2.3. Let X be a topological space and let $\mu : X \times X \rightarrow X$ be a continuous multiplication map with an identity element. Then (X, μ) is called an H -space.

As a consequence of Adams' Theorem, we have the following theorem:

Theorem 2.4. *The sphere S^n can be given an H -space structure if and only if $n = 0, 1, 3, 7$.*

The idea of the proof is to show that an H -space multiplication induces a map of Hopf invariant one. To do this we will use what is known as the *Hopf construction*:

Construction 2.5. Let $\mu : S^{n-1} \times S^{n-1} \rightarrow S^{n-1}$ be a map. We will construct a corresponding map $\hat{\mu} : S^{2n-1} \rightarrow S^n$. To construct such a map, note that we can identify $S^{2n-1} = \partial(D^{2n}) \cong \partial(D^n \times D^n) = D^n \times \partial(D^n) \cup \partial(D^n) \times D^n$. Additionally, we can think of S^n as the union of two disks D_+^n and D_-^n whose intersection is the equator. Then we can define the map $\hat{\mu}$ by its restrictions to $D^n \times \partial(D^n)$ and $\partial(D^n) \times D^n$. On the first, we can define $\hat{\mu}(x, y) = |x|\mu(x/|x|, y) \in D_+^n$ and we can define it similarly on the other component by letting it be $|y|\mu(x, y/|y|) \in D_-^n$.

This map is well defined because, on the intersection of the two components, we have $|x| = |y| = 1$ and the map is simply $\mu(x, y)$ in each component. Because $|\mu(x, y)| = 1$, this sits on the equator, which is where the two disks are identified and this is well-defined. It is also continuous on each component and thus continuous on the entire space. This is therefore the desired map.

We can now formulate the following claim:

Claim 2.6. *Let $\mu : S^{n-1} \times S^{n-1} \rightarrow S^{n-1}$ be an H -space multiplication map. Then $h(\hat{\mu}) = \pm 1$.*

Proof. The idea of the proof is in the following commutative diagram:

$$\begin{array}{ccc}
 C_f & \xrightarrow{\Delta} & C_f \wedge C_f \\
 \downarrow & & \downarrow \\
 C_f/S^n & \xrightarrow{\Delta} & C_f/D_+^n \wedge C_f/D_-^n \\
 \uparrow \Phi & & \uparrow \Phi \wedge \Phi \\
 D^{2n}/\partial(D^{2n}) \cong S^{2n} & \xrightarrow{\Delta} & (D^n \times D^n)/(\partial(D^n) \times D^n) \wedge (D^n \times D^n)/(D^n \times \partial(D^n)) \\
 & \searrow & \downarrow \rho_1 \wedge \rho_2 \\
 & & D^n/\partial(D^n) \wedge D^n/\partial(D^n)
 \end{array}$$

where Δ represents the diagonal map, Φ is induced by the inclusion of the $2n$ -cell, the ρ_i represent projection in the Cartesian product, and all other maps are quotient maps. The idea is then to take a generator $\alpha \in H^n(C_f)$. Then $\alpha \smile \alpha$ is the image of $\alpha \wedge \alpha$ under the induced map Δ^* in the top row. Note additionally that the maps in the left column induce isomorphisms on $H^{2n}(C_f)$ by construction. Specifically, the top map only collapses the n -cell, so it leaves a higher dimensional cohomology unaffected and the lower map Φ is the inclusion map.

Next, we claim that the maps ρ_i are actually homotopy equivalences. The homotopy inverse of ρ_i will be ι_i , the inclusion into the i -th term of the product. Additionally, because we are assuming that μ is an H -space multiplication, we know that Φ restricts to a homeomorphism on $D^n \times \{e\}$ and likewise on $\{e\} \times D^n$. This, the right column also consists of isomorphisms in cohomology. Thus, a generator $\alpha \otimes \alpha$ for $H^n(C_f) \otimes H^n(C_f)$ will map to a generator in the bottom row, which by definition implies that $\alpha \smile \alpha = \pm\beta$, so this is a map of Hopf invariant one. \square

This, combined with Adams' Theorem, proves the theorem about H -space multiplications.

An idea closely related to the H -space multiplication is that of a division algebra.

Definition 2.7. Let D be an algebra over a field. D is said to be a *division algebra* if, for any non-zero element $a \in D$ there exists a unique element $a^{-1} \in D$ satisfying $aa^{-1} = a^{-1}a = 1$. This differs from the definition of a field in that multiplication is not required to be commutative or associative.

The theorem about division algebras that we would like to prove is the following:

Theorem 2.8. *There exists a division algebra structure on \mathbb{R}^n if and only if $n = 1, 2, 4, 8$.*

The idea of this proof is to show that a division algebra structure induces an H -space structure on the unit sphere.

Proof. Suppose $\mu : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a division algebra map. Then define a multiplication on the sphere S^{n-1} by embedding it as the unit sphere in \mathbb{R}^n and letting $g(x, y) = \mu(x, y)/|\mu(x, y)|$. This is precisely the desired H -space structure.

We now claim that division algebras do exist for these dimension. For $n = 1$, this is simply \mathbb{R} endowed with its standard multiplication. Similarly, for $n = 2$, we have \mathbb{C} . For $n = 4$, we have the quaternions (denoted \mathbb{H}), which consist of numbers of the form $a + bi + cj + dk$. One interesting note is that this is not a commutative division algebra. Finally, we have, for $n = 8$, the Cayley octonions \mathbb{O} . This system is neither commutative nor associative, but has all other desired properties. \square

In addition to giving us information about algebraic structures, Adams' Theorem gives us information about the tangent bundles of spheres.

Definition 2.9. A manifold M is said to be *parallelizable* if τM is trivial.

