VIZING'S THEOREM AND EDGE-CHROMATIC GRAPH THEORY

ROBERT GREEN

Abstract. This paper is an expository piece on edge-chromatic graph theory. The central theorem in this subject is that of Vizing. We shall then explore the properties of graphs where Vizing's upper bound on the chromatic index is tight, and graphs where the lower bound is tight. Finally, we will look at a few generalizations of Vizing's Theorem, as well as some related conjectures.

Contents

1. Introduction & Some Basic Definitions

Definition 1.1. An **edge colouring** of a graph $G = (V, E)$ is a map $C : E \to S$, where S is a set of colours, such that for all $e, f \in E$, if e and f share a vertex, then $C(e) \neq C(f)$.

Definition 1.2. The **chromatic index** of a graph $\chi'(G)$ is the minimum number of colours needed for a proper colouring of G.

Definition 1.3. The **degree** of a vertex v, denoted by $d(v)$, is the number of edges of G which have v as a vertex. The maximum degree of a graph is denoted by $\Delta(G)$ and the minimum degree of a graph is denoted by $\delta(G)$.

Vizing's Theorem is *the* central theorem of edge-chromatic graph theory, since it provides an upper and lower bound for the chromatic index $\chi'(G)$ of any graph G. Moreover, the upper and lower bound have a difference of 1. That is, for all finite, simple graphs G, $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.

This theorem motivates the study of the properties of graphs where Vizing's lower bound holds (class one graphs) and graphs where the upper bound holds (class two graphs), and characterizations of each.

Date: August 28, 2015.

2. Vizing's Theorem

Definition 2.1. A colour c is **absent** at a vertex v if, given an edge colouring of G, no edge with v as a vertex receives the colour c. A colour c is **present** at a vertex v if, given an edge colouring of G, there exists an edge with v as a vertex that is assigned the colour c.

Theorem 2.2. ***(Vizing)*** *For any finite, simple graph G, $\Delta(G) \leq \chi'(G) \leq \Delta(G)+1$*

Proof. The lower bound, $\Delta(G)$, is trivial, since if G has a vertex v of degree d, then at least d edges share v as a vertex and cannot be coloured with less than d colours.

Now, suppose for contradiction that there exist counterexamples to Vizing's upper bound. Of these counterexamples, let G be a counterexample of minimal size – that is, if one edge of G is removed, G becomes $(\Delta(G)+1)$-edge-colourable.

Let $e = \{v, w_0\}$ be the edge that, if removed, reduces the chromatic index of G to $\Delta(G)+1$.

We construct a sequence of edges $\{v, w_0\}, \{v, w_1\}, \{v, w_2\}, ...$ and a sequence of colours $c_0, c_1, c_2, ...$ called a Kempe Chain as follows:

Let c_i be a colour absent at w_i. Let $\{v, w_{i+1}\}$ be an edge coloured c_i. The Kempe Chain stops at $k \in \mathbb{N}$ when either c_k is a colour absent at v, or c_k is already used on $\{v, w_j\}$ for $j < k$.

If c_k is absent at v, then we can reassign colours c_i to $\{v, w_i\}$ for $i \in [k]$ and we are done.

So now assume c_k is not absent at v. Let c_q be a colour absent at v (We know that this colour exists because we are allowing ourselves $\Delta+1$ colours where maximum degree is Δ). Then recolour $\{v, w_i\}$ for $i \in [j-1]$, and remove the colour from $\{v, w_j\}$. We now must find a way to colour $\{v, w_j\}$. Note that c_k is absent at both w_j and w_k.

Case 1: If c_k is absent at v, then colour $\{v, w_j\}$ with c_k.

Case 2: If c_q is absent at w_j, then colour $\{v, w_j\}$ blue.

Case 3: If c_q is absent at w_k, then colour $\{v, w_i\}$ with c_i for $j \leq i < k$ and colour $\{v, w_k\}$ with c_q (since none of the $\{v, w_i\}$, $j \leq i < k$ are coloured with either c_k or c_q).

If none of these conditions hold, then consider the subgraph G' of G consisting only of edges coloured with c_k or c_q, and their corresponding vertices. Note that the components of G' are either paths or cycles. Since none of the above conditions hold, v, w_j, and w_k must all be endpoints of paths, and so they cannot all be part of the same component.

