BUNDLES, STIEFEL–WHITNEY CLASSES, & BRAID GROUPS

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Abstract. We explain the basics of vector bundles and principal G-bundles, as well as describe the relationship between vector bundles and principal O_n-bundles. We then give an axiomatization of Stiefel–Whitney classes and derive some basic consequences. Finally, we apply the theory of bundles and Stiefel–Whitney to understand the space of unordered configurations of n distinct points in the plane.

Contents

0. Overview

There are two main goals of this paper. The first is to provide a survey of the theory of principal G-bundles, vector bundles, and Stiefel–Whitney classes. The purpose of this is to highlight the main ideas of the theory in order to give the unfamiliar reader a general understanding of how the different pieces fit together, while also reminding the experienced reader about the central ideas of the theory. The second goal is to use these tools to study the space $UC_n(\mathbf{C})$ of unordered configurations of n distinct points in the plane. We begin by outlining some preliminary definitions and notations, then in §§2 and 3 explain the basics of principal G-bundles and vector bundles. In §4, we explain the relationship between principal bundles, and vector bundles, universal principal bundles and classifying spaces. In § 5 we give an axiomatization of Stiefel–Whitney classes and derive some basic consequences from the axioms. In §6 we tie all of these subjects together to study braid groups, and the spaces $UC_n(\mathbf{C})$, which are the classifying spaces of the braid groups.

Prerequisites. We assume that the reader is familiar with basic algebraic topology: the fundamental group and covering spaces, basic homology and cohomology theory, and the definitions of the higher homotopy groups, i.e., the material from [7, Chs. 1–4] or [11, Chs. 1–20]. We also assume some familiarity with the basic terminology of category theory, namely, categories, functors, and natural transformations — we refer the unfamiliar reader to [16, Ch. 1] for an excellent introduction to the subject.

Date: November 21, 2015.

Acknowledgments. I would like to thank Nir Gadish for being a patient teacher and excellent mentor, Benson Farb for suggesting this project, and Peter May for making it possible for me to attend the REU program at the University of Chicago as well as making it such a pleasurable experience, and providing many helpful comments on this paper. I would also like to thank Randy Van Why for his helpful conversations about the material in this paper, and significant contributions to the ideas appearing in § 6. Finally, I would like to thank Janson Ng for many, many wonderful conversations while we were at UChicago and since.

1. Preliminaries

In this section we outline some of our notational conventions and recall some preliminary definitions that we assume the reader has familiarity with and use extensively throughout this paper. In particular, we are interested in actions of topological groups on spaces.

1.1. **Definition.** Suppose that G is a topological group. A right ***G-action*** on a space X is a continuous map $r: X \times G \longrightarrow X$, written $r(x, g) = xg$, so that $x1 = x$, where 1 is the identity of G, and $(xg)g' = x(gg')$ for all $x \in X$ and $g, g' \in G$. We call X a right ***G-space***. Left G-spaces are defined analogously.

We say that a G-action is ***free*** if $xg = x$ if and only if g is the identity, i.e., the action has no fixed points.

Suppose that X is a (right) G-space. The ***orbit space*** X/G is the quotient space of X under the equivalence relation $\sim$ on X given by saying that $x \sim xg$ for each $x \in X$ and all $g \in G$, i.e., the equivalence relation that associates all of the points in an orbit.

1.2. **Example.** For each positive integer n, the space $\mathbf{R}^n$ is a left $\mathrm{GL}_n(\mathbf{R})$-space, with the action given by left matrix multiplication.

1.3. **Definition.** Suppose that X and Y are G-spaces and that $f: X \longrightarrow Y$ is a continuous map. We say that f is ***G-equivariant***, or a ***G-map*** if f commutes with the G-actions on X and Y, i.e., for all $g \in G$ and $x \in X$ we have $f(xg) = f(x)g$.

1.4. **Definition.** Suppose that $f, f': X \longrightarrow Y$ are G-equivariant maps. We say that f and f' are ***G-homotopic*** if there exists a homotopy $h: X \times [0,1] \longrightarrow Y$ between f and f' so that h is G-equivariant, with respect to the G-action on $X \times I$ given by $(x,t)g \coloneqq (xg, t)$. Such a homotopy is called a ***G-homotopy***.

There are also a number of situations where the underlying point-set topology affects the theory of bundles. Here we recall some relevant definitions from point-set topology.

1.5. **Definition.** Let $\mathcal{U}$ be a collection of subsets of a space X. A collection $\mathcal{V}$ of subsets of X is called a ***refinement*** of $\mathcal{U}$ if every set in $\mathcal{V}$ is contained in a set in $\mathcal{U}$. We say that $\mathcal{V}$ is an ***open refinement*** of $\mathcal{U}$ if all of the sets in $\mathcal{V}$ are open in X.

1.6. **Example.** A space X is compact if and only if every open cover $\mathcal{U}$ of X has a finite open refinement $\mathcal{V}$ which covers X.

1.7. **Definition.** A collection $\mathcal{U}$ of subsets of a space X is called ***locally finite*** if each point $x \in X$ has a neighborhood that intersects only finitely many elements of $\mathcal{U}$.

1.8. **Definition.** A space X is ***paracompact*** if every open cover $\mathcal{U}$ of X has a locally finite open refinement $\mathcal{V}$ which covers X.

Milnor and Stasheff [12, Ch. 5 §8] also assume that paracompact spaces are Hausdorff, following the lead of Bourbaki [3, Pt. I Ch. 9 §1 Def. 1], which assumes that compact spaces are Hausdorff. We follow Munkres [13, §41] and do not assume the Hausdorff condition, as many students in the present generation learned topology from [13]. Detailed expositions on paracompactness can be found in [12, Ch. 5 §8; 13, §§39 and 41].

1.9. **Example.** As Husemöller [9, §4.9] notes, a Hausdorff space X is paracompact if and only if every open cover of X is numerable.

1.10. **Example.** Metrizable spaces are paracompact. Various proofs of this fact can be found in [13, Tm. 41.4; 14; 17].

1.11. **Example.** Manifolds are second-countable and regular, hence metrizable by the Urysohn metrization theorem. Thus manifolds are paracompact.

1.12. **Remark.** Milnor and Stasheff [12, §5.8] note that nearly all familiar spaces are paracompact and Hausdorff.

1.13. **Definition.** Suppose that X is a topological space and $f\colon X \longrightarrow \mathbf{R}$ is a continuous map. The ***support*** of f, denoted by $\operatorname{supp}(f)$, is the closure of the subspace $f^{-1}(\mathbf{R} \smallsetminus \{0\})$.

1.14. **Definition.** A ***partition of unity*** on a space X is a collection $\{\lambda_\alpha\}_{\alpha\in A}$ of continuous functions $\lambda_\alpha\colon X \longrightarrow [0,1]$ satisfying the following properties:

(1.14.a) the family of supports $\{\operatorname{supp}(\lambda_\alpha)\}_{\alpha\in A}$ is locally finite,
(1.14.b) and for all $x \in X$ we have $\sum_{\alpha\in A} \lambda_\alpha(x) = 1$.

1.15. **Definition.** An open cover $\{U_\alpha\}_{\alpha\in A}$ of a topological space X is called ***numerable*** if there exists a partition of unity $\{\lambda_\alpha\}_{\alpha\in A}$ so that $\operatorname{supp}(\lambda_\alpha) \subset U_\alpha$ for each $\alpha \in A$.

Finally, a bit of notation.

1.16. **Notation.** We often use the arrow "$\hookrightarrow$" to denote an inclusion or an embedding, the arrow "$\twoheadrightarrow$" to denote a surjection, and the arrow "$\xrightarrow{\sim}$" to denote an isomorphism.

2. Principal Bundles

Bundles are important tools in topology and geometry. In this section we begin our study of bundles by studying *principal G-bundles*, which are in many ways similar to covering spaces, but with the additional information of a topological group G acting on the covering space. We only outline the basic theory; more complete accounts can be found in [9, Ch. 4 §§2–4; 20, Ch. 14 §1].

2.1. **Definition.** Suppose that G is a topological group. A ***principal*** G-***bundle*** ξ over a topological space B consists of the following data:

(2.1.a) a right G-space $E(\xi)$,
(2.1.b) and a continuous surjection $\xi\colon E(\xi) \longrightarrow B$,

subject to the following axioms.

(2.1.1) The map ξ is *constant-on-orbits*: for each $x \in E$ and $g \in G$ we have $\xi(xg) = \xi(x)$.
(2.1.2) There exists a *numerable*† open cover $\{U_\alpha\}_{\alpha\in A}$ of B and a G-equivariant homeomorphism

$$\phi_\alpha\colon \xi^{-1}(U_\alpha) \xrightarrow{\sim} U_\alpha \times G$$

†Following the conventions of May [11, Ch. 23 §8], we require that the cover be numerable. In light of Example 1.9, this is always true when the base space is paracompact and Hausdorff. Since most of the results are typically proven when the base space is paracompact and Hausdorff, there is nothing lost assuming this.

making the following triangle commute

$$\begin{array}{ccc} \xi^{-1}(U_\alpha) & \xrightarrow[\phi_\alpha]{\sim} & U_\alpha \times G \\ & {\scriptstyle \xi}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_{U_\alpha}} & \\ & U_\alpha & \end{array},$$

where the G-action on $U_\alpha \times G$ is given by $(u, g)g' \coloneqq (u, gg')$. Such a map ϕ_α is called a ***local trivialization*** of ξ over U_α.