We claim the following result:

Theorem 2.10. *The sphere S^n is parallelizable if and only if $n = 0, 1, 3, 7$.*

Once again, we will seek to show that having a trivial tangent bundle induces an H -space multiplication on the sphere. We will make use of the following lemma:

Lemma 2.11. *Suppose M is a parallelizable n -manifold. Then there exist n tangent vector fields that are linearly independent at each point.*

Proof. The proof is clear - τM is homeomorphic to $M \times \mathbb{R}^n$, so we can simply take the vector fields given by $M \times \{e_i\}$ and achieve the desired result. \square

Using this lemma, we can use the action of $SO(n)$ on the sphere to induce an H -space multiplication.

Proof. Suppose S^{n-1} is parallelizable and embedded in \mathbb{R}^n . Then define v_1, \dots, v_{n-1} to be tangent vector fields at each point. We can apply the Gram-Schmidt process to $x, v_1(x), \dots, v_{n-1}(x)$ to assure that they are orthonormal at the point x . Then define a linear transformation $\alpha_x \in SO(n)$ to be the transformation sending the canonical basis e_1, \dots, e_n to $x, v_1(x), \dots, v_{n-1}(x)$. We claim that the map sending (x, y) to $\alpha_x(y)$ is an H -space multiplication. It clearly has identity e_1 and is continuous, so the claim holds. Therefore, this induces a map of Hopf invariant one and, by Adams' Theorem, we know that such a vector field can only exist if $n = 0, 1, 3, 7$, as claimed. \square

3. CHARACTERISTIC CLASSES

One partial solution to the Hopf invariant one problem comes from the theory of characteristic classes. The idea of characteristic classes is to assign an invariant to vector bundles over a given topological space. Because we have related the Hopf invariant to tangent bundles over spheres, we can compute characteristic classes associated to these vector bundles to show whether or not they are trivial.

3.1. Definition and construction.

Definition 3.1. Let h^* be a cohomology theory. A degree q *characteristic class* c of n -dimensional vector bundles ξ over a space X is a way of assigning a cohomology class $c(\xi) \in h^q(X)$ that is natural (that is, it commutes with morphisms of vector bundles).

For the purpose of this paper, we will be working with a specific characteristic class - the Stiefel-Whitney class.

Theorem 3.2. *Let ξ be as before. Then there are unique characteristic classes $w_k(\xi) \in H^k(X; \mathbb{Z}_2)$ satisfying the following four properties:*

- (1) $w_0(\xi) = 1$ and $w_k(\xi) = 0$ if $k > \dim \xi$
- (2) $w_1(\gamma_1) \neq 0$, where γ_1 is the canonical line bundle over $\mathbb{R}P^\infty$
- (3) $w_k(\xi \oplus \varepsilon) = w_k(\xi)$, where ε is the trivial line bundle
- (4) $w_k(\xi \oplus \eta) = \sum_{i+j=k} w_i(\xi) \smile w_j(\eta)$

These are called the Stiefel-Whitney classes of ξ . Additionally, every mod 2 characteristic class can be written uniquely as a polynomial in the Stiefel-Whitney classes.

To give a full proof of this statement would require machinery beyond the scope of this paper, but we can fairly easily describe the Stiefel-Whitney classes of line bundles: to each line bundle ξ , we can associate a homotopy class of maps $X \rightarrow BO(1) = \mathbb{R}P^\infty$. Let f be a representative of this homotopy class. It is a fact that $H^*(\mathbb{R}P^\infty; \mathbb{Z}_2) \approx \mathbb{Z}_2[\alpha]$ where α is of degree 1. We then let $w_1(\xi) = f^*(\alpha)$ be the pullback of the generator along the map. Letting $w_0(\xi) = 1$ and $w_k(\xi) = 0$ for $k > 1$ gives us precisely the desired characteristic classes.

Note that the choice of mod 2 cohomology is significant here - we can only define $w_1(\xi)$ uniquely because there is a unique generator for mod 2 cohomology. This is related to the fact that every vector bundle admits a unique mod 2 orientation.

Of particular interest to this paper is what is known as the total Stiefel-Whitney class, which has nicer properties surrounding direct sums.

Definition 3.3. Let ξ be a vector bundle. Define the *total Stiefel-Whitney class* $w(\xi) \in H^*(X; \mathbb{Z}_2)$ as the formal sum of all of its Stiefel-Whitney classes - that is, $w(\xi) = \sum_{i=0}^{\dim \xi} w_i(\xi)$.

The advantage of looking at total Stiefel-Whitney classes is the following result:

Lemma 3.4. *Let ξ and η be vector bundles over the same space. Then $w(\xi \oplus \eta) = w(\xi) \smile w(\eta)$.*

This leads us to an important application of Stiefel-Whitney classes. Let M be an arbitrary manifold with tangent bundle τM and normal bundle ν . (Note that a normal bundle in particular requires M to be embedded or immersed in \mathbb{R}^n for some n .) Then we have that $\tau M \oplus \nu$ is the trivial bundle over M . Thus, by the previous lemma we get that $w(\tau M) \smile w(\nu) = 1$. This result is sometimes referred to as *Whitney duality*. Because τM does not depend on an immersion, this allows us to place a lower bound on the dimension of the normal bundle by computing the Stiefel-Whitney class of the tangent bundle and finding its multiplicative inverse.

3.2. Real projective space. This also suggests a possible approach to the problem about spheres being parallelizable: if we can show that the tangent bundle to a sphere has a non-trivial Stiefel-Whitney class, then the sphere cannot be parallelizable.

Example 3.5. Consider the sphere S^n with the standard embedding in \mathbb{R}^{n+1} . Then its normal bundle associates to each point x the vector x , so it is trivial. Thus, by Whitney duality, we have $w(\tau S^n) = 1$.