In the component containing exactly one of these vertices, switch c_k with c_q. Then condition 2 or 3 must apply.

□

Definition 2.3. Graphs that can be coloured with Δ colours are called **class one graphs**. Graphs that require $\Delta+1$ colours are called **class two graphs**.

Naturally, these definitions provoke a lot of questions about how to tell whether a graph is class one or class two. As we will see below, it is far more likely for a random graph to be class one.

3. General Properties of Class One and Class Two Graphs

Theorem 3.1. ***(Erdős)*** *Almost all graphs are class one. That is, if U_n is the set of class one graphs on n vertices, and V_n is the set of all graphs on n vertices, then $\frac{|U_n|}{|V_n|} \to 1$ as $n \to \infty$.*

The proof of this theorem relies on two previous theorems by Erdős, the first being that almost all graphs are connected, and the second being that almost all graphs have a unique vertex of maximum degree. With these two, it suffices to show (as Vizing did) that every graph with a unique vertex of maximum degree must be class one. More specifically, he showed that every class two graph must have at least three vertices of maximum degree.

Definition 3.2. A graph is **regular** if every vertex has the same degree. A **k-regular graph** is a graph where every vertex has degree k.

Definition 3.3. A **perfect matching** on a graph $G = (V, E)$ is a subset $F \subset E$ such that for all $v \in V$, v appears as the endpoint of exactly one edge of F.

Theorem 3.4. *A regular graph on an odd number of vertices is class two*

Proof. Let G be a k-regular graph on $n = 2x + 1$ vertices, for some x. On a graph of odd order, $k = 2l$ for some l, otherwise the k-regularity condition is violated.

By the Handshake Lemma, G has $\frac{kn}{2} = l(2x + 1)$ edges.

Now, assume for contradiction that k colours can colour the edges of G.

$\frac{l(2x+1)}{k} = x + \frac{1}{2}$ (average edges per colour), so there is at least one colour that appears on $x+1$ edges. But G only has $2x+1$ vertices, so two of these same-coloured edges must share a vertex, and we have a contradiction. By Vizing's Theorem, the only other option for the chromatic index is $k + 1$.

□

Alternatively, this theorem is an immediate corollary of a much more complex theorem that states that a k-regular graph is class one if and only if it is one-factorable (can be written as the union of k perfect matchings). Clearly, a graph on an odd number of vertices cannot have even one perfect matching, let alone k of them.

Now we come to an important theorem of König regarding bipartite graphs.

Definition 3.5. A graph $G = (V, E)$ is **bipartite** if V can be partitioned into two sets V_1 and V_2 such that if $v, w \in V_i$, then $v \not\sim w$.

Theorem 3.6. ***(König)*** *If G is bipartite, then $\chi'(G) = \Delta(G)$*

Proof. Suppose, for contradiction, that counterexamples exist. Let G be a class two bipartite graph of minimum size.

Let $e = \{v, w\}$ be the edge that, if removed, transforms G into a class one graph. So $\chi'(G - e) = \Delta(G - e)$. Since $G - e$ can be coloured with $\Delta(G)$ colours, each of the $\Delta(G)$ colors must be assigned to an edge incident to either v or w (otherwise this color could be assigned to e producing a class one coloring of G).

Now, in $G - e$, $d(v) < \Delta(G)$ and $d(w) < \Delta(G)$, so there is a colour c_a absent at v, and a colour c_b absent at w. Then $c_a \neq c_b$, and furthermore, c_a is present at w and c_b is present at v.

Let P be a path of maximum length having initial vertex w whose edges are alternately coloured c_a and c_b. Note that $v \notin P$, since otherwise P has odd length, implying that the first and last edges of P are both coloured with c_a. This is a contradiction, since c_a is absent at v, so v is not incident with any edge coloured by c_a.

Interchanging the colours c_a and c_b on the edges of P produces a new edge colouring of Ge with $\Delta(G)$ colours in which neither v nor w is incident with an edge coloured by c_a.

Assigning $\{v, w\}$ to c_a then produces a class one edge colouring of G.

□

Definition 3.7. Let $G = (V, E)$ be a graph. An independent edge set $F \subset E$ is a set of edges where no two of them share any vertex. The **edge-independence number** $\alpha'(G)$ of G is the size of the largest independent edge set in G.

