THE TOPOLOGICAL APPROACH TO SOCIAL CHOICE

WEI HAN CHIA

Abstract. Social Choice Theory is a field in economics which studies the possibility of fair aggregation maps of choices. One particularly famous result is Arrow's Impossibility Theorem, which states that given a selected set of rational criteria for this aggregation map, no map exists which fulfills these criteria. However, developments in topological machinery have provided an alternative way of looking at social choice problems, such as the existence of continuous maps from profile spaces into preference spaces. This paper will attempt to present the problem of social choice in a topological setting, citing results derived from the topological standpoint, and link this approach to the discrete approach utilised in Arrow's original proof.

Contents

1. Introduction

Social Choice Theory is a field in economics which studies the methods of aggregating the set of choices generated by members of a group. This extends itself into resource allocation, decision making, as well as voting. In particular, the field has been strongly defined by the presence of numerous impossibility theorems, which present the non-existence of an ideal aggregation map.

While previous forays into this topic have dealt with mostly combinatorial questions, Chichilnisky and Heal presented a novel way of considering the problem of social choice, which generalizes the problem to a continuous set of choices. This paper will attempt to introduce how topology can be used to understand problems about social choice, and demonstrate the usefulness of various topological machinery in not only proving results, but providing intuition about the nature of the problem of social choice. In addition, we build upon the approach of Baryshnikov in utilizing topological machinery to generate a space out of finite choices, and adapt his proof of Arrow's Impossibility Theorem.

Date: September 1 2015.

Theorem 1.1. (Arrow's Impossibility Theorem) *Given any group of n-voters $n > 1$, with well-defined choices over 3 or more distinct alternatives, there does not exist an aggregation method which fulfills the following criteria:*

1. ***Pareto Efficiency (Weak):*** *A pareto efficient aggregation is one in which if every individual ranks one alternative over the other, then the resultant aggregated map also ranks that alternative over the other.*
2. ***Non-dictatorship:*** *An aggregation is said to be non-dictatorial when no individual can impose their set of preferences on the entire group.*
3. ***Independence of Irrelevant Alternatives:*** *An aggregation respecting independence of irrelevant alternatives is one in which the only factors affecting the rankings of two alternatives is their relative positions in each individual's preference. This means that adding in a choice to the system without changing the pre-existing rankings should mean that the aggregation ranks the two original choices in the same order.*

Arrow's Impossibility Theorem is the classical impossibility result, demonstrating that it is impossible to obtain a reasonable aggregation of preferences across a set of individuals over more than 3 choices. This theorem was the basis of the field of Social Choice, and the investigation of whether a method to aggregate a set of preferences across multiple individuals could exist, and what conditions such a map would have to fulfill. The development of topological machinery in considering this question has brought an interesting perspective to the problem. This paper will present the topological approach through a proof of both Arrow's Impossibility Theorem, as well as a more general theorem by Chichilnisky on continuous choice spaces, and explore the intuition gained from considering this question from a topological perspective.

While this paper will attempt to be as self-contained a discussion as possible, it assumes some prior knowledge of basic topological concepts, and hence much of the topological results utilized in the proofs will not be elaborated upon unless they afford important intuition.

2. Generalizing the Space of Preferences

To begin the discussion of social choice, we must first define the parameters upon which this decision is to be made. Namely, we must first characterize each individual's set of preferences in a topological space which can be worked in.

For the purposes of this paper, we will work with the common economic assumption that the space of choices X, is isomorphic to $\mathbb{R}^n$. To each individual, we further assign a utility function:

$$u : \mathbb{R}^n \to \mathbb{R}$$

Given this function, we can therefore characterize a preference ordering.

Definition 2.1. Given $x, y \in \mathbb{R}^n$.
We say that x is preferred to y, or $y \prec x \iff u(y) < u(x)$

Since any preference can be identified by a utility function, the space of preferences is therefore just the space of all utility functions. We will work with the simplification that all individuals have linear utility functions. [1] We will proceed to construct the space of preferences using this assumption.