This, unfortunately, does not help us with the problem, as it shows that any sphere could be parallelizable. Instead, we will need another approach. The idea is to focus on something with a more interesting cohomology ring - namely, real projective spaces. If we view $\mathbb{R}P^n$ as a quotient space of a sphere, then a sphere with trivial tangent bundle should induce a trivial tangent bundle on projective space. This will be the goal of this section. We will compute the Stiefel-Whitney class of real projective space in order to find which sphere are non-parallelizable. The central claim we will seek to prove is the following:

Claim 3.6. *The total Stiefel-Whitney class of real projective space is given by $w(\tau\mathbb{R}P^n) = (1 + \alpha)^{n+1} = \sum_{i=0}^n \binom{n+1}{i} \alpha^i$*

Proof. We will make use of the fact that $H^*(\mathbb{R}P^n; \mathbb{Z}_2) \approx \mathbb{Z}_2[\alpha]/(\alpha^{n+1})$. Let γ denote the canonical line bundle. Then, by definition, $w_1(\gamma) = \alpha$ and $w_i(\gamma) = 0$ for $i > 1$. Therefore $w(\gamma) = 1 + \alpha$. Let γ^\perp be the orthogonal complement of γ in \mathbb{R}^{n+1} . By the product formula, we have that $w(\gamma^\perp) = 1 + \alpha + \dots + \alpha^n$.

To finish the proof, we will need a few facts about Hom-bundles. Note that $\text{Hom}(\gamma, \gamma) = \varepsilon^1$, as the set of vector space homomorphisms from a line bundle to itself is trivial. Additionally, note that $\text{Hom}(\gamma, \varepsilon) = \gamma$. Next, we claim that $\tau\mathbb{R}P^n = \text{Hom}(\gamma, \gamma^\perp)$. To see this, let $(x, v) \in \mathbb{R}P^n$. Note that, if we consider this instead as a point in τS^n , then this is the same point as $(-x, -v)$. Thus, we can consider the line L_x through the origin containing x . Then there is a linear map $f : L_x \rightarrow L_x^\perp$ that totally determines the point in (x, v) . This correspondence shows that the tangent bundle is the desired Hom-bundle. Thus, using some of the basic properties of Hom-bundles, we arrive at the following result:

$$\begin{aligned} \tau\mathbb{R}P^n \oplus \varepsilon &= \text{Hom}(\gamma, \gamma^\perp) \oplus \text{Hom}(\gamma, \gamma) \\ &= \text{Hom}(\gamma, \gamma \oplus \gamma^\perp) \\ &= \text{Hom}(\gamma, \varepsilon^{n+1}) \\ &= \bigoplus_1^{n+1} \gamma \end{aligned}$$

Thus, by multiplicativity of total Stiefel-Whitney classes, we get $w(\tau\mathbb{R}P^n) = (w(\gamma))^{n+1} = (1 + \alpha)^{n+1}$, as claimed. \square

Using this fact, we can now see the following result, which gives us a partial solution to problem about parallelizable spheres:

Theorem 3.7. *Suppose that S^{n-1} is parallelizable. Then $n = 2^k$ for some integer k .*

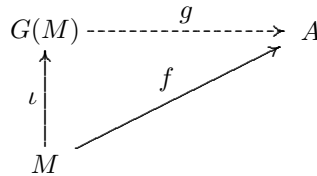
Proof. We will show that, for other values of n , the corresponding $(n - 1)$ -sphere will be non-parallelizable because the corresponding real projective space $\mathbb{R}P^{n-1}$ will have at least one non-trivial Stiefel-Whitney class. Suppose that $n \neq 2^k$. Then we claim that $\binom{n}{i}$ will be odd for some integer $0 < i < n$. This would imply the desired result, as there would be a non-trivial Stiefel-Whitney class. Let $i = 2^\ell$ be the largest power of 2 strictly less than n . We claim that $\binom{n}{i}$ is odd. Note that, modulo 2^ℓ , the numerator and the denominator will be identical and therefore will contain the same powers of two, so the desired result holds. \square

4. K-THEORY

The solution to the Hopf invariant one problem arises most easily in the field of topological K -theory. The idea of this approach is to once again consider vector bundles over a topological space X . As we will see later, it is most useful to consider complex vector bundles. Then, using the direct sum and tensor product operations discussed earlier, we would like to induce a ring structure on the set of vector bundles. Notably, the direct sum operation does not admit additive inverses, so these operations define a semi-ring rather than a ring, so we will appeal to what is known as the *Grothendieck construction* to form a ring. We will eventually explore the basic structure of this ring and show that it has a special periodicity property known as Bott periodicity. We will then construct natural operations called the *Adams operations*, which we will use to prove Adams' theorem.

4.1. **The ring $K(X)$.** We will begin by describing the Grothendieck construction mentioned in the introduction to this section.

Claim 4.1. *Let (M, \oplus) be an abelian monoid. Then there exists an abelian group $(G(M), +)$ along with a monoid homomorphism $\iota : M \rightarrow G(M)$ satisfying the following universal property: any monoid homomorphism $f : M \rightarrow A$ with A an abelian group factors through a group homomorphism $g : G(M) \rightarrow A$ such that $f = g \circ \iota$. Additionally, if M is equipped with a product \otimes that makes it a semi-ring, $G(M)$ inherits a product $*$ that makes it a ring.*



The idea of the construction is to take pairs of elements in what essentially amount to equivalence classes of formal differences. One motivating example is the monoid of natural numbers $(\mathbb{N}, +)$. The Grothendieck group $G(\mathbb{N})$ is precisely the integers $(\mathbb{Z}, +)$, which is in some sense the “least” group that contains the natural numbers. The elements of \mathbb{Z} can be thought of as equivalence classes of formal differences $[a - b]$, where two such formal differences are equivalent if they are actually equal as subtraction on the integers. With this in mind, we describe the general construction:

Construction 4.2. Let S be the set consisting all pairs (m_1, m_2) with $m_i \in M$. We construct a monoid T by defining the group operation $(m_1, m_2) + (n_1, n_2) =$

$(m_1 \oplus n_1, m_2 \oplus n_2)$. Next, we define an equivalence relation on T as follows: the pair (m_1, m_2) is equivalent to (n_1, n_2) if there exists some element $k \in M$, we have $m_1 \oplus n_2 \oplus k = m_2 \oplus n_1 \oplus k$. We then let $G(M)$ be the set of equivalence classes of T under this relation.