It goes without saying that no graph can have an independent edge set with size greater than $\frac{|V|}{2}$. It also does not take much work to observe that $\chi'(G) \geq \frac{|E|}{\alpha'(G)}$.

Theorem 3.8. *If $|E(G)| > \alpha'(G)\Delta(G)$, then G is class two.*

Proof. By assumption, and the observation above, we have the following inequality:

$$\chi'(G) \geq \frac{|E|}{\alpha'(G)} > \frac{\alpha'(G)\Delta(G)}{\alpha'(G)} = \Delta(G)$$

By Vizing's Theorem, $\chi'(G) = \Delta(G) + 1$.

□

4. The Petersen Graph and Other Snarks

Definition 4.1. A **bridge** is an edge of a graph that, if removed, increases the number of connected components of the graph.

Definition 4.2. A **snark** is a bridgeless, 3-regular, class two graph.

Snarks have some very interesting properties, especially the Petersen Graph. Snarks are quite hard to find or invent, and until 1946, the Petersen Graph was the only known Snark. Since then, many others have been discovered, and infinite families of snarks with similar characteristics have been constructed.

Figure 1 is the Petersen Graph.

Theorem 4.3. *The Petersen Graph is class two.*

Proof. Suppose, for contradiction, that there exists a three-edge-colouring of the Petersen Graph. Since the Petersen Graph contains a five-cycle (the outer one), all three colours must appear on this five-cycle. Call these colours c_1, c_2, c_3.

One of these colours will appear once on the five-cycle, whereas the others will appear twice. Assume without loss of generality that the colouring of the outer five-cycle goes c_1, c_2, c_3, c_1, c_2. This will determine the colours of the five edges connecting the inner five-cycle to the outer one.

Notice that this determines the colour of four of the edges of the inner five-cycle – two of which are adjacent to each other and receive the same colour. Thus we have a contradiction.

□

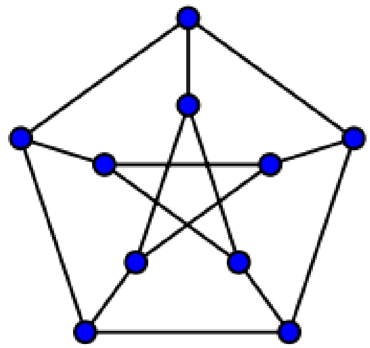

FIGURE 1. The Petersen Graph

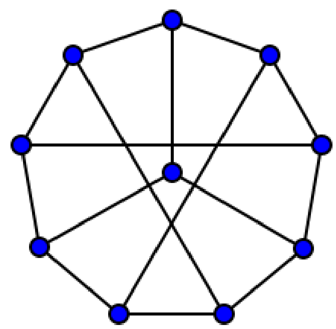

FIGURE 2. The Petersen Graph, drawn differently

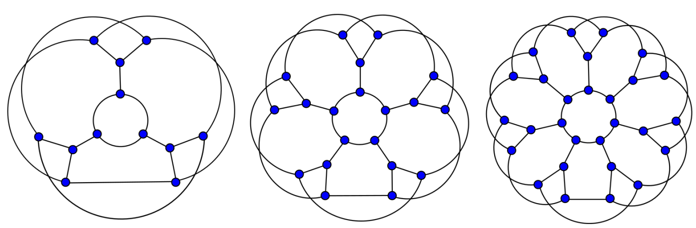

FIGURE 3. Flower Snarks

In 1975, Rufus Isaacs constructed an infinite family of snarks, called the Flower Snarks. Up until this point, only three snarks had been known to the mathematical community (See figure 2).

Construction of the Flower Snarks started with the Petersen Graph (shown above). We then replace the central vertex with a three-cycle to form J_3, the three-petaled Flower Snark. J_5, J_7, and so on, are constructed by adding two petals to the graph and two vertices to the central cycle. Proving that these graphs are snarks goes the same way as the above proof for the Petersen Graph.

J_3, J_5, and J_7 are pictured in figure 3.

5. Generalizations and Conjectures Regarding Vizing's Theorem

Of course, as mathematicians, we are never satisfied when we solve a specific problem (in this case, Vizing's Theorem). Rather, we seek to continue learning about the subject, see if we can relate it to anything else, and try to make our results as general as possible.