Definition 2.2. $u_1 \sim u_2 \iff \forall x \in \mathbb{R}^n, u_1(x) = \lambda u_2(x)$ for some $\lambda > 0$.

Definition 2.3. We define a space of preferences, P for a given choice space X, to be the set of equivalent linear utility functions $u(x)$ defined on all $x \in X$.

Proposition 2.4. *The space of preferences for a given choice space $X = \mathbb{R}^n$ is homeomorphic to S^{n-1}.*

Proof. Given any utility function, we can identify this linear utility function with its gradient, which will be a vector in $\mathbb{R}^n$, or a straight line through the origin.

Now from 2.2, we see that every set of positively linearly related vectors are associated to a family of equivalent utility functions, which can be expressed simply by any single vector. Given that we are interested purely in the ordinal nature of preferences, we can represent every family of equivalent utility functions with its normalized vector. The resultant space of vectors is clearly equivalent to S^{n-1}. The association of each equivalent set of preferences to its vector is therefore the required homeomorphism (ψ) from the space of preferences to S^{n-1}. □

This characterization is particularly powerful in the topological approach, given the depth of study which has gone into understanding the homology of spheres and relevant maps to and from spheres.

However, the above process only works for a continuous space of preferences. A complete study of the space of preferences will therefore require the characterization of preferences over a finite set of choices, reminiscent of the original phrasing of Arrow's Impossibility Theorem. To do so, we introduce the idea of a nerve.

Definition 2.5. Given an open cover U of X where $U_i \in U \ \forall i \in I$ (where I represents some index set), the nerve of U, N_U is the simplicial complex with vertex set $\{v_i | i \in I\}$, and where an n-simplex $[v_1, ..., v_n]$ belongs to N_U if $\bigcap_{i=1}^{n} U_i \neq \varnothing$.

Given any suitable open cover, we can proceed to generate an abstract simplicial complex. This simplicial complex can therefore be utilized to investigate the relevant homologies of the space.

Example 2.6. Given a set of discrete choices $X = \{1, 2, 3\}$ and an ordering $>$ on X, the space of all possible preferences $P = \{\sigma(1) > \sigma(2) > \sigma(3)\}$ where σ represents all possible permutations on X.

Now let $\mu \in \{+, -\}$. For $i > j$ we define sets $U_{ij}^{+} = \{p \in P \mid i >_p j\}$ and $U_{ij}^{-} = \{p \in P \mid i <_p j\}$

The sets U_{ij}^{μ} form an open cover of the discrete space of preferences, and we can proceed to generate the nerve of P, N_P from this open cover.

The vertices of N_P can therefore be identified as $\{ij \mid i \neq j\}$, corresponding to each set U_{ij}^{μ}, and 2-simplexes corresponding to the actual set of feasible preferences.

Definition 2.7. More generally, given any discrete space of choices X, we have a discrete space of preferences P represented by the set of all possible orderings of these choices. We can likewise construct the sets $U_{ij}^{+} = \{p \in P \mid i >_p j\}$ and

[1] It is possible to generate the same space of preferences with all possible utility functions, by looking at indifference surfaces [Chichilnisky 4]

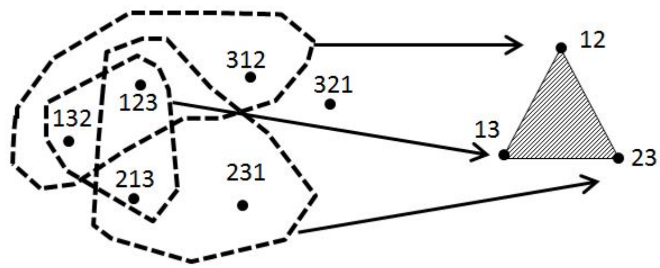


FIGURE 1. Constructing a simplicial complex by open covers

$U_{ij}^{-} = \{p \in P \mid i <_p j\}$ in a similar fashion by considering the pairwise ordering of two distinct alternatives.