This inherits a group operation from $(T, +)$, as the addition operation is compatible with the equivalence relation. This is precisely the Grothendieck group $G(M)$. The Grothendieck group has identity element $[(e, e)]$ and the inverse of $[(m_1, m_2)]$ is $[(m_2, m_1)]$.

To see that this also inherits a product operation, we define:

$$[(m_1, m_2)] \times [(n_1, n_2)] = [((m_1 \otimes m_2) \oplus (n_1 \otimes n_2), (m_1 \otimes n_2) \oplus (m_2 \otimes n_1))]$$

The operation is defined in this way precisely so that it is distributive over the group operation. In particular, we have defined a ring $G(M)$, as desired.

We are now ready to define the K -theory of a topological space.

Definition 4.3. Let X be a topological space. Let $\text{Vect}(X)$ be the semi-ring of all equivalence classes of vector bundles over X equipped with the direct sum and tensor product operations. We define the K -theory of X as the Grothendieck ring $K(X) = G(\text{Vect}(X))$. The elements of $K(X)$ are written as $[\xi - \eta]$ and referred to as *virtual bundles* over X .

Remark 4.4. Note that we could have defined an analogous construction using real vector bundles. This ring is generally referred to as $KO(X)$ because of its connection with the orthogonal group. Because of this, some sources might rarely refer to $K(X)$ as $KU(X)$.

Notation 4.5. Because ε is the multiplicative identity, it is conventional to write $[\varepsilon^n] = n$.

Example 4.6. Suppose $X = \{*\}$, the set of a single element. Then there is exactly one vector bundle in each dimension (the trivial one). Then $m \otimes n = mn$ and $m \oplus n = m + n$, so $K(X) \approx \mathbb{Z}$.

We should first recall the following fact:

Proposition 4.7. *We can think of $K(-)$ as a contravariant functor from the category of topological spaces \mathbf{Top} to the category of rings \mathbf{Ring} . Contravariance comes from the fact that $\mathcal{E}(-)$ is a contravariant set-valued functor from \mathbf{Top} to \mathbf{Vect} .*

As with many cohomology theories, it is often useful to talk about a reduced cohomology for based spaces, which we will define in the following way:

Definition 4.8. Let X be a based topological space. Let $d : K(X) \rightarrow \mathbb{Z}$ send a vector bundle to the dimension of its fiber over the basepoint. Define the *reduced K -theory* $\tilde{K}(X)$ of a topological space to be $\ker(d)$.

This defines a contravariant functor from the category \mathbf{Top}^* of based topological spaces to \mathbf{Rng} (that is, the category of rings, not necessarily with identity).

Example 4.9. Consider again $X = \{*\}$. Then the kernel of the dimension map is the single zero-dimensional vector bundle, so the reduced K -theory of a singleton is the zero ring.

Example 4.10. Let $X = S^0$, the set of two points. Then we can assign separate bundles to each component. Fix one of the components to be the basepoint. Then the kernel of the dimension map will have one bundle in each dimension over the other component, so $\tilde{K}(X) \approx \mathbb{Z}$.

We will define a couple of the basic constructions that we will need in K -theory. First is the notion of an external product.

Definition 4.11. Define the *external product* $\mu : K(X) \otimes K(Y) \rightarrow K(X \times Y)$ to be the map given by $\mu(\alpha \otimes \beta) = \rho_1^*(\alpha)\rho_2^*(\beta)$, where ρ_i^* are the homomorphisms induced by the projections.

It is possible to define a similar reduced external product, but we will first need more machinery. For the purpose of this next lemma, because we are working with pointed spaces, CX will denote the reduced cone $(X \times I)/((X \times \{1\}) \cup (\{*\} \times I))$ and ΣX will denote the reduced suspension, which is the double reduced cone:

Lemma 4.12. *Let $A \subset X$ be closed. Consider the maps $A \hookrightarrow X \rightarrow X/A$ given by inclusion and projection. Then the induced sequence of \tilde{K} -rings is exact. Furthermore, this can be extended to a long exact sequence:*

$$\dots \rightarrow \tilde{K}(\Sigma(X/A)) \rightarrow \tilde{K}(\Sigma X) \rightarrow \tilde{K}(\Sigma A) \rightarrow \tilde{K}(X/A) \rightarrow \tilde{K}(X) \rightarrow \tilde{K}(A)$$

Proof. Let ι be the inclusion and let ρ be the projection. That $\text{im}\rho^* \subset \ker\iota^*$ is fairly straightforward. Note that the map $\rho \circ \iota$ factors through A/A and is therefore trivial, so $\iota^* \circ \rho^*$ is also trivial. The opposite inclusion will take more work to show. Let $\eta : E \rightarrow X \in \ker\iota^*$. This means that η restricts to a trivial vector bundle over A . Define a homeomorphism $h : p^{-1}(A) \rightarrow A \times \mathbb{C}^n$. Then define the quotient space E/\sim by quotienting out by the relation $h^{-1}(x, v) \sim h^{-1}(y, v)$. This induces a projection into X/A . We need to show that there is a way of choosing h such that this is a vector bundle. This amounts to showing that h extends to a trivialization of η over a neighborhood $U \supset A$. Assuming our base space is “sufficiently nice” (e.g., connected, normal, etc.), then there is such a neighborhood that deformation retracts onto A , which would give us a trivialization as a pullback bundle. Otherwise, we can make an argument using partitions of unity to show that such a trivialization exists. This means that η is the image of a vector bundle over X/A , so the sequence is exact, as claimed.