An open cover as in (2.1.2) is called a ***bundle atlas***.

2.2. **Example.** For any space B and any topological group G, the space $B \times G$ is a right G-space with the action $(b, g)g' \coloneqq (b, gg')$. The projection $\mathrm{pr}_B \colon B \times G \longrightarrow B$ onto B, yields a principal G-bundle over B called the ***trivial G-bundle***.

2.3. **Observation.** It is an immediate consequence of Definition 2.1 that if $\xi \colon E \longrightarrow B$ is a principal G-bundle, then G acts *freely* on E, so ξ factors through the quotient map $q \colon E \twoheadrightarrow E/G$ and induced a continuous bijection $E/G \longrightarrow B$. Since ξ and q are open maps, they are quotient maps, the continuous bijection $E/G \longrightarrow B$ is a homeomorphism.

2.4. **Example.** For each positive integer n there is a free $\mathbf{Z}/2$ action on S^n given by the antipodal map sending $x \longmapsto -x$. Real projective n-space $\mathbf{RP}^n$ is the orbit space of S^n under this action, and the projection $p_n \colon S^n \twoheadrightarrow \mathbf{RP}^n$ is a principal $\mathbf{Z}/2$-bundle (where we equip $\mathbf{Z}/2$ with the discrete topology). The elements of $\mathbf{RP}^n$ are unordered pairs $\{x, -x\}$, where $x \in S^n$.

2.5. **Example.** Suppose that $n \geq 2$ is an integer and consider the sphere S^{2n-1} as a subspace of $\mathbf{C}^n$. There is a free S^1 action on S^{2n-1} given by coordinate-wise scalar multiplication. The orbit space under this action is the complex projective space $\mathbf{CP}^{n-1}$, and the orbit map $p \colon S^{2n-1} \twoheadrightarrow \mathbf{CP}^{n-1}$ has the structure of a principal S^1-bundle. In the special case that $n = 2$, since $\mathbf{CP}^1 \cong S^2$, this is the Hopf fibration $S^1 \hookrightarrow S^3 \twoheadrightarrow S^2$.

2.6. **Definition.** Suppose that $\xi \colon E \longrightarrow B$ and $\xi' \colon E' \longrightarrow B'$ are principal G-bundles. A ***morphism of principal G-bundles***, or simply a ***bundle map*** $\xi' \longrightarrow \xi$, consists of a G-equivariant map $\bar{f} \colon E' \longrightarrow E$ and a continuous map $f \colon B' \longrightarrow B$ making the following square commute

$$\begin{array}{ccc} E' & \xrightarrow{\bar{f}} & E \\ {\scriptstyle \xi'}\big\downarrow & & \big\downarrow{\scriptstyle \xi} \\ B' & \xrightarrow[f]{} & B \end{array}.$$

We often denote a bundle map by $(\bar{f}, f) \colon \xi' \longrightarrow \xi$.

2.7. **Example.** For each positive integer n, the inclusion of S^n into the equator of S^{n+1} is $\mathbf{Z}/2$-equivariant, and induces an inclusion $\mathbf{RP}^n \hookrightarrow \mathbf{RP}^{n+1}$ given by taking $\mathbf{Z}/2$-orbits. Since the inclusions respect the $\mathbf{Z}/2$-action, the square

$$\begin{array}{ccc} S^n & \hookrightarrow & S^{n+1} \\ {\scriptstyle p_n}\big\downarrow & & \big\downarrow{\scriptstyle p_{n+1}} \\ \mathbf{RP}^n & \hookrightarrow & \mathbf{RP}^{n+1} \end{array}$$

commutes, hence the two inclusions define a map of principal $\mathbf{Z}/2$-bundles.

We can also construct new bundles from old bundles, probably the most important construction being the *pullback bundle.* To explain this construction, we first must recall how to construct pullbacks of topological spaces.

2.8. **Recollection.** Suppose that we have continuous maps of topological spaces $f\colon X \longrightarrow Z$ and $p\colon Y \longrightarrow Z$. The ***pullback*** $X \times_Z Y$ is the terminal space equipped with maps so that the square

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{\bar{f}} & Y \\ {\scriptstyle f^\star p}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ X & \xrightarrow[f]{} & Z \end{array}$$

commutes. The pullback can be regarded as the following subset of the product $X \times Y$

$$X \times_Z Y = \{(x, y) \in X \times Y \mid f(x) = p(y)\},$$

with the subspace topology.

2.9. **Definition.** Suppose that $\xi\colon E \longrightarrow B$ is a principal G-bundle and that f is a continuous map $B' \longrightarrow B$. We form the pullback $B' \times_B E$ in **Top**. Regarding the pullback $B' \times_B E$ as a subset of the product $B' \times E$ as above, we give a G-action on $B' \times E$ by setting

$$(b', e)g \coloneqq (b', eg)$$

for all $g \in G$ and $(b', e) \in B' \times E$. This action is well-defined by the above construction of the pullback and because the map ξ is constant-on-orbits. If $\{U_\alpha\}_{\alpha\in A}$ is a bundle atlas for ξ with trivializations $\{\phi_\alpha\}_{\alpha\in A}$, then setting $U'_\alpha \coloneqq f^{-1}(U_\alpha)$ gives an open cover $\{U'_\alpha\}_{\alpha\in A}$ of B', and

$$f^\star\xi\colon B' \times_B E \longrightarrow B'$$

is trivial over each U'_α. We call $f^\star\xi$ the bundle ***induced by*** f, or the ***pullback along*** f, and denote the total space of this bundle by $E(f^\star\xi)$.

The amazing fact about the pullback is the following homotopy theorem.

2.10. **Theorem** ([20, Tm. 14.3.3]). *Suppose that $\xi\colon E \longrightarrow B$ is a principal G-bundle and that $f_0, f_1\colon B' \longrightarrow B$ are homotopic maps. Then the induced bundles $f_0^\star\xi$ and $f_1^\star\xi$ are isomorphic.*

This theorem tells us that we can transform the rather rigid data of a principal bundle into purely homotopical data. Moreover, homotopies between base spaces can be lifted, giving *homotopic bundle maps,* with the appropriate notion of a homotopy of bundle maps.

2.11. **Definition.** Suppose that $\xi\colon E \longrightarrow B$ and $\xi'\colon E' \longrightarrow B'$ are principal G-bundles, and that Suppose that

$$(\bar{f}_0, f_0), (\bar{f}_1, f_1)\colon \xi' \longrightarrow \xi$$

are bundle maps. A ***bundle homotopy*** from $(\bar{f}_0, f_0)$ to $(\bar{f}_1, f_1)$ is a pair of homotopies $(\bar{h}, h)$, where $\bar{h}$ is a G-homotopy from $\bar{f}_0$ to $\bar{f}_1$, and h is a homotopy from f_0 to f_1, with the property that for each $t \in [0, 1]$ the pair $(\bar{h}|_{E'\times\{t\}}, h|_{B'\times\{t\}})$ is a bundle map.

2.12. **Theorem** ([20, Tm. 14.3.4]). *Suppose that $\xi\colon E \longrightarrow B$ and $\xi'\colon E' \longrightarrow B'$ are principal G-bundles, $(\bar{f}_0, f_0)\colon \xi' \longrightarrow \xi$ is a bundle map, and f_0 is homotopic to a map $f_1\colon B' \longrightarrow B$ via a homotopy h. Then there exists a G-equivariant map $\bar{f}_1\colon E' \longrightarrow E$ so that $(\bar{f}_1, f_1)$ is a bundle map, as well as a G-homotopy $\bar{h}$ between $\bar{f}_0$ and $\bar{f}_1$ so that $(\bar{h}, h)$ is a bundle homotopy.*

3. Vector Bundles

In this section we continue our study of bundles by studying *vector bundles*, which are in many ways similar to covering spaces, but with the additional data of a vector space structure on each fiber. Vector bundles appear throughout topology and differential geometry, among many other areas, and are the basis of *topological K-theory,* a generalized cohomology theory widely used in geometric and low-dimensional topology. We only outline the basic theory; more complete accounts can be found in [8, Ch. 1; 9, Ch. 3 §§1–4; 12, Ch. 2; 20, Ch. 14 §2].

The definition of a vector bundle is very similar to that of a principal G-bundle.

3.1. **Definition.** A ***real vector bundle*** ξ over a topological space B, called the ***base space***, consists of the following data:

(3.1.a) a topological space E, called the ***total space***, equipped with a continuous surjection $\xi\colon E \longrightarrow B$,

(3.1.b) and for each $b \in B$, the structure of a finite-dimensional real vector space on the fiber $\xi^{-1}(b)$.

subject to the following axiom.

(3.1.1) There exists a *numerable*[‡] open cover $\{U_\alpha\}_{\alpha\in A}$ of B, a natural number n_α for each $\alpha \in A$, and a homeomorphism $\phi_\alpha\colon \xi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbf{R}^{n_\alpha}$ so that the triangle

$$\begin{array}{ccc} \xi^{-1}(U_\alpha) & \xrightarrow[\phi_\alpha]{\sim} & U_\alpha \times \mathbf{R}^{n_\alpha} \\ & {\scriptstyle \xi}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_{U_\alpha}} & \\ & U_\alpha & \end{array}$$

commutes, and, moreover, for any $b \in U_\alpha$ the restriction of ϕ_α to the fiber $\xi^{-1}(b)$ is a linear isomorphism of real vector spaces. Such a map ϕ_α is called a ***local trivialization*** of ξ over U_α.