As in the example, we can then identify the vertices of N_P as $\{ij \mid i \neq j\}$, and we implement a simplicial structure by adding an n-simplex when the sets corresponding to the n distinct vertices U_{ij}^{+} (for distinct ij) have a non-empty intersection.

The structure formed after constructing all possible simplexes is the desired representation of our space of preferences. If we have a space with n distinct choices, then any n-simplex in our final structure can be represented by a coherent preference (i.e. a valid set of orderings of n different choices).

Proposition 2.8. *In the case of 3 alternatives, the simplicial complex N_P generated from choice set X where $|X| = 3$ is homotopy equivalent to S^1.*

Proof. We notice that this simplicial complex contains a boundary of a 2-simplex that does not bound an 2-simplex. This is constructed by the vertices $\{12, 23, 31\}$ Each of these vertices is connected to each other by construction, but do not bound a 2-simplex as their intersection is empty (they do not form a coherent preference). In addition, we notice that combinatorially, the only other impossible set of preferences is given by $\{32, 21, 13\}$. However, it follows from the picture below that the resultant structure is equivalent to S^1, since the two impossible cycles share a boundary. □

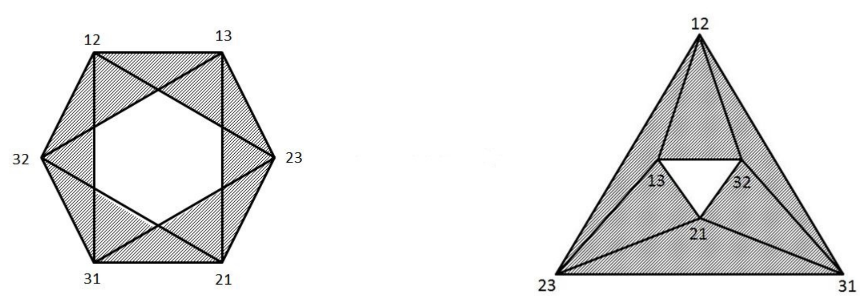


FIGURE 2. The Simplicial Complex Constructed over a Preference Space with 3 Choices

This proposition illustrates a more general statement, that any simplicial complex N_P generated from choice set X where $|X| = n$ is homotopy equivalent to S^{n-2}. This statement is proven by Baryshnikov (Baryshnikov, 1993).

Two things are worth noting here. First, finding an impossibility result for the space for the case of 3 choices is sufficient to prove an impossibility result for any space with more than 3 choices. And second, that the generator of H_1 (the first homology group of S^1) corresponds to a set of choices which are cyclic and therefore cannot form a coherent preference, a phenomenon that underlies the Concordet paradox [2]

3. Profiles and Aggregation Maps

Having shown that the space of preferences is in both the continuous and discrete case, homeomorphic to some n-sphere, we proceed to study the structure of the relevant profiles.

Definition 3.1. A profile r of preferences P is a k-tuple determined by the number of "voters" k. I.e. given k voters, a profile $r = (p_1, ..., p_k)$ where $p_i \in P\ \forall 1 \leq i \leq k$. Therefore, $r \in P^k$.

Proposition 3.2. *In the case of a choice made over a continuous space X,*
$P^k \simeq S^{n-1} \times ... \times S^{n-1} = (S^{n-1})^k$

Proof. We simply use the same homeomorphism (ψ) defined in the proof of 2.4 on every k entry of $r \in P^k$. This clearly forms a homotopy equivalence $(\psi_1, ...\psi_k)$ between each entry of P^k and $(S^{n-1})^k$, since ψ is a homotopy equivalence between P and S^{n-1}. □

Once again, we would like to be able to analyze the discrete conditions through a similar framework. While every profile can be likewise represented as the k-th cartesian product of a preference space with itself, this set does not provide any useful topological intuition. To develop this intuition, it is necessary to once again apply the concept of a nerve, generated as follows:

Definition 3.3. We define a vector $\overline{\mu}$ to be a vector $\{\mu_1, ..., \mu_n\}$ where $\mu_i \in \{+, -\}$ Now given a finite set of choices, we define $U_{ij}^{\overline{\mu}} = \left\{r = (p_1, ..., p_k) \mid p_i \in U_{ij}^{\mu_i}\right\}$

Now the set of all $U_{ij}^{\overline{\mu}}$ forms a cover for P^k. Constructing the nerve of this covering N_{P^k} results in a complicated simplicial structure, simply by virtue of the large number of different possible profiles available even in the finite setting. Since we are interested in maps which go from our space of profiles into the space of preferences, we are interested in induced maps on the homologies of N_{P^k} and N_P. Therefore, we can only look at the homology of N_{P^k} up to dimensions of $n - 2$ (as N_P is homeomorphic to S^{n-2}).

Proposition 3.4. *Given a set of 3 choices, $H^1(N_{P^2}) = \mathbb{Z}^2$*

Proof. We first prove that the generator of $H_1(N_{P^2})$ is defined by the cycles $[12*, 23*, 31*]$ and $[*12, *23, *31]$. These cycles are in $H_1(N_{P^2})$ because they represent a set of incoherent preferences and therefore do not bound a 2-simplex.

[2]The Concordet paradox is a two-person voting paradox for 3 persons and 3 choices, in which a majority aggregation rule is unable to be achieved when cyclic preferences exist.

Next, we prove that any set of incoherent preferences can be generated by these cycles. We do this by considering the fact that boundaries between these structures are shared. Consider the cycle $[1221, 2323, 3131]$. This cycle has a shared boundary with $[1221, 2323, 3113]$ as in the picture below. A similar method will demonstrate that this is true for any possible permutation of the orders in the second position. This allows us to conclude that $[12*, 23*, 31*]$ is a generator. Using the same proof in the case of $[*12, *23, *31]$ allows us to conclude that these two cycles are a generator of all possible cycles representing incoherent preferences.

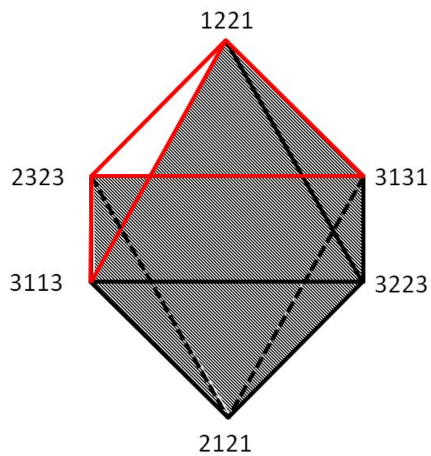


FIGURE 3. The Shared Boundaries of [1221,2323,3131] and [1221,2323,3113]

Now to conclude that $H_1(N_{P^2}) = \mathbb{Z}^2$, it remains to prove that these two generators are independent. This follows by considering the projection map of each preference from $N_{P^2} \to N_P$. □

This illustration once again underlies the more general statement, that given a set of n choices and k individuals, the homology groups of N_{P^k} are 0 up to $n-2$ and $H^{n-2}(N_{P^k}) = \mathbb{Z}^k$. The details of this proof can once again be found in Baryshnikov (Baryshnikov, 1993).

Having illustrated the topological structure of both the space of profiles and preferences, we now concern ourselves with a mathematical representation of what an aggregation map represents, and list certain common assumptions about these aggregation maps.

Definition 3.5. An aggregation map is a function $f : P^k \to P$. Which is often assumed to fulfill certain rational assumptions about human behavior.

We will list below the mathematical interpretations of the original assumptions used by Arrow in the description of his impossibility theorem.

(1) **Pareto Efficiency (Weak):** An aggregation map is said to be pareto efficient if for any profile $r = \{p_1, ..., p_k\}$ such that in every preference p_i, $x \prec y$, $f(r) \mapsto p$ such that in the aggregated preference p, $x \prec y$.