To extend this, we can look at the following sequence of maps:

$$A \hookrightarrow X \hookrightarrow X \cup CA \hookrightarrow (X \cup CA) \cup CX \hookrightarrow ((X \cup CA) \cup CX) \cup C(X \cup CA) \hookrightarrow \dots$$

where each step is obtained by putting a cone over everything except the previous cone. We can deformation retract the cones to a single point, so in particular we get homotopy equivalences $X \cup CA \simeq X/A$, $(X \cup CA) \cup CX \simeq \Sigma A$, etc. □

Example 4.13. Let $X = A \vee B$, where $A \vee B$ is the *wedge product* given by $A \times \{b\} \cup \{a\} \times B$. Then $X/A = B$ and we can use our long exact sequence. Furthermore, the long exact sequence splits and we get a split short exact sequence:

$$0 \rightarrow \tilde{K}(B) \rightarrow \tilde{K}(X) \rightarrow \tilde{K}(A) \rightarrow 0$$

Because this splits, we get $\tilde{K}(X) \approx \tilde{K}(A) \oplus \tilde{K}(B)$.

The long exact sequence will also allow us to construct a “reduced external product” as follows:

Construction 4.14. We wish to construct a product $\hat{\mu} : \tilde{K}(X) \otimes \tilde{K}(Y) \rightarrow \tilde{K}(X \wedge Y)$, where $X \wedge Y$ denotes the *smash product* given by $(X \times Y)/(X \vee Y)$. We will use our long exact sequence with $X \times Y$ and $X \vee Y$. This gives:

$$\dots \rightarrow \tilde{K}(\Sigma(X \times Y)) \rightarrow \tilde{K}(\Sigma(X \vee Y)) \rightarrow \tilde{K}(X \wedge Y) \rightarrow \tilde{K}(X \times Y) \rightarrow \tilde{K}(X \vee Y)$$

By the previous example, we note that $\tilde{K}(X \vee Y) \approx \tilde{K}(X) \oplus \tilde{K}(Y)$. Additionally, noting that $\Sigma(X \vee Y) = \Sigma X \vee \Sigma Y$, we can get a similar isomorphism for $\tilde{K}(\Sigma(X \vee Y))$. Next, we claim that the last map in the sequence splits. We can construct a map $\tilde{K}(X) \oplus \tilde{K}(Y) \rightarrow \tilde{K}(X \times Y)$ by simply taking the homomorphisms ρ_i induced by projection to get a map $(a, b) \rightarrow \rho_1(a) + \rho_2(b)$. The first map splits in the exact same way. Thus, the short exact sequence splits and we get $\tilde{K}(X \times Y) \approx \tilde{K}(X \wedge Y) \oplus \tilde{K}(X) \oplus \tilde{K}(Y)$.

Recall that the external product constructed above gives a map $\mu : K(X) \otimes K(Y) \rightarrow K(X \times Y)$. Let $a \in \tilde{K}(X)$ and $b \in \tilde{K}(Y)$. Then, because the reduced K -theories are ideals of the unreduced K -ring, we can apply μ to get an element $\mu(a, b) \in K(X \times Y)$. We claim that this product actually restricts to 0 in $K(X \times Y)$. This follows from the fact that the reduced K -theory is obtained as the kernel of the reduction homomorphism $K(X) \rightarrow K(x_0)$. Because this product is 0, it lies in the kernel of the homomorphism $K(X \times Y) \rightarrow K(x_0 \times y_0)$ and it is an element of $\tilde{K}(X \times Y)$. Then, by our short exact sequence, this pulls back to a unique element of $\tilde{K}(X \wedge Y)$. This is precisely the element $\hat{\mu}(a, b)$.

4.2. Clutching functions and Bott periodicity. One of the most important K -rings is that of the 2-sphere. The reason for this importance will be discussed later in this section, but for now we will attempt to compute it. At the heart of this is the idea of a *clutching function*. The idea is to take vector bundles over the 2-spheres and make them trivial on each hemisphere individually. Then the idea is that the vector bundle will be completely determined by its behavior on the equator, which we can think of as being classified by an endomorphism from S^1 to itself.

Definition 4.15. Let $S^k = D_+^k \cup D_-^k$, with $D_+^k \cap D_-^k = S^{k-1}$. Let $f : S^{k-1} \rightarrow GL_n(\mathbb{C})$. Define a vector bundle E_f by the quotient $(D_+^k \times \mathbb{C}^n \sqcup D_-^k \times \mathbb{C}^n) / \sim$ with \sim identifying $(x, v) \in \partial D_+^k \times \mathbb{C}^n$ with $(x, f(x) \cdot v) \in \partial D_-^k \times \mathbb{C}^n$. This defines an n -dimensional vector bundle over S^k . f is called a *clutching function* for the bundle E_f .

