FINITE CONNECTED H-SPACES ARE CONTRACTIBLE

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Abstract. The non-Hausdorff suspension of the one-sphere S^1 of complex numbers fails to model the group's continuous multiplication. Moreover, finite connected H-spaces are contractible, and therefore cannot model infinite connected non-contractible H-spaces. For an H-space and a finite model of the topology, the multiplication can be realized on the finite model after barycentric subdivision.

Contents

1. Introduction

Given an infinite topological space X with a multiplication law, we wish to examine possible finite models of the space and the map. When the multiplication is continuous, what is needed is a finite space weakly equivalent to X that admits a continuous multiplication. We begin in Section 2 by introducing the algebraic structures based on continuous multiplication, namely, topological groups. We take up the simplest example of a connected, non-contractible topological group, namely S^1. In Section 3, we show that the multiplication on the model of the circle given by the non-Hausdorff suspension is discontinuous. We then move on in Section 4 to H-spaces, structures more general than topological groups but preserving the notions of continuous multiplication and identity points. Section 5 is devoted to combinatorial analysis of these spaces, leading to the conclusion that a finite H-space cannot model S^1 or any non-contractible connected H-space. Finally, in Section 6, we note that it is possible to model the multiplication on an H-space by a product on the space's (iterated) barycentric subdivision.

Date: August 28, 2015.

2. Topological groups

The interaction of group multiplication with a space's topology is captured in the following definition.

Definition 2.1. A *topological group* is a group that is also a T_0 topological space in which the multiplication map given by $(x, y) \mapsto x \cdot y$ and the inverse map given by $x \mapsto x^{-1}$ are continuous.

Proposition 2.2. *Let H be a group that is also a T_0 topological space. Then H is a topological group if and only if the map $\rho : H \times H \to H : (x, y) \mapsto x \cdot y^{-1}$ is continuous.*

Proof. Suppose H is a topological group. The functions $f : H \times H \to H \times H : (x, y) \mapsto (x, y^{-1})$ and $g : H \times H \to H : (a, b) \mapsto a \cdot b$ are then continuous, and so $\rho = g \circ f$ is as well.

Conversely, suppose ρ is continuous. First, the map v taking x to x^{-1} is equal to the composition of the continuous maps ρ and $h : H \to H \times H : x \mapsto (e, x)$, and is therefore itself continuous. Second, the product map g is continuous because it equals the composition of the continuous functions ρ and f. □

Example 2.3. $(\mathbb{Z}, +)$
When equipped with the order topology, this is a T_1 space. Consider the open interval (a, b), an arbitrary basis element for $(\mathbb{Z}, +)$. Define $\rho : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} : (x, y) \mapsto x - y$. For the pre-image, we have $\rho^{-1}(a, b) = \{(x, y) : a < x - y < b\} = \{(x, y) : a + y < x < b + y\}$. This pre-image is the union over all y of the corresponding open sets $(a + y, b + y) \times (y - 1, y + 1)$, and is therefore open. Thus ρ is continuous.

Example 2.4. $(\mathbb{R}, +)$
The continuity of $\rho : \mathbb{R} \to \mathbb{R} : (x, y) \mapsto x - y$ in the usual topology is a standard fact of analysis.

Example 2.5. $(\mathbb{R}_+, \times)$
The continuity of the quotient operation $q : \mathbb{R} \times (\mathbb{R} - \{0\}) \to \mathbb{R}$ is a standard fact of analysis. For $\mathbb{R}_+$, construct the continuous ρ by restricting q's domain to $\mathbb{R}_+ \times \mathbb{R}_+$ and its range to $\mathbb{R}_+$.

Example 2.6. $(S^1, \times)$
The beauty of the algebra of these numbers is that their multiplication is the same as the addition of real numbers (complex numbers on the unit circle are written as exponentials, and their multiplication is given by the addition of the exponents). That $(S^1, \times)$ is a topological group follows from the fact that $(\mathbb{R}, +)$ is.

We take S^1 as our main example. We are interested in finite models of S^1 that can be equipped with continuous multiplication.