(2) **Non-dictatorship:** An aggregation map is said to be non-dictatorial if $\nexists\, p_k$ such that $\forall\, p, f(p, ..., p_k, ..., p) \mapsto p_k$.
(3) **Independence of Irrelevant Alternatives:** If $\overline{p_i}$ is a preference obtained by inserting a new option randomly into some preference p_i such that $x \prec y$ in both p_i and $\overline{p_i}$, $x \prec y$ in $f(p_1, ..., p_n) \implies x \prec y$ in $f(\overline{p_1}, ..., \overline{p_n})$.

It is important here to add an additional condition which is relevant when considering the case of a continuous choice space. We naturally want the aggregation map to be a continuous one, not only because of the mathematical usefulness of this assumption, but also because it reflects a certain stability in the aggregation map, which would be natural to expect in any ideal collective decision making process.

4. A Topological Impossibility Result

We are now equipped to present proofs of the impossibility results in Social Choice theory. However, rather than begin with a proof of Arrow's impossibility theorem, we begin with a proof of impossibility of an aggregation map in a continuous setting, presented by Chichilnisky. This theorem is not only useful because of its generality in the continuous setting, but also because of the intuition it affords to the topological approach to proving social choice theorems.

Theorem 4.1. (Chichilnisky) *Given a set of continuous choices and an aggregation map:*

$$f : P^k \to P$$

satisfying the following conditions

(1) Stability, i.e. f is continuous
(2) Pareto Efficient, i.e. $x \prec y$ in p_i $\forall i \implies x \prec y$ in $f(p_1, ..., p_k)$
(3) Weak Positive Association, if $f(p_1, ..., p_i, ..., p_k) = -\lambda p_i$ for some $i = 1, ..., k$, $\lambda > 0$, and some $(p_1, ..., p_k) \in P^k$, then $f(-p_i, ..., p_i, ..., -p_i) \neq \lambda p_i$ for any $\lambda > 0$.

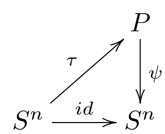
Then this map is homotopic to a dictatorial map:

$$f_d : P^k \to P$$

$$f_d(p_1, ..., p_d, ..., p_k) \mapsto p_d$$

The intuition behind the first two conditions is readily understandable. The assumption of weak positive association is reasonable because one would expect that if the societal aggregation rule chose a result opposite of a single individual's preference given some preferences by other individuals, it would not make sense for this same aggregation to choose that individual's preference when all other individuals had the opposite preference.

We earlier defined a homeomorphism ψ from $P \to S^n$. We now need to identify a continuous map τ from $S^n \to P$. This will be easily identified through the following commutative diagram.



τ here is simply the inverse of ψ.

Let f be a aggregation map which fulfills the conditions in 4.1. Now we can define the map ϕ through following commutative diagram:

$$\begin{array}{ccc} P^k & \xrightarrow{f} & P \\ {\scriptstyle \psi_i,\ldots,\psi_k}\uparrow & & \downarrow{\scriptstyle \tau} \\ (S^n)^k & \xrightarrow{\phi} & S^n \end{array}$$

It is clear that ϕ should be a continuous map from $(S^n)^k \to S^n$, such that

$$\phi(x_1, ..., x_n) \mapsto \tau(f(\psi_1(x_1), ..., \psi_2(x_2)))$$

Fixing $x_0 \in S^n$, we define

$$G_i = \{(x_1, ..., x_k) \mid z_j = z_0 \forall\, j \neq i\}$$

Since each G_i is homeomorphic to a sphere S^n, we can utilize the following topological notion of degree.

Definition 4.2. For a map from $S^n \to S^n$, the induced map

$$f_* : H_n(S^n) \to H_n(S^n)$$

is a homomorphism from an infinite cyclic group $((\mathbb{Z}))$ to itself, and so $f_*(\alpha) = d\alpha$ for some integer d depending on f. We call this integer d the degree of the map f.