An open cover as in (3.1.1) is called a ***bundle atlas***.

We say that a real vector bundle ξ is an ***n-plane bundle*** if each fiber $\xi^{-1}(b)$ for $b \in B$ has the structure of an n-dimensional real vector space, and a ***line bundle*** in the special case that $n = 1$.

3.2. **Example.** For any space B and any natural number n, the projection $\mathrm{pr}_B\colon B \times \mathbf{R}^n \longrightarrow B$ has the structure of an n-plane bundle. The vector space structure on each fiber is given by

$$\alpha(b,u) + \beta(b,v) \coloneqq (b, \alpha v + \beta u),$$

for $\alpha, \beta \in \mathbf{R}$ and $u, v \in \mathbf{R}^n$. This bundle is called the ***trivial n-plane bundle.***

The following example is one of the most important examples of vector bundles, and is integral to the axiomatization of Stiefel–Whitney classes in §5.

3.3. **Example.** Define a line bundle ℓ_n over $\mathbf{R}\mathrm{P}^n$ in the following manner. The total space $E(\ell_n)$ is the subset of $\mathbf{R}\mathrm{P}^n \times \mathbf{R}^{n+1}$ consisting of pairs $(\{\pm x\}, u)$, where u lies on the line in $\mathbf{R}^{n+1}$ spanned by x, and the projection $\ell_n\colon E(\ell_n) \longrightarrow \mathbf{R}\mathrm{P}^n$ is given by sending $(\{\pm x\}, u) \longmapsto \{\pm x\}$. The fiber over $\{\pm x\}$ is simply the line in $\mathbf{R}^{n+1}$ spanned by x. We endow each fiber with its usual 1-dimensional real vector space structure. We call ℓ_n is called the ***canonical line bundle*** over $\mathbf{R}\mathrm{P}^n$. In particular, since $\mathbf{R}\mathrm{P}^1 \cong S^1$, the total space of ℓ_1 is an open Möbius band.

A generalization of this example is given by the *Stiefel and Grassmanian manifolds.*

[‡]Again, following May [11, Ch. 23 §1] we assume that the cover is numerable.

3.4. **Definition.** An ***orthonormal n-frame*** in $\mathbf{R}^k$ is a n-tuple of linearly independent orthonormal vectors in $\mathbf{R}^k$, i.e., an orthonormal basis for a n-dimensional subspace of $\mathbf{R}^k$. The set of all orthonormal n-frames in $\mathbf{R}^k$ is a subset of the n-fold product $\mathbf{R}^k \times \cdots \times \mathbf{R}^k$, called the ***Stiefel manifold***, denoted $\mathrm{St}_n(\mathbf{R}^k)$.

There is a natural action of the real orthogonal group O_n on the Stiefel manifold $\mathrm{St}_n(\mathbf{R}^k)$ given by multiplying each vector in the an orthonormal n-frame by an orthogonal matrix. The ***Grassmann manifold***, or simply ***Grassmannian***, $\mathrm{Gr}_n(\mathbf{R}^k)$ is the orbit space $\mathrm{St}_n(\mathbf{R}^k)/\,\mathrm{O}_n$. The elements of $\mathrm{Gr}_n(\mathbf{R}^k)$ are associated with n-dimensional subspaces of $\mathbf{R}^k$. Moreover, the projection $\mathrm{St}_n(\mathbf{R}^k) \longrightarrow \mathrm{Gr}_n(\mathbf{R}^k)$ is a principal O_n-bundle.

3.5. **Warning.** The notation $V_n(\mathbf{R}^k)$ is very commonly used instead of $\mathrm{St}_n(\mathbf{R}^k)$ for the Stiefel manifold, but we prefer the notation $\mathrm{St}_n(\mathbf{R}^k)$ as we find it more evocative. Likewise, the notation $G_n(\mathbf{R}^k)$ is commonly used for the Grassmanian $\mathrm{Gr}_n(\mathbf{R}^k)$, but, again, we prefer the notation $\mathrm{Gr}_n(\mathbf{R}^k)$ as we find it more evocative, and clearer in the context of this paper as we always use G to denote a (topological) group.

3.6. **Example.** For each positive integer k, there is a homeomorphism $\mathrm{Gr}_1(\mathbf{R}^k) \cong \mathbf{R}\mathrm{P}^k$.

It is a standard result [12, Lem. 5.1] that the Grassmannian manifolds are compact topological manifolds. Moreover, Grassmanians have a beautifully combinatorial cell structure; for a description see [12, Ch. 6]. We can start to see the connection between principal bundles and vector bundles by constructing a vector bundle from the principal O_n-bundle $\mathrm{St}_n(\mathbf{R}^k) \longrightarrow \mathrm{Gr}_n(\mathbf{R}^k)$.

3.7. **Example.** Construct a ***canonical n-plane bundle*** $\gamma_{n,k}$ over $\mathrm{Gr}_n(\mathbf{R}^k)$ as follows. Regarding elements of $\mathrm{Gr}_n(\mathbf{R}^k)$ as n-dimensional subspaces of $\mathbf{R}^k$, the total space is

$$E(\gamma_{n,k}) \coloneqq \{(V, v) \,|\, v \in V \in \mathrm{Gr}_n(\mathbf{R}^k)\},$$

topologized as a subspace of $\mathrm{Gr}_n(\mathbf{R}^k) \times \mathbf{R}^k$. The projection $\gamma_{n,k} \colon E(\gamma_{n,k}) \longrightarrow \mathrm{Gr}_n(\mathbf{R}^k)$ is defined by $\gamma_{n,k}(V, v) = V$, and the fiber over a n-plane $V \in \mathrm{Gr}_n(\mathbf{R}^k)$ has the obvious vector space structure. Since the proof that the vector bundle $\gamma_{n,k}$ is locally trivial is not the case of primary interest, we omit it — a proof can be found in [12, Lem. 5.2].

The canonical n-plane bundles $\gamma_{n,k}$ are integral to the relationship between vector bundles and principal bundles. To begin to explain this relationship, we need to discuss *morphisms of vector bundles*. Just like with principal G-bundles, morphisms are given by commutative squares, however, with vector bundles we have two reasonable choices for the morphisms.

3.8. **Definition.** Suppose that $\xi \colon E \longrightarrow B$ and $\xi' \colon E' \longrightarrow B'$ are vector bundles. A ***bundle morphism*** $\xi' \longrightarrow \xi$ consists of a pair $(\bar{f}, f)$ of maps making the following square commute

$$\begin{CD} E' @>{\bar{f}}>> E \\ @V{\xi'}VV @VV{\xi}V \\ B' @>>{f}> B, \end{CD}$$

where $\bar{f}$ is linear on each fiber. We call a bundle morphism $(\bar{f}, f)$ a ***bundle map*** if $\bar{f}$ is a linear isomorphism on each fiber.

Milnor and Stasheff show that any n-plane bundle admits a bundle map to the canonical bundle $\gamma_{n,k}$, at least when k is large enough.

3.9. **Lemma** ([12, Lem. 5.3]). *For any n-plane bundle ξ over a base space B, there exists a bundle map $\xi \longrightarrow \gamma_{n,k}$, for sufficiently large k.*

There are two categories at play here: the category of vector bundles and bundle morphisms, and the category of vector bundles and bundle maps. For the rest of this section we present a few methods to construct new vector bundles from old — we will see that both the category of vector bundles and bundle morphisms and the category of vector bundles and bundle maps are important to consider for these constructions.

The main construction that we are interested in is a direct sum operation for vector bundles which is called the *Whitney sum*. We only present the constructions that are relevant to giving an axiomatization of Stiefel–Whitney classes (which is the goal of §5), but almost all of the structures on vector spaces, such as images, (co)kernels, tensor products, and exterior powers, lift to suitable structures of vector bundles, defined fiber-wise. Probably the most elementary construction is restriction to a subspace.

3.10. **Definition.** Suppose that $\xi\colon E \longrightarrow B$ is a vector bundle and that B' is a subspace of B. Defining $E' \coloneqq \xi^{-1}(B')$, we see that the restriction of ξ to E' gives a vector bundle $\xi|_{B'}\colon E' \longrightarrow B'$, with the property that the fiber of $\xi|_{B'}$ at $b \in B'$ is $\xi^{-1}(b)$. We call $\xi|_{B'}$ the ***restriction of ξ to B'***.