3. Failure of the non-Hausdorff suspension of S^1

Our standard four-point model of S^1, the non-Hausdorff suspension, is incompatible with continuous complex multiplication. The model in the complex numbers is pictured in the following diagram. An arrow pointing from one element to another says that the element being pointed to is greater than the other. The far-right and far-left points are identical.

$$i \longleftarrow -1 \longrightarrow -i \longleftarrow 1 \longrightarrow i$$

Proposition 3.1. *In the complex numbers, the non-Hausdorff suspension $\mathbb{S}^1$ of the zero-sphere gives discontinuous multiplication.*

Proof. We have $(i, i) > (-1, i)$, but $i \cdot i = -1 < -1 \cdot i$. □

4. Finite H-spaces

This and the following two sections review work of Stong [5].

Definition 4.1. Let X be a topological space, $e \in X$ a basepoint. Equip (X, e) with continuous product map $\phi : X \times X \to X$. Write xy for $\phi(x, y)$. We now say (X, e) is an *H-space* if the maps $\theta_1 : X \to X : x \mapsto xe$ and $\theta_2 : X \to X : x \mapsto ex$ are homotopic to the identity map.

A topological group is an H-space in which multiplication by e is the identity map, so that e is an algebraic identity element. Of course, for a space to be a group also requires that the multiplication satisfy certain other algebraic rules.

Before proving propositions about H-spaces, we modify the definitions of minimal finite spaces and cores to respect basepoints. Recall that an *upbeat point* in a finite space is a point x such that there exists $y > x$ with all points $z > x$ also necessarily satisfying $z \geq y$. A *downbeat point* x' is the analogue where there is only one point directly under x'. A *beat point* is a point that is upbeat or downbeat. A *minimal finite space* X is a finite T_0 space with no beat points. A *core* of a finite space X is a subspace Y that is minimal and a deformation retract of X. Now, we present the analogous notions for based spaces.

Definitions 4.2. A based finite space (X, x) is *minimal* if it satisfies the T_0 axiom and has no beat points except possibly x. A *core* of a finite space (X, x) is a subspace (Y, x) that is minimal and a deformation retract of X.

When we reduce a based space to its core, we wish not to delete the basepoint, and hence we modify the definitions.

Now we prove the basic facts about general and finite H-spaces that we will use in the proofs of the main results.

If two H-spaces sharing the same set of points have homotopic product maps, we identify those H-spaces. This equivalence of H-spaces gives the following homotopy invariance property.

Proposition 4.3. *If (X, e) and (Y, f) are homotopy equivalent, then H-space structures on (X, e) correspond bijectively to H-space structures on (Y, f).*

We will prove our major proposition, 5.2, for minimal spaces only. Proposition 4.3 will then allow us to establish a general negative result on the possibility of finite H-space models of S^1.

We continue with another important fact.

Proposition 4.4. *Let (X, e) be a minimal finite H-space. Then $\theta_1 : X \to X : x \mapsto xe$ and $\theta_2 : X \to X : x \mapsto ex$ are equal to the identity map.*

Proof. Since (X, e) is a minimal finite space, any map from X to itself that is homotopic to the identity is the identity. □

5. The major result

The following proposition provides the structure for the proof of the major result, Proposition 5.2. Note that to say that x is *upbeat* (respectively, *downbeat*) *under* (*over*) *a point* y means x is upbeat (downbeat) and y is the immediate successor (predecessor) of x.

Proposition 5.1. *Let* (X, e) *be a minimal finite space,* $x \in X$*. Then*

(i) x *is less than each of two distinct maximal points, or*
(ii) x *is maximal, or*
(iii) x *is upbeat under a maximal point (so* $x = e$*)*

and

(i') x *is greater than each of two distinct maximal points, or*
(ii') x *is minimal, or*
(iii') x *is downbeat over a minimal point (so* $x = e$*)*

Proof. Suppose for contradiction that the set A of points that do not satisfy any of (i), (ii), (iii) is nonempty, and let a be a maximal element of A. Since a is not maximal in X, there exists $z \in X$ such that $z > a$. Let $B = \{x : x > a\} \subset X$. If B contains a point z' other than z, then a satisfies (i). If not, then a satisfies (iii). Either way, we have arrived at a contradiction with the fact that a is in A. We conclude that A must be empty. Similarly, every point must satisfy one of (i'), (ii'), (iii'). □

Here is the combinatorial result that gives the main conclusions about finite H-spaces.