The degree of f fulfills the following properties:

(1) $\deg(id) = 1$
(2) $\deg(f) = 0$ if f is not surjective.
(3) degree of a map is homotopy invariant

We now establish certain facts resulting from the degree.

Proposition 4.3. *The degree of any restricted map $\phi \restriction_{G_i}$ is either 0 or 1*

Proof. We first notice a fact resulting from the pareto condition. Given some set of choices $(x_1, x_0, ...x_0)$ $x_0, x_1 \in S^n$, the resultant aggregated map x must lie within the shortest circular segment on S^n between x_0 and x_1 $(C(x_1, x_0))$. If the resultant aggregated map was not within this segment, there would exist some y and z so that $x.y > x.z$, but $x_0.y < x_0.z$ and $x_1.y < x_1.z$, which contradicts pareto efficiency.

Now we consider $\phi \restriction_{G_i} (x_0, ..., -x_0, ..., x_0)$. If this does not map to $-x_0$, then the fact above gives us that $\phi \restriction_{G_i}$ can never map to $-x_0$. So $\phi \restriction_{G_i}$ is not surjective, and its degree is 0.

Now if this maps to $-x_0$, then by continuity, for any point z near $-x_0$, it should map to a point on the circular segment between z and x_0. We also know that the profile $(x_0, ..., x_0)$ must map to x_0, by our axiom of pareto efficiency. This, together with the property of continuity applied to points on each circular segment, establishes that the degree of this map should be 1 (Chichilnisky, 1982). □

Now we can define

$$D = \{(x, ..., x) \mid x \in X\}$$

to be the diagonal. D is likewise homeomorphic to S^n, and since f is pareto efficient, it should respect unamity (i.e. $f(p, ..., p) = p$). It therefore follows that the restriction of ϕ to D, $\phi \restriction_D$ has degree 1 since it is the identity.

We can now define the inclusion map.

$$in_{G_i} : S^n \to (S^n)^k$$

where

$$in_{G_i}(x) \mapsto (x_1, ..., x, ..., x_n) \text{ where } x_j = x_0 \ \forall j \neq i$$

The diagonal inclusion is defined similarly,

$$in_D(x) \mapsto (x, ..., x)$$

This allows us to represent the maps $\phi \restriction_D = \phi \circ in_D$ and $\phi \restriction_{G_i} = \phi \circ in_{G_i}$.

Proposition 4.4. *in_D is homotopic to $\sum_{i=1}^k in_{G_i}$.*

Proof. We first notice that $\sum_{i=1}^k in_{G_i}$ is not always a point in $(S^n)^k$. However, we recall that when defining the space of preferences, we identified an equivalence between linear utility functions based on their gradient. It follows that we can present the point identified by $\sum_{i=1}^k in_{G_i}$ as a point in $(S^n)^k$ by utilizing this equivalence.

Now given any $z \in S^n$, notice that $\sum_{i=1}^k in_{G_i}(z) \sim (\frac{(k-1)x_0+z}{|(k-1)x_0+z|}, ..., \frac{(k-1)x_0+z}{|(k-1)x_0+z|})$. Now $(k-1)x_0$ is a constant, this can be continuously deformed to $(z, ...z)$, and so the two maps are homotopic. □

Now since degree is homotopy invariant, we have $deg(\phi \circ in_D) = \sum_{i=1}^k deg(\phi \circ in_{G_i}) = 1$. Now from 4.3, we must have that $deg(\phi \circ in_{G_d}) = 1$ for some d, and $deg(\phi \circ in_{G_i}) = 0$ for all $i \neq d$.

Without loss of generality, say $d = 1$. Also note that since we chose x_0 arbitrarily when defining in_{G_i}, then by continuity of ϕ, for any $x \in S^n$:

$$\phi(x, -x, ..., -x) = x \tag{4.5}$$

Now consider any $(x_1, ..., x_k) \in (S^n)^k$. By the weak positive association condition, if

$$\phi(x_1, ..., x_k) = -x_1,$$

it follows that

$$\phi(x_1, -x_1, ..., -x_1) \neq x_1$$

However, this contradicts equation 4.5, so we have that $\phi(x_1, ..., x_k) \neq -x_1$.