3.11. **Definition.** Suppose that $\xi\colon E \longrightarrow B$ is a vector bundle and that f is a continuous map $B' \longrightarrow B$. We form the pullback $B' \times_B E$ in **Top**. Regarding the pullback $B' \times_B E$ as a subset of the product $B' \times E$, we give a vector space structure on each fiber $(f^\star\xi)^{-1}(b)$ by defining

$$\alpha(b', x) + \beta(b', y) \coloneqq (b', \alpha x + \beta y),$$

where $\alpha, \beta \in \mathbf{R}$ and $x, y \in E$ are elements in the fiber of ξ over $f(b')$, which is a vector space because ξ is a vector bundle. Then the map $\bar{f}\colon B' \times_B E \longrightarrow E$ given by the pullback sends the vector space $(f^\star\xi)^{-1}(b)$ isomorphically to $\xi^{-1}(f(b))$. If $\{U_\alpha\}_{\alpha\in A}$ is a bundle atlas for ξ with trivializations $\{\phi_\alpha\}_{\alpha\in A}$, then setting $U'_\alpha \coloneqq f^{-1}(U_\alpha)$ gives an open cover $\{U'_\alpha\}_{\alpha\in A}$ of B, and defining a homeomorphsim

$$\phi'_\alpha\colon U'_\alpha \times \mathbf{R}^{n_\alpha} \xrightarrow{\sim} (f^\star\xi)^{-1}(U'_\alpha)$$

by $\phi'_\alpha(b, x) \coloneqq (b, \phi_\alpha^{-1}(f(b), x))$ gives a trivialization of the cover. It is clear that gives

$$f^\star\xi\colon B' \times_B E \longrightarrow B'$$

the structure of a vector bundle. We call $f^\star\xi$ the bundle ***induced by f***, or the ***pullback along f***, and denote the total space of this induced bundle by $E(f^\star\xi)$.

3.12. **Example.** Suppose that $\xi\colon E \longrightarrow B$ is a vector bundle and that B' is a subspace of B. Let ι denote the inclusion $B' \hookrightarrow B$. Then $\xi|_{B'} \cong \iota^\star\xi$.

In analogy with principal G-bundles, homotopic maps between base spaces induce isomorphic vector bundles.

3.13. **Theorem.** *Suppose that $\xi\colon E \longrightarrow B$ is a vector and that $f_0, f_1\colon B' \longrightarrow B$ are homotopic maps. Then the induced bundles $f_0^\star\xi$ and $f_1^\star\xi$ are isomorphic.*

Again, theorem tells us that we can transform the rather rigid data of a vector bundle into purely homotopical data. Moreover, homotopies between base spaces can be lifted, giving *homotopic bundle maps,* which are defined in the same way they were for principal bundles.

3.14. **Definition.** Suppose that $\xi\colon E \longrightarrow B$ and $\xi'\colon E' \longrightarrow B'$ are vector bundles, and that Suppose that

$$(\bar{f}_0, f_0), (\bar{f}_1, f_1)\colon \xi' \longrightarrow \xi$$

are bundle maps. A ***bundle homotopy*** from $(\bar{f}_0, f_0)$ to $(\bar{f}_1, f_1)$ is a pair of homotopies $(\bar{h}, h)$, where $\bar{h}$ is a homotopy from $\bar{f}_0$ to $\bar{f}_1$, h is a homotopy from f_0 to f_1, so that for each $t \in [0,1]$ the pair $(\bar{h}|_{E' \times \{t\}}, h|_{B' \times \{t\}})$ is a bundle map.

3.15. **Theorem.** *Suppose that $\xi\colon E \longrightarrow B$ and $\xi'\colon E' \longrightarrow B'$ are vector bundles,*

$$(\bar{f}_0, f_0)\colon \xi' \longrightarrow \xi$$

is a bundle map, and f_0 is homotopic to a map $f_1\colon B' \longrightarrow B$ via a homotopy h. Then there exists a map $\bar{f}_1\colon E' \longrightarrow E$ so that $(\bar{f}_1, f_1)$ is a bundle map, along with a homotopy $\bar{h}$ between $\bar{f}_0$ and $\bar{f}_1$ so that $(\bar{h}, h)$ is a bundle homotopy.

We can also use pullback bundles to define a direct sum operation on vector bundles, which on fibers is just the direct sum of the vector spaces. The most illuminating way to do this is to first define the *product* of vector bundles, and then define the direct sum as a pullback of the product.

3.16. **Definition.** Given vector bundles $\xi\colon E \longrightarrow B$ and $\xi'\colon E' \longrightarrow B'$, we can form the ***product bundle*** $\xi \times \xi'$ whose base space is $B \times B'$ and total space is $E \times E'$. The projection $E \times E' \longrightarrow B \times B'$ is simply the product map $\xi \times \xi'$. The vector space structure on the fiber $(\xi \times \xi')^{-1}(b, b')$ is given as the product of the vector space structures on the individual fibers, and the bundle atlas on $\xi \times \xi'$ is the obvious one.

Notice that since the product of vector spaces agrees with the direct sum, the fiber of $\xi \times \xi'$ over (b, b') is isomorphic to the direct sum of the fiber of ξ over b with the fiber of ξ' over b'. In this sense, the product vector bundle can be seen as an *external direct sum.*

The product $\xi \times \xi'$ is clearly the categorical product in the category of vector bundles and bundle morphisms. However, the product projections are not isomorphisms on each fiber, so the product projections are bundle morphisms, but *not* bundle maps. Thus, in order to be able to do standard constructions such as taking products, it often makes sense to work in the category of vector bundles and bundle morphisms, rather than bundle maps.

3.17. **Definition.** Suppose that ξ and ξ' are vector bundles a space B. Let $\Delta\colon B \hookrightarrow B \times B$ denote the diagonal embedding. The induced bundle $\Delta^*(\xi \times \xi')$ is called the ***Whitney sum*** of ξ and ξ', denoted by $\xi \oplus \xi'$.

In a similar manner to forming that product, or external direct sum, of two vector bundles $\xi\colon E \longrightarrow B$ and $\xi'\colon E' \longrightarrow B'$, we can form an *external tensor product* $\xi \boxtimes \xi'$ of vector bundles, where the fiber of $\xi \boxtimes \xi'$ over (b, b') is the tensor product of the fiber of ξ over b with the fiber of ξ' over b'. Having done this, the *tensor product* $\xi \otimes \xi'$ of vector bundles over the same base space is defined as the pullback $\Delta^*(\xi \boxtimes \xi')$ as in Definition 3.17, though there are point-set topological considerations that one must consider. The basic idea is the fiber-wise operations pass to operations on bundles, when suitably topologized. We refer the interested reader to [9, Ch. 3 §8; 12, Ch. 3; 20, Ch. 14 §§2 and 5] for more thorough treatments.

4. Relating Principal Bundles & Vector Bundles

We are now ready to describe the amazing relationships between vector bundles and principal bundles, and how this theory lets us translate the much more strict data of bundle maps into homotopical data. This is the result of a very general phenomenon discussed in [11, Ch. 23 §8]; we only provide a brief summary of this phenomenon here.

One might hope that given a topological group G, there is some sort of *universal* principal G-bundle ξ_G with the property that any principal G-bundle ξ admits a bundle map $\xi \longrightarrow \xi_G$, which is unique in an appropriate sense. Assuming mild additional hypotheses, this is a reasonable thing to ask for.

4.1. **Definition.** A principal G-bundle $\xi_G \colon EG \longrightarrow BG$ is called ***universal*** if any principal G-bundle ξ admits a G-bundle map $\xi \longrightarrow \xi_G$ which is unique up to bundle homotopy. The space BG is called the ***classifying space*** of G.

Amazingly, universal bundles exist, and, in principle, we should provide a construction of a universal principal G-bundle $\xi_G \colon EG \longrightarrow BG$, but to avoid unnecessary technical details, we only summarize a common construction using simplicial spaces. Regarding a topological group G as a one-object topological groupoid, we form the nerve NG of G, which is a simplicial space. The geometric realization $|NG|$ of NG is a model for the classifying space BG. In this setting, the realization of the simplicial space whose n-simplices are given by G^{n+1} is a model for EG. An early source for this construction is [18].

From the definition, it is immediate that universal G-bundles are unique up to bundle homotopy. This is just one way of characterizing the universal principal G-bundle, though there are many others. One extremely convenient characterization is given by the total space being contractible.

4.2. **Theorem.** *A principal G-bundle $\xi \colon E \longrightarrow B$ is universal if and only if E is contractible.*

There is an extremely interesting formal consequence of the existence of universal principal G-bundles; informally, isomorphism classes of principal G-bundles are classified by homotopy classes of maps of the base space into BG. To state this formally, we must first introduce a bit of notation and terminology.

4.3. **Notation.** Write $h\mathbf{Top}$ for the ***homotopy category*** of topological spaces, whose objects are topological spaces, and a morphism $X \longrightarrow Y$ is a homotopy class of maps $X \longrightarrow Y$.

4.4. **Definition.** For any topological space B, let $P_G(B)$ be the set of isomorphism classes of numerable principal G-bundles over B. Given a map of topological spaces $f \colon B' \longrightarrow B$, sending the isomorphism class of a bundle $\xi \colon E \longrightarrow B$ to the isomorphism class of it pullback $f^\star\xi \colon E(f^\star\xi) \longrightarrow B'$ defines a set-map

$$P_G(f) \colon P_G(B) \longrightarrow P_G(B').$$

In light of Theorem 3.13, the assignment $f \longmapsto P_G(f)$ is homotopy invariant. Moreover, it is obviously functorial, hence these data assemble into a well defined functor $P_G \colon h\mathbf{Top}^{op} \longrightarrow \mathbf{Set}$.

4.5. **Notation.** Suppose that C is a category and $c, c' \in C$. We write $C(c, c')$ for the collection of morphisms $c \longrightarrow c'$ in C. We write $C(-, c') \colon C^{op} \longrightarrow \mathbf{Set}$ for the ***contravariant functor represented by*** c'. The assignment on morphisms is given as follows: given a morphism $f \colon x \longrightarrow y$ in C, the morphism $C(f, c') \colon C(y, c') \longrightarrow C(x, c')$ is given by pre-composition by f, i.e., the set-map that sends a morphism $f' \colon y \longrightarrow c'$ to the composite $f' \circ f \colon x \longrightarrow c'$.