Proposition 5.2. *Let (X, e) be a minimal finite H-space. Then e is both maximal and minimal under the associated order* $\leq$*.*

Knowing that e must satisfy one of the conditions (i), (ii), (iii) and one of (i'), (ii'), (iii'), we proceed to eliminate from possibility all pairs of conditions except the pair consisting of (ii) and (ii').

Remark 5.3. From the definition of the order on a finite T_0 space [3] we can deduce the appropriate order on a product of two finite spaces. Let $(a, b), (c, d) \in X \times Y$ where X and Y are finite T_0 spaces. Then $(a, b) \leq (c, d)$ if and only if $a \leq c$ and $b \leq d$. If any of the two inequalities in the factor spaces is strict, then the inequality in the product space is strict as well.

The point e does not satisfy (i).

Lemma 5.4. *Let* m *and* m' *be maximal points in* X *with* $m, m' > e$*. Then* $m = m'$*.*

Proof. Since $m' > e$ and $m > e$, we have

$$(m, m') > (m, e) \tag{5.5}$$

$$(m, m') > (e, m'). \tag{5.6}$$

Now we apply the continuous (order-preserving) product ϕ to each of (5.5) and (5.6). We obtain

$$mm' \geq me = m \tag{5.7}$$

$$mm' \geq em' = m' \tag{5.8}$$

Because the right-hand sides of the inequalities in (5.7) and (5.8) are maximal, we deduce

$$m = mm' = m'. \tag{5.9}$$

□

The point e does not satisfy (i'). This is true for perfectly symmetric reasons.

The point e does not satisfy both (ii') and (iii). We show that if it did, then X would have infinitely many subsets, in contradiction to the fact that X is finite.

Suppose for contradiction that e satisfies (ii') and (iii), i.e., e is both minimal and upbeat under a maximal point.

Our claim is that for every integer $r \geq 0$, X contains a subset

$$D_r = \{e = u_0, u_1, ..., u_r; m_0, ..., m_{r-1}\} \tag{5.10}$$

with all u_i minimal in X and all m_i maximal in X, and such that all of the following conditions hold:

(a) For i between 0 and $r-1$ (inclusive), the only points in X less than m_i are u_i and u_{i+1}.
(b) m_0 is the only point in X that is greater than u_0.
(c) For i between 1 and $r-1$ (inclusive), the only points in X greater than u_i are m_{i-1} and m_i.
(d) For i between 0 and $r-1$ (inclusive), $xm_i = m_ix = m_i$ if x is m_k or u_k with $k \leq i$.
(e) For i between 0 and r (inclusive), $xu_i = u_ix = u_i$ if $x = m_k$ with $k < i$ or $x = u_k$ with $k \leq i$.
(f) For every $x \in X$ not in D_r, $xm_i = x = xu_i$ and $m_ix = x = u_ix$.

For $r = 0$, we have the set $D_0 = \{e = u_0\}$. It contains no m_i. Conditions (a) – (d) and (f) are vacuously satisfied. For condition (e), the first option (involving m_i) is vacuously satisfied, and the second demands only that we check $ee = e$. That equation is true in the minimal space (X, e) because multiplication by e is homotopic to the identity, but in fact it is true for any H-space. Multiplication by e is homotopic to the identity through maps from (X, e) to (X, e). That is, the fact that e is the space's basepoint means the only allowed intermediate maps take e to itself.

Now assume X contains a set D_k of the form in (5.10). We show that there are an additional maximal point m_k and an additional minimal point u_{k+1} such that D_{k+1} (of the form (5.10)) satisfies (a) through (f).

First we show that it satisfies (a), (b), (c).

For $k = 0$, the assumption that e is upbeat under a maximal point gives a unique m_k, namely the point under which e is upbeat.

For $k > 0$, we know that $u_k \neq e$ and that u_k is not maximal (being less than m_{k-1}). So u_k is less than each of two distinct maximal points. By (a), only one of those is in D_k.