The first lemma about clutching functions is that they are homotopy invariant:

Lemma 4.16. *Let $f, g : S^{k-1} \rightarrow GL_n(\mathbb{C})$ be homotopic maps. Then the vector bundles E_f and E_g are equivalent.*

Proof. We can construct another vector bundle E_H over $S^k \times I$ using an analogous construction and thinking of the homotopy as a clutching function of its own. Then E_f and E_g are the restrictions of this bundle to $S^k \times \{0\}$ and $S^k \times \{1\}$. We can construct an isomorphism of vector bundles using the homotopy as follows: for each $\alpha \in S^k$, we can define a neighborhood $U_\alpha \times I$ over which E_H is trivial. This is because we can cover $\{\alpha\} \times I$ by open neighborhoods in which the vector bundle is trivial. Then, by compactness of $[0, 1]$, we can pass to a finite subcover and look at the intersection of the first two coordinate neighborhoods. Then, we can take finitely many of the U_α (because S^k is compact) to cover all of the cylinder.

This gives an isomorphism over each of the finitely many neighborhoods, so these bundles are isomorphic. \square

Note that this is extremely similar to the argument outlined for the proof of (1.11). The importance of clutching functions lies in the following proposition:

Proposition 4.17. *The map $\Phi : [S^{k-1}, GL_n(\mathbb{C})] \rightarrow \text{Vect}_n(S^k)$ sending a homotopy class of clutching functions to the corresponding vector bundle is a bijection.*

The reasoning behind this is precisely what was said in the introduction to this section: any bundle can be trivialized over each of the hemispheres individually, so we need only see how they agree or differ at the equator to fully characterize a vector bundle.

Example 4.18. Consider complex vector bundles over S^1 . We claim that $\text{Vect}_n(S^1)$ contains only one element. To see this, note that $[S^0, GL_n(\mathbb{C})]$ contains only one element because S^0 is discrete and $GL_n(\mathbb{C})$ is path connected, so all maps between them are homotopic. Thus, all complex vector bundles over S^1 are trivial.

In order to figure out what the ring $K(S^2)$ looks like, we will use the following fact:

Claim 4.19. *Let H be the canonical line bundle over $\mathbb{C}P^1 = S^2$. Then $(H \otimes H) \oplus 1 = H \oplus H$.*

Proof. The proof will be clear when we consider the corresponding clutching functions. Because S^1 is the complex unit sphere, we can consider clutching functions as the restrictions of maps $\mathbb{C} \rightarrow GL_2(\mathbb{C})$. The left hand side corresponds to the clutching function:

$$f(z) = \begin{pmatrix} z^2 & 0 \\ 0 & 1 \end{pmatrix}$$

while the right hand side corresponds to the clutching function:

$$g(z) = \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}$$

Thus, we need only show that these two maps are homotopic. We can construct a path between the two matrices in $GL_2(\mathbb{C})$, which induces a homotopy between the two maps. \square

We now claim the following:

Claim 4.20. *The ring $K(S^2)$ is equal to the polynomial ring on one generator $\mathbb{Z}[H]/(H - 1)^2$.*

The existence of the relation $(H - 1)^2 = 0$ follows from the previous fact, but we will delay a discussion of why there are no other relations. The importance of $K(S^2)$ lies in a remarkable theorem of Bott. We will give multiple formulations of the theorem, though we will give only an indication of the proof.

Theorem 4.21. *(Bott periodicity) The external multiplication map $\mu : K(X) \otimes K(S^2) \rightarrow K(X \times S^2)$ is an isomorphism.*

Remark 4.22. There is a corresponding theorem for KO -theory. However, instead of S^2 , real Bott periodicity gives us an isomorphism for S^8 . We also get an isomorphism with S^8 if we deal with quaternionic K -theory, which is denoted $KSp(X)$ due to its connection with the symplectic group.

Proof. The idea of the proof is fairly straightforward - to define clutching functions for vector bundles over $X \times S^2$ and use analytic methods to reduce the possible clutching functions to only the linear case. The way to define these clutching functions is fairly straightforward - we can consider a vector bundle $p : E \rightarrow X$ and then construct a new vector bundle $p : E \times S^2 \rightarrow X \times S^2$ by taking the product $E \times D^2$ and attaching two copies by some map $f : E \times S^1 \rightarrow E \times S^1$.

One important fact that arises during the proof is that the only property of $K(S^2)$ that is needed is that it is an ideal of $\mathbb{Z}[H]/(H-1)^2$, so the isomorphism factors through $K(X) \otimes \mathbb{Z}[H]/(H-1)^2$, implying that this must, in fact, be $K(S^2)$. \square

There is a corresponding statement for reduced K -theory:

Theorem 4.23. (*Bott periodicity*) *The external multiplication map $\hat{\mu} : \tilde{K}(X) \otimes \tilde{K}(S^2) \rightarrow \tilde{K}(X \wedge S^2)$ is an isomorphism.*

Note that $S^n \wedge X = \Sigma^n X$, the space obtained by applying the reduced suspension operation n times. An easy consequence of this fact is that $\tilde{K}(S^{2n}) \approx \mathbb{Z}$ and $\tilde{K}(S^{2n+1}) = 0$ by induction. Thus, we can restate the previous result as $\tilde{K}(X) \approx \tilde{K}(\Sigma^2 X)$. Applying this result to our long exact sequence gives the following periodic exact sequence:

$$\begin{array}{ccccc}
 \tilde{K}(X/A) & \longrightarrow & \tilde{K}(X) & \longrightarrow & \tilde{K}(A) \\
 \uparrow & & & & \downarrow \\
 \tilde{K}(\Sigma A) & \longleftarrow & \tilde{K}(\Sigma X) & \longleftarrow & \tilde{K}(\Sigma(X/A))
 \end{array}$$

This is extremely similar to the long exact sequence for ordinary cohomology. We will denote $\tilde{K}^0(X) = \tilde{K}(X)$ and $\tilde{K}^{-n}(X) = \tilde{K}(\Sigma^n X)$. We will define a relative K -theory by $\tilde{K}^*(X/A) = \tilde{K}^*(X, A)$, as in ordinary cohomology. We can define this for positive degrees using Bott periodicity, thus creating a \mathbb{Z} -graded ring $\tilde{K}^*(X)$.