Given categories C and D and functors $F, G \colon C \longrightarrow D$, we denote a natural transformation α from F to G by $\alpha \colon F \Longrightarrow G$.

Now we are ready to formally state the classification theorem. In order to avoid unnecessary technical detail for readers unfamiliar with formal properties of pullbacks, we only give a sketch of the proof, which provides sufficient detail for the reader familiar with these formalities to check the remaining details.

4.6. **Theorem.** *Suppose that G is a topological group. The contravariant functor P_G is representable, with representing object the classifying space BG, i.e., there is a natural isomorphism*

$$P_G \cong h\mathbf{Top}(-, BG).$$

For the reader with little experience in category theory, morally what Theorem 4.6 says is that the data of an isomorphism class of a numerable principal G-bundle over space B is "the same" as the data of a homotopy class of maps $B \longrightarrow BG$. This is an amazing result as it translates the data of a homeomorphism into seemingly weaker homotopical data. This theorem also explains the name *classifying space* — BG classifies principal G-bundles. Moreover, from basic category theory this implies that BG is unique up to unique homotopy equivalence.

Before we provide a sketch of the proof of Theorem 4.6, let us take a slight diversion in order to explain how Theorem 4.6 fits into the theory.

4.7. **Definition.** Suppose that X is a right G-space and that Y is a left G-space. The ***balanced product*** $X \times_G Y$ is the quotient of $X \times Y$ under the equivalence relation $(xg, y) \sim (x, gy)$, for $(x, y) \in X \times Y$ and $g \in G$. Alternatively, if we convert Y into a right G-space by setting $yg := g^{-1}y$, then $X \times Y$ is the orbit space of the right G-space $X \times Y$ under the diagonal action.

Given right G-spaces X and X', left G-spaces Y and Y', and a pair of G-equivariant maps $\phi\colon X \longrightarrow X'$ and $\psi\colon Y \longrightarrow Y'$, taking orbit spaces induces a map

$$\phi \times_G \psi\colon X \times_G Y \longrightarrow X' \times_G Y',$$

called the ***balanced product*** of the maps ϕ and ψ.

4.8. **Observations.** Here are some basic properties of the balanced product.

(4.8.a) For any right G-space X, we have $X \times_G * \cong X/G$, where $*$ is the one-point space.

(4.8.b) Regarding G as a left G-space, for any right G-space X, the right action of G on itself by multiplication makes $X \times_G G$ into a right G-space, and the action map $X \times G \longrightarrow X$ induces a G-equivariant homeomorphism $X \times_G G \xrightarrow{\sim} X$.

4.9. **Construction.** Suppose that $\xi\colon E \longrightarrow B$ is a principal G-bundle, and that F is a *free* left G-space. Taking the quotient of the projection $\mathrm{pr}_E\colon E \times F \longrightarrow E$ gives a map

$$q := \mathrm{pr}_E/G\colon E \times_G F \longrightarrow E/B \cong B$$

Then the fiber of q over every point of B is F. Moreover, a local trivialization

$$\phi\colon \mathrm{pr}_E^{-1}(U) \longrightarrow U \times G$$

of ξ yields a local trivialization of q via the map

$$q^{-1}(U) = \mathrm{pr}_E^{-1}(U) \times_G F \xrightarrow[\phi \times_G \mathrm{id}_F]{\sim} (U \times G) \times_G F \cong U \times F\,.$$

We call $q\colon E \times_G F \longrightarrow G$ the ***associated fiber bundle with structure group*** G ***and fiber*** F.

An extremely important point of this theory, whose technical details we suppress in order to maintain concise, is that with the correct formal definition of a *fiber bundle with structure group G and fiber F,* every fiber bundle $E' \longrightarrow B$ is equivalent to one of the form $E \times_G F \longrightarrow E/G \cong B$ for some principal G-bundle $E \longrightarrow B$. The conclusion is that for any fiber F with a free G-action, the set $P_G(B)$ is (naturally) isomorphic to the set of equivalence classes of bundles over B with structure group G and fiber F.

The key insight into the theory that we are interested lies in the following very specific example of the general phenomenon.

4.10. **Example** ([11, Ch. 23 §8]). Consider the case of Construction 4.9 when G is the orthogonal group O_n, and the fiber F is the space $\mathbf{R}^n$ with the usual left O_n-action given by left matrix multiplication. Fiber bundles over a base space B with structure group O_n and fiber

$\mathbf{R}^n$ are real n-plane bundles. Hence Construction 4.9 gives an explicit way of connecting the theory of vector bundles with theory of principal bundles.

Now we return to our original path and present a sketch of the proof of Theorem 4.6.

Sketch of the proof of Theorem 4.6. Let $\xi_G \colon EG \longrightarrow BG$ be the universal principal G-bundle. Since homotopic maps $B \longrightarrow BG$ have isomorphic induced bundles by Theorem 2.10, the map

$$\mu_B \colon h\mathbf{Top}(B, BG) \longrightarrow P_G(B)$$

defined by sending the homotopy class of a map f to the isomorphism class of the induced bundle $f^\star \xi_G$ is well-defined. Moreover, it is immediate from the definitions and basic properties of pullbacks that the components μ_B assemble into a natural transformation

$$\mu \colon h\mathbf{Top}(-, BG) \Longrightarrow P_G.$$

Suppose that $\xi \colon E \longrightarrow B$ is a numerable principal G-bundle. By the universal property of the universal bundle $\xi_G \colon EG \longrightarrow BG$, there exists a G-bundle map $(\bar{f}, f) \colon \xi \longrightarrow \xi_G$ which is unique up to bundle homotopy. Define a map

$$\eta_B \colon P_G(B) \longrightarrow h\mathbf{Top}(B, BG)$$

by sending ξ to the homotopy class of $f \colon B \longrightarrow BG$. It is an immediate consequence of the definitions that the components η_B assemble into a natural transformation

$$\eta \colon P_G \Longrightarrow h\mathbf{Top}(-, BG).$$

Again, it is an immediate consequence of the definitions that the composites $\mu_B \eta_B$ and $\eta_B \mu_B$ are the identity. ☐

The rest of the section is devoted to an extremely important class of examples, which are a special case of the general theory. In § 3 we saw that the projection $\mathrm{St}_n(\mathbf{R}^k) \longrightarrow \mathrm{Gr}_n(\mathbf{R}^k)$ provided a principal O_n-bundle, and constructed canonical n-plane bundles $\gamma_{n,k}$. Under the general framework of Construction 4.9, the associated fiber bundles of the principal O_n-bundles $\mathrm{St}_n(\mathbf{R}^k) \longrightarrow \mathrm{Gr}_n(\mathbf{R}^k)$ are the canonical bundles $\gamma_{n,k}$. Another important aspect of this theory that we saw was that any n-plane bundle admits a bundle map to $\gamma_{n,k}$ for sufficiently large k. What this suggests is that $k \longrightarrow \infty$, the n-plane bundle $\gamma_{n,k}$ becomes universal. To state this precisely, let us first recall how to take *colimits* of a sequence of topological spaces.

4.11. **Recollection.** Suppose that we have a sequence of spaces ordered by inclusion, as displayed below

$$X_1 \hookrightarrow X_2 \hookrightarrow X_3 \hookrightarrow \cdots. \tag{4.11.1}$$

The ***colimit*** of the sequence (4.11.1), denoted by $\mathrm{colim}_{n\geq 1} X_n$ is the space whose underlying set is the union $\bigcup_{n\geq 1} X_n$, topologized in the following manner: a subset $U \subset \mathrm{colim}_{n\geq 1} X_n$ is open (resp., closed) if and only if $U \cap X_n$ is open (resp., closed) for each $n \geq 1$.

4.12. **Example.** A CW-complex X is the colimit $\mathrm{colim}_{n\geq 0} X_n$, where X_n is the n-skeleton of X.

4.13. **Example.** Infinite real projective space $\mathbf{R}\mathrm{P}^\infty$ is the colimit $\mathrm{colim}_{n\geq 1} \mathbf{R}\mathrm{P}^n$. Similarly, infinite complex projective space $\mathbf{C}\mathrm{P}^\infty$ is the colimit $\mathrm{colim}_{n\geq 1} \mathbf{C}\mathrm{P}^n$.

4.14. **Notation.** Write $\mathbf{R}^\infty$ for vector space whose vectors are infinite sequences $(x_1, x_2, \ldots)$, where only finitely many x_i are nonzero, with addition and scalar multiplication defined component-wise. As a topological space, $\mathbf{R}^\infty$ is the colimit $\mathrm{colim}_{n\geq 1} \mathbf{R}^n$.

4.15. **Definition.** The ***infinite Stiefel manifold*** $\mathrm{St}_n(\mathbf{R}^\infty)$ is the set of orthonormal n-frames in $\mathbf{R}^\infty$ topologized as the colimit $\mathrm{colim}_{k\geq 0}\, \mathrm{St}_n(\mathbf{R}^{n+k})$. Similarly, the ***infinite Grassmannian*** $\mathrm{Gr}_n(\mathbf{R}^\infty)$ is the orbit space $\mathrm{St}_n(\mathbf{R}^\infty)/\,\mathrm{O}_n$, where the O_n-action on $\mathrm{St}_n(\mathbf{R}^\infty)$ is given by multiplying each n-frame by an orthogonal matrix. Alternatively, $\mathrm{Gr}_n(\mathbf{R}^\infty)$ is the set of n-dimensional linear subspaces of $\mathbf{R}^\infty$, topologized as the colimit $\mathrm{colim}_{k\geq 0}\, \mathrm{Gr}_n(\mathbf{R}^{n+k})$.