In order for D_{k+1} to satisfy (c), we cannot have a choice of multiple maximal points to call m_k. In the case $k = 0$, this needed uniqueness property has already been shown. In the case $k > 0$, in which we know that there exists a maximal point outside D_k, the uniqueness follows from the combination of (f) with the procedure of the previous subsection.

Existence and uniqueness of u_{k+1} now follow by the analogous argument, using $m_k \neq e$.

One can now see that D_{k+1} satisfies (a), (b), (c). We finally show that it satisfies (d), (e), (f).

To verify (d) for D_{k+1}, we substitute $x = m_k$ in the assumption (f) for D_k. Likewise, to verify (e), we substitute $x = u_{k+1}$.

Finally, let us verify (f) for D_{k+1}. We will show $xm_k = x = xu_k$. The derivation of the other identity uses the analogous argument.

Suppose x is not in D_{k+1}. We obtain immediately $xm_k \geq xu_k = x$. We now proceed by induction.

In the base case, where x is maximal, we find from the above that $xm_k = x$.

Now, for the inductive step, consider the point w, supposing that for every $y > w$, $ym_k = y$.

For any such y, by continuity of ϕ, we have $y = ym_k \geq wm_k$.

Thus, either w is upbeat or $wm_k = w$. The former is false because X is a minimal space (no point other than e can be upbeat) with $xe \neq e$ for $x \neq e$. So $wm_k = w = wu_k$. This completes the verification of (f).

We now see that if e satisfied (ii') and (iii), we would be able to construct infinitely many distinct subsets of X, contradicting the fact that X is finite.

The point e does not satisfy both (ii) and (iii'). This possibility is ruled out in the same way as the possibility (ii') and (iii).

The point e does not satisfy both (iii) and (iii'). This possibility is conceptually similar to the last, because it just replaces maximality by the situation of being upbeat under a maximal point. The technicalities of the demonstration are slightly different, but offer negligible additional insight.

The point e satisfies (ii) and (ii'). This is the only remaining possible pair of conditions. The proof of Proposition 7.2 is now complete.

6. Inviability of finite H-space models of non-contractible connected spaces

Theorem 6.1. *Let X be a finite space, $e \in X$. There exists an H-space structure on (X, e) only if $\{e\}$ is a deformation retract of e's connected component in X, so that the component of e is contractible.*

Proof. Since (X, e) is homotopy equivalent to its core (Y, e), Proposition 4.3 says that there is an H-space structure on (X, e) only if there is one on (Y, e).

Because (Y, e) is a minimal finite space, it is an H-space only if e is both maximal and minimal in Y under the associated order $\leq$, i.e., $\{e\}$ is a path component of Y.

In finite spaces, path components are the same as connected components. So, $\{e\}$ is a path component of Y only if it is a component of Y.

If $\{e\}$ is a component of Y (the core), then $\{e\}$ is the core of e's component in X.

A core of a component is a deformation retract of the component. Thus the result is established. □

Corollary 6.2. *A connected finite space X is an H-space only if it is contractible.*

7. Modeling the multiplication after barycentric subdivision of the finite space

The following simplicial approximation theorem for finite spaces is due to Hardie and Vermeulen:

Theorem 7.1. *Let A and B be finite T_0 spaces, $f : |\mathcal{K}(A)| \to |\mathcal{K}(B)|$ a continuous map. Then there exist an integer n and a continuous map $g : A^{(n)} \to B$ such that $|\mathcal{K}(g)| \simeq f$.*

The theorem implies that for an H-space X with product ϕ and finite model Y, there exist an integer n and a continuous map $\mu : (Y \times Y)^{(n)} \to Y$ such that $|\mathcal{K}(\mu)| \simeq \phi$.

Acknowledgments. I would like to thank my graduate mentor, Alan Chang, for reading closely several drafts of this paper and giving detailed feedback, including references to examples of his own expository mathematical writing. I would also like to thank Peter May for organizing the Chicago REU, presenting the theory of finite spaces, answering mathematical questions, proposing problems for students to work on and topics for students to expose in papers, and giving feedback on a late draft of this paper. Finally, I appreciate the efforts of those who transcribed and edited Dr. May's talks and made the written versions available online.

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