The effects on the level of induced maps between S^n allow us to conclude that given preferences $(p_1, ..., p_k)$ we likewise have $f(p_1, ..., p_k) \neq -p_1$. Now it follows that given any d, $tp_d + (1-t)f(p_1, ..., p_k) \neq 0$.

Proof. (Chichilnisky) We can therefore define the following map H on $P^k \times [0, 1]$ to be

$$H(p_1, ..., p_k, t) = \frac{tp_d + (1-t)f(p_1, ..., p_k)}{||tp_d + (1-t)f(p_1, ..., p_k)||} \tag{4.6}$$

Now it is easy to see that any $H(p_1, ..., p_k, t) \in P$ for any t, and by construction,

$$H(p_1, ..., p_k, 0) = f(p_1, ..., p_k)$$

and

$$H(p_1, ..., p_k, 1) = p_d.$$

we have the desired homotopy between the aggregation map f and a dictator map. □

This proof highlights the value of the topological approach. With a set of rather simple conditions, we can demonstrate that any map is homotopic to a dictator map. This enables us to study the actual mathematical effects of the assumptions we make in generating impossibility results, and provides an easier way to understand them. However, to obtain a better intuition behind social choice theory, we turn towards the proof of the more practical Arrow's Impossibility Theorem.

5. A Proof of Arrow's Impossibility Theorem

We refer to the definitions provided in Section 3 to provide a statement of Arrow's Impossibility Theorem.

Theorem 5.1. *Given any group of n-voters $n > 1$, with well-defined choices over 3 or more distinct alternatives, there does not exist an aggregation map f which fulfills the following criteria:*

(1) Pareto Efficiency (Weak)
(2) Non-dictatorship
(3) Independence of Irrelevant Alternatives

Using the concept of the nerve, it has been established that the nerve of possible preferences, N_P is homeomorphic to S^{n-2} and the nerve of possible profiles N_{P^K} has $H_{n-2}(N_{P^k}) = \mathbb{Z}^k$ (Baryshnikov, 2000). Given this, we identify the H_{n-2} as the homology group of interest, given that $H_{n-2}(N_P) = \mathbb{Z}$.

To investigate this relationship, we want to have some way of identifying maps from N_{P^k} to N_P with the actual aggregation map. Axiom (3) gives us this relationship.

Proposition 5.2. *Any aggregation map which satisfies the independence of irrelevant alternatives axiom is a simplicial map from N_{P_k} to N_P*

Proof. Axiom (3) tells us that any ranking of two alternatives only depends on their relative rankings in each preference, and not on any other alternatives. This means that every vertex in N_{P_k}, which is identified with some subset of the open cover of P_k, $U_{ij}^{\overline{\mu}}$, can be unambiguously mapped to some vertex in N_P, because each vertex is defined purely based on a relation between two elements, and these relations are independent of each other.

Now to define a proper simplicial map, we require that if the set of vertices in N_{P_k} span a simplex, then their image in N_P also span a simplex. However, this follows because the vertices in N_{P_k} would only span a simplex if there is exists a valid profile, but such a profile would consist of valid preferences from each individual, and hence it must be represented by a set of vertices in N_P which spans a simplex. □

We therefore see the value of the independence of irrelevant alternatives axiom in helping to flesh out our topological modeling of the social choice problem. In particular, it enables us to move from the finite case to a general topological space, and in conjunction, the more abstract n-2th homology group. It is now clear how we can proceed to prove Arrow's impossibility theorem using axioms (1) and (2). To do so, we re-use the topological framework provided by Chichilnisky.