Like at each finite stage, the projection $\mathrm{St}_n(\mathbf{R}^\infty) \longrightarrow \mathrm{Gr}_n(\mathbf{R}^\infty)$ is a principal O_n-bundle, and by taking the colimit, this bundle becomes the universal principal O_n-bundle, so $\mathrm{Gr}_n(\mathbf{R}^\infty)$ is the classifying space of O_n. One particularly easy way to see this is to show that $\mathrm{St}_n(\mathbf{R}^\infty)$ is contractible, a fun exercise which we leave to the reader.

In light of Construction 4.9 and Example 4.10, the associated fiber bundle to the universal principal O_n-bundle $\mathrm{St}_n(\mathbf{R}^\infty) \longrightarrow \mathrm{Gr}_n(\mathbf{R}^\infty)$ will also have a universal property. Because the description is enlightening, we provide a description of the fiber bundle associated to the principal O_n-bundle $\mathrm{St}_n(\mathbf{R}^\infty) \longrightarrow \mathrm{Gr}_n(\mathbf{R}^\infty)$; it is the colimit of the canonical n-plane bundles $\gamma_{n,k}$.

4.16. **Construction.** In analogy with Example 3.7, construct a ***canonical n-plane bundle*** γ_n over $\mathrm{Gr}_n(\mathbf{R}^\infty)$ as follows. Regarding elements of $\mathrm{Gr}_n(\mathbf{R}^\infty)$ as n-dimensional subspaces of $\mathbf{R}^\infty$, the total space is

$$E(\gamma_n) \coloneqq \{(V, v) \,|\, v \in V \in \mathrm{Gr}_n(\mathbf{R}^\infty)\},$$

topologized as a subspace of $\mathrm{Gr}_n(\mathbf{R}^\infty) \times \mathbf{R}^\infty$. The projection $\gamma_n\colon\ E(\gamma_n) \longrightarrow \mathrm{Gr}_n(\mathbf{R}^\infty)$ is defined by $\gamma_n(V, v) = V$, and the fiber over a n-plane $V \in \mathrm{Gr}_n(\mathbf{R}^\infty)$ has the obvious vector space structure.

Applying the general theory from this section to this very specific example, we get the universal property of the canonical n-plane bundle from the universal property of the universal O_n-bundle $\mathrm{St}_n(\mathbf{R}^\infty) \longrightarrow \mathrm{Gr}_n(\mathbf{R}^\infty)$.

4.17. **Proposition.** *Every n-plane bundle ξ admits a bundle map $\xi \longrightarrow \gamma_n$ which is unique up to bundle homotopy.*

5. Stiefel–Whitney Classes Classes & Universal Bundles

In this section we give an axiomatic definition of invariants of vector bundles called *Stiefel–Whitney classes*. Stiefel–Whitney classes are cohomology classes of the base space, and provide an incredible amount of information about vector bundles. Most of material in this section can be found in [12, §4].

5.1. **Notation.** Suppose that X is a space and A is a commutative ring. We write $H^\star(X; A)$ for the cohomology ring of X with coefficients in A and with multiplication given by the cup product.

Throughout this section, all cohomology is with coefficients in $\mathbf{Z}/2$. With this notation in hand we are ready to describe a set of axioms that completely characterize Stiefel–Whiteny classes.

5.2. **Axioms.** ***Stiefel–Whitney classes*** are cohomology classes $w_k(\xi) \in H^k(B; \mathbf{Z}/2)$ for each natural number k assigned to each vector bundle $\xi\colon\ E \longrightarrow B$ satisfying the following axioms.

(5.2.a) The class $w_0(\xi)$ is the identity element of $H^0(B; \mathbf{Z}/2)$, and if ξ is a n-plane bundle, then $w_k(\xi) = 0$ for $k > n$.

(5.2.b) *Naturality:* If $(\bar{f}, f)\colon\ \xi' \longrightarrow \xi$ is a bundle map, then for each $k \in \mathbf{N}$ we have

$$w_k(\xi') = f^\star w_k(\xi),$$

where $f^\star\colon\ H^k(B; \mathbf{Z}/2) \longrightarrow H^k(B'; \mathbf{Z}/2)$ is the induced map on cohomology.

(5.2.c) *Whitney sum formula:* If ξ and ξ' are vector bundles on the same base space B, then

$$w_k(\xi \oplus \xi') = \sum_{i=0}^{k} w_i(\xi) \smile w_{k-i}(\xi'),$$

where $\smile$ denotes the cup product on the cohomology ring $H^\star(B;\mathbf{Z}/2)$.

(5.2.d) The line bundle ℓ_1 on $\mathbf{RP}^1 \cong S^1$ has the property that $w_1(\ell_1) \neq 0$.

It is not obvious that there exists cohomology classes $w_k(\xi)$ satisfying these axioms, or that, assuming existence, these axioms uniquely characterize cohomology classes satisfying the axioms. Since we are more interested in the properties of Stiefel–Whitney classes rather than the proofs of their existence and uniqueness, we refer the reader to [12, Ch. 7] for the proof of uniqueness of Stiefel–Whitney classes, which uses the cohomology of Grassmann manifolds (see Definition 3.4) and projective space, and [12, Ch. 8] for a proof of the existence of Stiefel–Whitney classes, which makes use of Steenrod squares. An alternative proof utilizing the Leray–Hirsch theorem can be found in [8, Ch. 3 §1]. Yet another proof of these facts can be found in [9, Ch. 17 §§1–6].

Assuming that Stiefel–Whitney classes exist and are unique, we can derive some immediate consequences from the axioms.

5.2.1. **Corollary.** *If $\xi \cong \xi'$, then $w_k(\xi) = w_k(\xi')$ for all natural numbers k.*

Proof. This follows immediately from axiom (5.2.b) because $H^k(B;\mathbf{Z}/2) \cong H^k(B';\mathbf{Z}/2)$ for all $k \in \mathbf{N}$. □

5.2.2. **Corollary.** *If ε is a trivial vector bundle, then $w_k(\varepsilon) = 0$ for $k > 0$.*

Proof. Since ε is trivial, there exists a bundle map from ε to a vector bundle over the one-point space $*$. Since $H^k(*;\mathbf{Z}/2) = 0$ for $k > 0$, axiom (5.2.b) implies that $w_k(\varepsilon) = 0$ for $k > 0$. □

5.2.3. **Corollary.** *If ε is the trivial vector bundle over a base space B, and ξ is any other vector bundle over B, then $w_k(\varepsilon \oplus \xi) = w_k(\xi)$ for all $k \in \mathbf{N}$.*

Proof. This follows immediately from the previous corollary and the Whitney sum formula (5.2.c). □

There is another cohomology ring which is not generally isomorphic to $H^\star(B;\mathbf{Z}/2)$ that we can associate to any space. In this ring we can define a *total Stiefel–Whitney class*, which is a compact way of expressing of the Stiefel–Whitney classes w_k all at once.

5.3. **Definition.** For a topological space B, write $H^{\star\star}(B;\mathbf{Z}/2) \coloneqq \prod_{k\geq 0} H^k(B;\mathbf{Z}/2)$. Regarding the elements of the abelian group $H^{\star\star}(B;\mathbf{Z}/2)$ as formal power series $\sum_{k\geq 0} a_k t^k$, where $a_k \in H^k(B;\mathbf{Z}/2)$, we can make $H^{\star\star}(B;\mathbf{Z}/2)$ into a ring by giving it the following product structure: the product on $H^{\star\star}(B;\mathbf{Z}/2)$, suggestively denoted by $\smile$, is given by

$$\left(\sum_{k=0}^{\infty} a_k t^k\right) \smile \left(\sum_{k=0}^{\infty} b_k t^k\right) \coloneqq \sum_{k=0}^{\infty} t^k \sum_{i=0}^{k} a_i \smile b_{k-i}.$$

The ***total Stiefel–Whitney class*** of a n-plane bundle ξ over B is defined to be the element $w(\xi) \coloneqq \sum_{k\geq 0} w_k(\xi)$.

5.4. **Remarks.** First notice that the product $\smile$ on $H^{\star\star}(B;\mathbf{Z}/2)$ is commutative since we are working modulo 2. Second, the Whitney sum formula Axiom 5.2.c can be compactly expressed by saying that $w(\xi \oplus \xi') = w(\xi) \smile w(\xi')$ in $H^{\star\star}(B;\mathbf{Z}/2)$.

Now comes one of the major theorems that ties all of the theory together — the Stiefel–Whitney classes of the canonical n-plane bundles γ_k generate the $\mathbf{Z}/2$-cohomology ring of $\mathrm{Gr}_n(\mathbf{R}^\infty)$.

5.5. **Theorem** ([12, Thm. 7.1]). *The cohomology ring* $H^\star(\mathrm{Gr}_n(\mathbf{R}^\infty); \mathbf{Z}/2)$ *is given by the polynomial algebra* $\mathbf{Z}/2[w_1(\gamma_n), \dots, w_n(\gamma_n)]$.