We can define an external product $\tilde{K}^i(X) \otimes \tilde{K}^j(Y) \rightarrow \tilde{K}^{i+j}(X \wedge Y)$ as the previously defined external product of $\Sigma^i X$ and $\Sigma^j Y$. Then $\tilde{K}^*(X)$ inherits a product structure as the composition of the external product $\tilde{K}^*(X) \otimes \tilde{K}^*(X) \rightarrow \tilde{K}^*(X \wedge X)$ and the pullback of the diagonal map $\Delta : X \rightarrow X \wedge X$. For relative K -theory, we can define a similar map $\tilde{K}^*(X, A) \otimes \tilde{K}^*(X, B) \rightarrow \tilde{K}^*(X, A \cup B)$ by taking a pullback of the relative diagonal map $\Delta : X/(A \cup B) \rightarrow (X/A) \times (X/B)$ induced by inclusion into each factor.

Example 4.24. Suppose that $X = A \cup B$, where A and B are compact and contractible. Then the product $\tilde{K}^*(X) \otimes \tilde{K}^*(X) \rightarrow \tilde{K}^*(X)$ is trivial. To see this, note that, because A and B have the homotopy type of a point, and the reduced K -theory of a point is the zero ring, our long exact sequence tells us that $\tilde{K}^i(X) \approx \tilde{K}^i(X, A)$. Therefore, our product can be thought of as factoring through

the maps:

$$\tilde{K}^*(X) \otimes \tilde{K}^*(X) \rightarrow \tilde{K}^*(X, A) \otimes \tilde{K}^*(X, B) \rightarrow \tilde{K}^*(X, A \cup B) \rightarrow \tilde{K}^*(X)$$

where the first map is the isomorphism, the second is the relative external product, and the final one is the homomorphism induced by projection. Because $A \cup B = X$, this factors through the zero ring, so the multiplication is trivial, as claimed.

To end this section, we will discuss an equivalent definition of the Hopf invariant using K -theory. The construction is entirely analogous to the construction using ordinary cohomology given in (2.1) and can be shown to be equivalent. We give here the definition:

Definition 4.25. Let $f : S^{4n-1} \rightarrow S^{2n}$ and let C_f be as before. Note that $C_f/S^{2n} = S^{4n}$. Additionally, $\tilde{K}^{-1}(S^{2n}) = \tilde{K}^{-1}(S^{4n}) = 0$. Therefore, we have the following short exact sequence in K -theories:

$$0 \rightarrow \tilde{K}(S^{4n}) \rightarrow \tilde{K}(C_f) \rightarrow \tilde{K}(S^{2n}) \rightarrow 0$$

Let α be the image of a generator of $\tilde{K}(S^{4n})$ in $\tilde{K}(C_f)$. Let η be a generator for $\tilde{K}(S^{2n})$. We claim that η pulls back uniquely to an element β of $\tilde{K}(C_f)$. Suppose that it is non-unique. Then there exist β_1, β_2 that map to η , so $\beta_1 - \beta_2$ is in the kernel and therefore the image of the previous map, implying that it has dimension $4n$, a contradiction because β has dimension $2n$. Thus, we can pull η back to an element $\beta \in \tilde{K}(C_f)$.

Then, because multiplication is trivial in $\tilde{K}(S^{2n})$, we know that β^2 is in the kernel of the second map, so it is in the image of the first map. Because $\tilde{K}(S^{4n})$ is generated by α , we conclude that $\beta = h\alpha$. The integer h is called the *Hopf invariant* of the map f . Note, as before, that it is defined only up to sign based on choice of generator.

4.3. Adams operations. With all of the machinery of K -theory in place, we are set to finally prove Adams' theorem about maps of Hopf invariant one. To do so, we will construct special natural operations in the K -ring called Adams operations satisfying certain properties:

Theorem 4.26. *There exist ring homomorphisms $\psi^k : K(X) \rightarrow K(X)$, called Adams operations, satisfying the following properties:*

- (1) (Naturality) For any map $f : X \rightarrow Y$, $\psi^k \circ f^* = f^* \circ \psi^k$
- (2) For any line bundle L , $\psi^k(L) = L^{\otimes k}$
- (3) For any integers k, ℓ , the relation $\psi^k \circ \psi^\ell = \psi^\ell \circ \psi^k = \psi^{k\ell}$
- (4) For any prime p , $\psi^p(\alpha) \equiv \alpha^p \pmod{p}$

In order to construct these operations, we will appeal to the exterior power operation on vector bundles. The reason for this is that the exterior power turns direct sums into tensor products, which will be useful for our construction.

Construction 4.27. Let E be a vector bundle that can be written as the direct sum of line bundles $\{L_i\}$. First, define a polynomial $\lambda_t(E) = \sum_i \Lambda^i(E)t^i \in K(X)[t]$. Note that this sum is well-defined because the exterior power is zero for powers higher than the dimension of E , so the sum terminates. This has the property that $\lambda_t(E_1 \oplus E_2) = \sum_i \sum_j \Lambda^i(E_1) \otimes \Lambda^j(E_2)t^{i+j} = \lambda_t(E_1)\lambda_t(E_2)$. Thus, because E is the direct sum of line bundles, $\lambda_t(E) = \prod_i (1 + L_i t)$. Letting $t_i = L_i t$, this can be written as $\prod_i (1 + t_i) = \sum_i \sigma_i$, where σ_i is the complete symmetric

polynomial in i variables. Thus the coefficient of t^j in λ_t is $\sigma_t(L_1, \dots, L_n)$. It is a fact (sometimes referred to as the *fundamental theorem of symmetric polynomials*) that any symmetric polynomial can be written as a polynomial in the complete symmetric polynomials. Thus, we can say $t_1^k + t_2^k + \dots + t_n^k = s_k(\sigma_1, \dots, \sigma_k)$. Then $s_k(\Lambda^1(E), \dots, \Lambda^k(E)) = L_1^k + \dots + L_n^k$.