A question one might ask about Vizing's Theorem is, "Why is the upper bound on the chromatic index $\Delta + 1$, of all things?"

The next theorem generalizes Vizing's Theorem to multigraphs (graphs that can have more than one edge between two vertices, and can have edges whose endpoints are the same vertex).

Definition 5.1. The **maximum multiplicity** of a graph $\mu(G)$ is the maximum number of edges connecting the same pair of vertices.

Theorem 5.2. *For any multigraph,* $\chi'(G) \leq \Delta(G) + \mu(G)$

For simple (non-multi) graphs, this theorem simplifies to Vizing's Theorem, $\chi'(G) \leq \Delta(G) + 1$

After the publication of this incredibly complex theorem, Claude Shannon (known mainly for his work in Information Theory) came up with an upper bound for the chromatic index of a multigraph purely in terms of Δ:

Theorem 5.3. *For any multigraph G,* $\chi'(G) \leq \frac{3\Delta(G)}{2}$

Proof. Suppose, for contradiction, that counterexamples exist. Let G be a counterexample of minimum size.

Let $\Delta(G) = \Delta$ and $\mu(G) = \mu$ and $\chi'(G) = k$.

Because G is minimal, for every edge $f \in E$, $\chi'(G - f) = k - 1$.

By Theorem 5.2, $k \leq \Delta + \mu$. By assumption, $k > \frac{3\Delta}{2}$.

Let v and w be vertices of G such that there are μ edges joining them. Let e be one of the edges joining v and w.

Then $\chi'(G - e) = k - 1$, so there exists a $(k-1)$-edge coloring of $G - e$. Now, the number of colors not used in coloring the edges incident to v is at least $(k - 1) - (\Delta - 1) = k - \Delta$. The same is true of edges incident to w.

Each of these $k - \Delta$ or more colors not used to color edges incident to v must be used to color edges incident to w, and vice versa – otherwise e can be coloured easily.

Thus, the number of colours used to colour the edges incident to v and w (except e) is $2(k - \Delta) + \mu - 1$.

By assumption, $\frac{3\Delta}{2} < k$, and by Theorem 5.2, $k \leq \Delta + \mu$, so we have $\frac{\Delta}{2} < \mu$.

Thus $2k - (\frac{3\Delta}{2}) - 1 < k - 1$ and $\frac{3\Delta}{2} > k$, a contradiction. □

Another generalization of Vizing's Theorem has been proposed (a generalization to hypergraphs, rather than multigraphs). However, this generalization has yet to be proven.

Definition 5.4. A **hypergraph** $H = (V, E)$ is defined by its vertex set V and its edge set E. For graphs, E is confined to be a subset of the two-vertex elements of $\wp(V)$. However, for hypergraphs, E can be any subset of $\wp(V)$ – that is, an edge can connect more than two vertices.

Definition 5.5. A uniform hypergraph is a hypergraph whose edges all have the same size. A k-uniform hypergraph is a hypergraph whose edges all have size k.

Conjecture 5.6. *Let H be a simple k-uniform hypergraph, and assume that every set of $k-1$ vertices is contained in at most r edges. Then there exists an $(r+k-1)$-edge-coloring so that any two edges which share $k-1$ vertices have distinct colors.*

This conjecture simplifies to Vizing's Theorem in the case where $k = 2$.

Conjecture 5.7. *Every Snark is contractible to the Petersen Graph. That is, every snark can be reduced to the Petersen graph by deleting certain edges and contracting others.*

This was conjectured by Tutte, and recently four mathematicians named Robertson, Sanders, Seymour, and Thomas announced that they had discovered a proof. As of August 2015, this proof remains unpublished. This theorem is exceptionally significant, as it would characterize snarks much more strongly than the "bridgeless 3-regular class two" definition. In addition to this, it provides yet another proof of the Four Colour Theorem (every planar graph can be vertex-coloured by at most four colours).

Remark 5.8. (Etymology of *Snark*)
Snarks were named in 1976 by a mathematician named Martin Gardner, after the elusive creature of Lewis Carrol's nonsense poem, *The Hunting of the Snark*.

Acknowledgments. I would like to thank my mentor, Foling Zou, for her guidance in the readings which helped me to produce this paper. I would also like to thank Professor Babai for his advice and lessons, which helped me to choose the topic of this paper.

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