6. Configuration Spaces & Braid Groups

In this section we explore *configuration spaces*, which have been studied by many, for example, Arnol'd [1, 2] and Segal [19], and have ties to classical homotopy theory. Recently, these spaces have been studied by Church–Farb [5] and Church–Ellenberg–Farb [4], as well as others, and have led to many new discoveries and the theory of *representation stability*. Configuration spaces are intimately related to *braid groups*, as the configuration spaces are the classifying spaces of the braid groups. We present the very basics of these subjects; the interested reader should see [10, 15] for more detailed expository accounts. We conclude by reviewing computations of Fuchs [6] on the cohomology of braid groups, and explain how this relates to vector bundles over configuration spaces and a problem posed by Farb. The material from this section is mostly joint work with Randy Van Why, and I would like to once again thank him for his helpful discussions regarding this material.

6.1. **Definition.** Suppose that X is a topological space and n is a positive integer. The ***ordered configuration space*** $\mathrm{OC}_n(X)$ is the set

$$\mathrm{OC}_n(X) \coloneqq \{ (x_1, \dots, x_n) \in X^n \mid x_i \neq x_j \text{ for } i \neq j \}$$

of n-tuples of distinct points in X, topologized as a subspace of the n-fold product X^n.

6.2. **Definition.** There is a natural action of the symmetric group Σ_n on $\mathrm{OC}_n(X)$ given by permuting the order of the coordinates. The ***unordered configuration space*** $\mathrm{UC}_n(X)$ is the orbit space $\mathrm{OC}_n(X)/\Sigma_n$ under the symmetric group action. Elements of $\mathrm{UC}_n(X)$ are unordered sets of n distinct points of X.

6.3. **Example.** If M is a manifold, then the n-fold product M^n is a manifold. The space $\mathrm{OC}_n(M)$ is open in M^n, hence is also a manifold. Since the symmetric group Σ_n acts freely on $\mathrm{OC}_n(M)$, the orbit space $\mathrm{UC}_n(M)$ is also a manifold. In light of Example 1.11 we see that $\mathrm{UC}_n(M)$ is paracompact.

There are many definitions for the *braid group*, a number of which can be found in [10, §§1.1–1.3]. Since we are concerned with their relation to configuration spaces, we simply define them to be the fundamental groups of configurations in the plane.

6.4. **Definition.** Suppose that n is a positive integer. The ***braid group*** B_n is the fundamental group of the unordered configuration space $\mathrm{UC}_n(\mathbf{C})$. The ***pure braid group*** P_n is the fundamental group of $\mathrm{OC}_n(\mathbf{C})$.

6.5. **Remark.** As Arnol'd [1] notes, the space $\mathrm{UC}_n(\mathbf{C})$ is homeomorphic to the space $\mathrm{Poly}_n(\mathbf{C})$ of monic square-free complex polynomials. The elements of $\mathrm{Poly}_n(\mathbf{C})$ are sets of roots of monic square-free polynomials, which are precisely sets of n unordered points in $\mathbf{C}$, moreover, the topology given on $\mathrm{Poly}_n(\mathbf{C})$ agrees with the topology on $\mathrm{UC}_n(\mathbf{C})$.

What is perhaps more important is the interpretation of what these groups are. A loop in $\mathrm{OC}_n(\mathbf{C})$ based at the point $(1, \dots, n)$ is an n-tuple of loops $(\beta_1(t), \dots, \beta_n(t))$ so that for all

$t \in [0,1]$ and $i \neq j$ we have $\beta_i(t) \neq \beta_j(t)$. A loop in $\mathrm{UC}_n(\mathbf{C})$ based at the point $\{1,\dots,n\}$ is a set of paths $\{\beta_1(t),\dots,\beta_n(t)\}$ so that

$$\{\beta_1(0),\dots,\beta_n(0)\} = \{1,\dots,n\} = \{\beta_1(1),\dots,\beta_n(1)\},$$

and for all $t \in [0,1]$ and $i \neq j$ we have $\beta_i(t) \neq \beta_j(t)$. That is, a loop in $\mathrm{UC}_n(\mathbf{C})$ is a collection of paths that collectively start and end at $\{1,\dots,n\}$, but the individual paths do not necessarily start and end at the same point.

6.6. **Observation.** Notice that the orbit map $\mathrm{OC}_n(\mathbf{C}) \longrightarrow \mathrm{UC}_n(\mathbf{C})$ satisfies the hypothesis of [7, Prop. 1.40(a)], hence is a normal covering map. Thus $\mathrm{OC}_n(\mathbf{C})$ is aspherical if and only if $\mathrm{UC}_n(\mathbf{C})$ is. Also notice that there is a natural fibration $p\colon \mathrm{OC}_{n+1}(\mathbf{C}) \twoheadrightarrow \mathrm{OC}_n(\mathbf{C})$ given by forgetting the last coordinate. Each fiber of p is homeomorphic to $\mathbf{C} \smallsetminus \{1,\dots,n\}$. Since the spaces $\mathbf{C} \smallsetminus \{1,\dots,n\}$ are aspherical, the long exact sequence in homotopy shows that all of the spaces $\mathrm{OC}_n(\mathbf{C})$ are aspherical.

6.7. **Observation.** Since the configuration space $\mathrm{OC}_n(\mathbf{C})$ is path-connected and locally path-connected, by basic covering space theory [7, Prop. 1.40(c)] the pure braid group is a subgroup of the braid group, and the symmetric group Σ_n is isomorphic to B_n/P_n. Let q_n denote the quotient map $B_n \twoheadrightarrow \Sigma_n$, and let $\rho_n\colon \Sigma_n \hookrightarrow \mathrm{O}_n$ denote the regular representation sending a permutation to its permutation matrix. Applying the classifying space functor, the composite map $\rho_n q_n$ yields a classifying map

$$B(\rho_n q_n)\colon \mathrm{UC}_n(\mathbf{C}) \simeq BB_n \longrightarrow B\mathrm{O}_n \simeq \mathrm{Gr}_n(\mathbf{R}^\infty),$$

which we simply denote by f_n from now on.

Fuchs [6, Remark 1.2] notes that the pullback bundle $f_n^\star \gamma_n$ of the canonical n-plane bundle over $\mathrm{Gr}_n(\mathbf{R}^\infty)$ can be described as follows. For simplicity we write ξ_n for $f_n^\star\gamma_n$ and E_n for the total space $E(f_n^\star\gamma_n)$. The total space E_n is the orbit space of $\mathrm{OC}_n(\mathbf{C}) \times \mathbf{R}^n$ under the diagonal action of the symmetric group Σ_n. The elements of E_n are unordered sets

$$\{(z_1,r_1),\dots,(z_n,r_n)\}$$

where the z_i are pairwise distinct complex numbers, and the r_i are real numbers. The projection $E_n \longrightarrow \mathrm{UC}_n(\mathbf{C})$ is given by forgetting the real numbers r_i, formally,

$$\{(z_1,r_1),\dots,(z_n,r_n)\} \longmapsto \{z_1,\dots,z_n\}.$$

Even though Fuchs is able to describe the bundle, he never gives an explicit geometric or algebraic description of the classifying map f_n, and no explicit description appears to be in the literature. Because the space $\mathrm{UC}_n(\mathbf{C})$ is so fundamental, Farb posed the following problem.

6.8. **Problem** (Farb)**.** Give an explicit algebraic or geometric description of the classifying map $f_n\colon \mathrm{UC}_n(\mathbf{C}) \longrightarrow \mathrm{Gr}_n(\mathbf{R}^\infty)$. In other words, describe a universal way of constructing a *specific* n-dimensional subspace of $\mathbf{R}^\infty$ from a set of n distinct points in $\mathbf{C}$ (or, equivalently from a square-free monic degree n polynomial).

One might naïvely propose a solution to Problem 6.8 by simply sending a set $\{z_1,\dots,z_n\}$ to the free real vector space on $\{z_1,\dots,z_n\}$, considered as a subspace of $\mathbf{R}^\infty$. Unfortunately, this approach is too naïve as it does not specify a *specific* n-dimensional subspace of $\mathbf{R}^\infty$. The rest of this section is devoted to answering Problem 6.8. The main tools we use come from Fuchs' [6] computation of the cohomology of $\mathrm{UC}_n(\mathbf{C})$, so before proceeding any further we briefly review Fuchs method of computation.

The main tools of Fuchs' calculation are a cell structure on the one-point compactification $\mathrm{UC}_n^*(\mathbf{C})$ and an application of Poincaré–Lefschetz duality. The cell structure on $\mathrm{UC}_n^*(\mathbf{C})$ is combinatorial in nature. Specifically, it uses *compositions*, which are like partitions, where order matters.

6.9. **Definition.** Suppose that n and k are positive integers. A ***composition of** n **of length** k* is a k-tuple $(m_1, \ldots, m_k)$ of positive integers so that $m_1 + \cdots + m_k = n$. We write $C(n)$ for the set of all compositions of n.