We claim that defining the Adams operations to be $\psi^k(E) = s_k(\Lambda^1(E), \dots, \Lambda^k(E))$ is the desired operation. One way to see this is to appeal to the Splitting Principle, which tells us that we can inject into another space such that any vector bundle can be written as the direct sum of line bundles, which we know this construction to be accurate for. The Splitting Principle also implies uniqueness of the construction by similar reasoning. We need only verify that it has all of the properties. Naturality is inherited from the exterior power operation, which is itself natural. That these are, in fact, ring homomorphisms also follows from the splitting principle, as we know these to be true for line bundles and we can split any direct sum of vector bundles into a direct sum of line bundles. Multiplicativity follows from similar reasoning and the fact that exterior powers take sums to products. The commutativity of ψ^k and ψ^ℓ also follows from splitting, as exponentiation of line bundles satisfies this property. Finally, the modular property follows from the fact that $(L_1 + \dots + L_n)^p \equiv L_1^p + \dots + L_n^p \pmod{p}$ by some elementary facts about binomial coefficients. Thus, these are in fact the desired operations.

Note that we can also define Adams operations for reduced K -theory by realizing it as the kernel of maps in K -theory. The Adams operations in reduced K -theory should have the same properties as in non-reduced K -theory because reduced K -theory is an ideal of the K -ring. We then have the following lemma:

Lemma 4.28. *The Adams operations $\psi^k : \tilde{K}(S^{2n}) \rightarrow \tilde{K}(S^{2n})$ are multiplication by k^n .*

Proof. We will prove this by induction on n . For $n = 1$, let $\alpha = H - 1$ be a generator. Then $\psi^k(\alpha) = \psi^k(H - 1) = H^k - 1 = (1 + \alpha)^k - 1 = k\alpha$ because $\alpha^2 = 0$. Next, we can use Bott periodicity to induct because the external product $\tilde{K}(S^2) \otimes \tilde{K}(S^{2n-2}) \rightarrow \tilde{K}(S^{2n})$ is an isomorphism. Any $\gamma \in \tilde{K}(S^{2n})$ can be written as a product $\alpha \otimes \beta$. By the inductive hypothesis, we have $\psi^k(\alpha \otimes \beta) = k\alpha \otimes k^{n-1}\beta = k^n(\alpha \otimes \beta) = k^n\gamma$. Thus, by induction, the lemma holds for all n . \square

We can now prove Adams' Theorem:

Theorem 4.29. *(Adams) Let $f : S^{4n-1} \rightarrow S^{2n}$ be a map with $h(f) = \pm 1$. Then $n = 1, 2, 4$.*

Proof. Let α, β be as defined in the definition of the Hopf invariant. Then $\psi^k(\alpha) = k^{2n}\alpha$ by the lemma and naturality and $\psi^k(\beta) = k^n\beta + \mu_k\alpha$ for some $\mu_k \in \mathbb{Z}$. Then, by composing Adams operations of different degree, we get:

$$\psi^k(\psi^\ell(\beta)) = \psi^k(\ell^n\beta + \mu_\ell\alpha) = (k\ell)^n\beta + (\mu_k\ell^n + \mu_\ell k^{2n})\alpha$$

Because switching the order that the Adams operations are applied doesn't change the result, we can compare the coefficients on α to get the equality $(k^{2n} - k^n)\mu_\ell = (\ell^{2n} - \ell^n)\mu_k$. Consider the case $\ell = 2, k = 3$. Note that $\psi^2(\beta) \equiv \beta^2 = h\alpha \pmod{2}$, so $\mu_2\alpha \equiv h\alpha \pmod{2}$, implying that μ_2 is odd if the Hopf invariant of the map is ± 1 (or any odd number). Therefore, 2^n divides the right hand side of the equation

while the only factor that can be even on the left hand side is $3^n - 1$. Thus, our result follows from the following fact from number theory: Let $n \in \mathbb{N}$. If $2^n | (3^n - 1)$, then $n = 1, 2, 4$.

To prove this fact, we will appeal to induction on the largest power of 2 dividing n . Suppose $n = 2^i m$. Then the highest power of 2 dividing $3^n - 1$ is 2^{i+2} for $i > 0$ and 2 for $i = 0$. We will first prove two base cases. For $i = 0$, we have that n is odd, so $3^n \equiv 3 \pmod{4}$ by Euler's Theorem. For $i = 1$, we have that $n = 2m$. This gives $3^{2m} - 1 = (3^m - 1)(3^m + 1)$. The highest power of 2 dividing the first factor is 2 (because this is the $i = 0$ case) and the highest power dividing the second is 4 (because m is odd), so 2^3 divides the entire thing. We can use similar reasoning to induct: We can reduce the problem to $n = 2m$, where the largest power of 2 dividing m is 2^{i-1} . Then $3^n - 1 = (3^m - 1)(3^m + 1)$. The first factor is divisible by 3^{i+1} by the inductive hypothesis. Because m is even, the second factor is divisible only by 2, so 2^{i+2} divides the entire thing and the result holds by induction. We then have that if $2^n | (3^n - 1)$, then $n \leq i + 2$, which implies that $2^i \leq i + 2$ and $i \leq 2$. Thus $n \leq 4$ and, by inspection, the only solutions are $n = 1, 2, 4$, as desired. \square

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