6.10. **Construction** ([6, Remark 3.2])**.** The cell structure on $\mathrm{UC}_n^*(\mathbf{C})$ has a single 0-cell, the point ∞. For each composition $\sigma = (m_1, \ldots, m_k)$ of n, let L_σ denote the subset of $\mathrm{UC}_n(\mathbf{C})$ consisting of elements $\{z_1, \ldots, z_n\}$ so that the points $z_1, \ldots, z_n$ lie on k distinct vertical lines in $\mathbf{C}$, where m_i of the points $z_1, \ldots, z_n$ lie on the i^{th} line, counting from left to right. Notice that the sets L_σ, for $\sigma \in C(n)$ are disjoint and partition $\mathrm{UC}_n(\mathbf{C})$. The $(n+k)$-cells are indexed by compositions of n of length k, and the image of the interior of the cell corresponding to a composition σ is the set L_σ.

The main results of Fuchs' computation are the following.

6.11. **Theorem** ([6, Thm. 4.8])**.** *For each positive integer n and nonnegative integer k, the rank of the group* $H^k(\mathrm{UC}_n(\mathbf{C}); \mathbf{Z}/2)$ *is equal to the number of ways that n can be partitioned into n − k (nonnegative) powers of* 2*. Hence each* $H^k(\mathrm{UC}_n(\mathbf{C}); \mathbf{Z}/2)$ *has a set of distinguished generators indexed by the partitions of n into n − k powers of* 2*.*

6.12. **Theorem** ([6, Thm. 5.2])**.** *For each natural number k, the Stiefel–Whitney class* $w_k(\xi_n)$ *is the sum of the distinguished generators of* $H^k(\mathrm{UC}_n(\mathbf{C}); \mathbf{Z}/2)$.

To make a convenient bundle atlas for ξ_n we expand the sets L_σ that appear in Construction 6.10 to open sets by replacing the vertical lines with disjoint *vertical strips* in $\mathbf{C}$.

6.13. **Definition.** A ***vertical strip*** V in $\mathbf{C}$ is an open set of the form

$$V = \{ z \in \mathbf{C} \mid a < \mathrm{Re}(z) < b \} \cong (a, b) \times \mathbf{R},$$

for real numbers $a < b$.

With this definition we are ready to construct a bundle atlas for ξ_n.

6.14. **Construction.** Construct an open cover for $\mathrm{UC}_n(\mathbf{C})$ in the following manner. For each composition $\sigma = (m_1, \ldots, m_k)$ of n, let U_σ denote the subset of $\mathrm{UC}_n(\mathbf{C})$ of elements $\{z_1, \ldots, z_m\}$ with the property that there exist k disjoint open vertical strips $V_1, \ldots, V_k$ in $\mathbf{C}$, so that $\mathrm{Re}(v_i) < \mathrm{Re}(v_{i+1})$, for all $v_i \in V_i$ and $v_{i+1} \in V_{i+1}$, m_i elements of $\{z_1, \ldots, z_m\}$ lie in V_i, and all of the elements of $\{z_1, \ldots, z_m\}$ lying in V_i have distinct imaginary part, for each i. To see that the sets U_σ are open, notice that for any element $\{z_1, \ldots, z_n\}$ of U_σ, for sufficiently small $\varepsilon > 0$, the set

$$B_\varepsilon(z_1) \cup \cdots \cup B_\varepsilon(z_n),$$

where $B_\varepsilon(z_i)$ denotes the open ball of radius ε in $\mathbf{C}$ centered at z_i, is an open neighborhood of $\{z_1, \ldots, z_n\}$ contained in U_σ. Moreover, for each $\sigma \in C(n)$ we have $L_\sigma \subset U_\sigma$, so the sets U_σ cover $\mathrm{UC}_n(\mathbf{C})$.

Now extend the open cover $\{U_\sigma\}_{\sigma \in C(n)}$ of $\mathrm{UC}_n(\mathbf{C})$ to a bundle atlas for ξ_n as follows. For each composition $\sigma = (m_1, \ldots, m_k)$ of n, there is a linear order $\leq_\sigma$ on a set $\{z_1, \ldots, z_n\} \in U_\sigma$ given in the following manner.

(6.14.a) First choose a k vertical strips $V_1, \ldots, V_k$ as above, so that m_i of the points $z_1, \ldots, z_n$ lie in V_i.

(6.14.b) Within each vertical strip V_i order the elements of $\{z_1, \dots, z_n\}$ lying in V_i by saying that $z_j \leq_\sigma z_\ell$ if and only if $\mathrm{Im}(z_j) \leq \mathrm{Im}(z_\ell)$, and equality holds if and only if $j = \ell$. Notice that this defines a linear order on the elements of $\{z_1, \dots, z_n\}$ in each V_i since all of the elements of $\{z_1, \dots, z_n\}$ in V_i have distinct imaginary part.

(6.14.c) Extend the linear orders on the elements of $\{z_1, \dots, z_n\}$ in each V_i to a linear order on $\{z_1, \dots, z_n\}$ by saying that if $z_j \in V_i$ and $z_{j'} \in V_{i'}$ with $i < i'$, then $z_j <_\sigma z_{j'}$.

(6.14.d) Finally, notice that this ordering on $\{z_1, \dots, z_n\}$ is independent of the choice of vertical strips $V_1, \dots, V_k$ satisfying the above hypotheses.

Now define a homeomorphism $\phi_\sigma \colon U_\sigma \times \mathbf{R}^n \xrightarrow{\sim} \xi_n^{-1}(U_\sigma)$ sending $(\{z_1, \dots, z_n\}, (r_1, \dots, r_n))$ to the set of pairs (z_i, r_j), where r_j is paired with the z_i which is the j^{th} (smallest) element of $\{z_1, \dots, z_n\}$ in the linear order $\leq_\sigma$. Moreover, it is clear that ϕ_σ is a continuous.

The inverse bijection is given in the following manner. For each $\sigma \in C(n)$ there is a map $\boldsymbol{r}_\sigma \colon \xi_n^{-1}(U_\sigma) \longrightarrow \mathbf{R}^n$ given by sending a set of pairs $\{(z_1, r_1), \dots, (z_n, r_n)\}$ to the vector

$$\boldsymbol{r}_\sigma(\{(z_1, r_1), \dots, (z_n, r_n)\})$$

in $\mathbf{R}^n$ whose i^{th} coordinate is the real number r_j so that z_j is the i^{th} smallest element of the set $\{z_1, \dots, z_n\}$ under the order relation $\leq_\sigma$. Moreover, $\boldsymbol{r}_\sigma$ is linear on each fiber of ξ_n. The inverse of ϕ_σ is given by the assignment

$$\{(z_1, r_1), \dots, (z_n, r_n)\} \longmapsto (\{z_1, \dots, z_n\}, \boldsymbol{r}_\sigma(\{(z_1, r_1), \dots, (z_n, r_n)\})),$$

which is clearly continuous, so ϕ_σ is a homeomorphism.

Now we can use an adapted version of the proof of [12, Lem. 5.3] to construct the classifying map $f_n \colon \mathrm{UC}_n(\mathbf{C}) \longrightarrow \mathrm{Gr}_n(\mathbf{R}^\infty)$, giving an answer to Problem 6.8.

6.15. **Construction.** Since $\mathbf{C}$ is a manifold, in light of Examples 1.9, 1.11 and 6.3, we can choose a partition of unity $\{\lambda_\sigma\}_{\sigma \in C(n)}$ so that $\mathrm{supp}(\lambda_\sigma) \subset U_\sigma$, for each $\sigma \in C(n)$. Define a map $\chi_\sigma \colon E_n \longrightarrow \mathbf{R}^n$ for each $\sigma \in C(n)$ by setting

$$\chi_\sigma(y) \coloneqq \begin{cases} \lambda_\sigma \xi_n(y) \boldsymbol{r}_\sigma(y), & y \in \xi_n^{-1}(U_\sigma) \\ 0, & y \notin \xi_n^{-1}(U_\sigma)\,. \end{cases}$$

Notice that each χ_σ is continuous because $\lambda_\sigma \xi_n$ vanishes outside of $\xi_n^{-1}(U_\sigma)$.

Now we combine all of these maps χ_σ together to define a map $E_n \longrightarrow \mathbf{R}^\infty$. First notice that there is a linear order on $C(n)$ given by ordering compositions of length k lexicographically, and extending these orders to $C(n)$ by saying that compositions of length k are less than compositions of length $k+1$, for each k. Because we are interested in constructing a *specific* plane in $\mathbf{R}^\infty$, we use the order relation on $C(n)$ to order the factors in the direct sum $\bigoplus_{\sigma \in C(n)} \mathbf{R}^n$, and define a map $\chi \colon E_n \longrightarrow \bigoplus_{\sigma \in C(n)} \mathbf{R}^n$ by

$$\chi(y) \coloneqq (\chi_\sigma(y))_{\sigma \in C(n)},$$

i.e., χ is the map induced by the maps χ_σ by the universal property of the product. Now extend χ to a map $\bar{\chi} \colon E_n \longrightarrow \mathbf{R}^\infty$ by post-composing the map χ with the standard inclusion

$$\bigoplus_{\sigma \in C(n)} \mathbf{R}^n \hookrightarrow \mathbf{R}^\infty.$$

The classifying map $f_n \colon \mathrm{UC}_n(\mathbf{C}) \longrightarrow \mathrm{Gr}_n(\mathbf{R}^\infty)$ is given by sending the element $\{z_1, \dots, z_n\}$ of $\mathrm{UC}_n(\mathbf{C})$ to the image of the fiber $\xi_n^{-1}(\{z_1, \dots, z_n\})$ under $\bar{\chi}$, which is an n-dimensional subspace of $\mathbf{R}^\infty$.

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