Given 5.2, we investigate the following relationship between the n-2th homology groups of N_P and N_{P^k}. In particular, we realize that the aggregation map f induces a map from the n-2th homology groups of N_P and N_{P_k}. To finish the proof, we require an additional lemma.

Lemma 5.3. *Given any map from $N_P \to N_P$, the degree of the map is either 0 or 1.*

Proof. Here, we prove this statement for the case of 3 choices. A detailed proof for the situation with more than 3 choices can be found in Baryshnikov (Baryshnikov, 1993).

We first notice that if the map is not surjective, then its degree is 0. We therefore deal with the situation when the map is surjective.

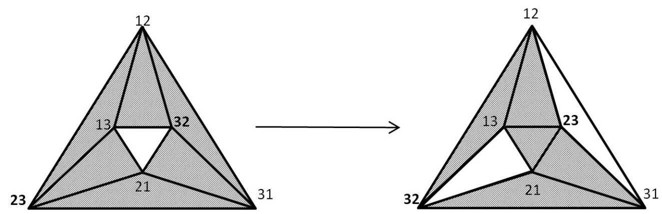


FIGURE 4. An Inversion Leading to a Non-Surjective Map

By our construction of the nerve, each point can only map to itself or its opposite order, i.e [12] can only be mapped to [12] or [21]. Now if [12] maps to [12], and we have an inversion in any of the other two coordinates, we get a non-surjective map, since a valid preference gets map to an invalid preference. So if we fix the mapping of [12], the only surjective map is the identity map.

Now if [12] maps to [21], by similar reasoning, all of the other vertices should be inverted. However, this map is homotopic to the identity map as well. So its degree should also be 1, and we are done. □

Here, we are interested in maps which are the composition of an inclusion map and our desired aggregation map. Now say an individual's inclusion map $f \circ in_{G_i}$ has degree 1. This map is therefore either dictatorial, or the inverse. Now if this map is the inverse, given an individual's preference, it should always map to the opposite preference. However, this means that the aggregation map takes [1212...] to [21], which clearly contradicts the axiom of pareto efficiency. Therefore the only possible degree 1 map created from a composition of an inclusion map and our desired aggregation map is the identity map.

Proof. (Arrow's Impossibility Theorem) The axiom of pareto efficiency gives us that the map induced by $f \circ in_D$ on H_{n-2} is the identity map. We also realize that the map induced by $f \circ in_D$ is homotopic to the map induced by $\sum_{i=1}^{n} f \circ in_{G_i}$, where in_{G_i} is defined similarly (as was the case in section 4). Now we have that the sum of degrees of these maps are 1, and by 5.3, since each of these maps must have either degree 0 or 1, only one of these induced maps must have degree 1.

Since degree is homotopy invariant, and the simplicial maps we are investigating are homotopic to the actual maps on the underlying space, we know that a map $f \circ in_{G_\alpha}$ that has degree 1 is the identity map, and therefore is dictatorial. We can therefore conclude that there must exist a dictator in the above scenario. □

The usage of topological machinery therefore provides a simple proof of Arrow's Impossibility Theorem. Certain striking features should be identified about this approach.

First, the nature of the n-th homology groups of the simplicial complex generated by the N_P and their relation to the impossibility of preferences. We see that the space generated by the simplicial complex has a n-2 dimensional hole corresponding to a cyclic set of preferences. The fact that this space is not contractible underlies the impossibility statement, and provides a powerful visual representation of the contradiction one would face in finding a suitable aggregation, even in a finite case.

Second, the ability to move between statements on a homological level and maps from the actual space helps to simplify the question notably. In most cases, we actually only consider whether a map is surjective, or whether it is the identity map. This simplicity stands in stark contrast to the complex reasoning necessary in the original combinatorial proof of Arrow's Impossibility Theorem, and further demonstrates the usefulness of the topological approach.

6. Concluding Remarks

Topology has indeed been used widely in the field of economics, and extends itself to more than just Social Choice. Differential topology and Morse theory features strongly in arguments about critical points in economies, a sophistication of the original approach to general equilibrium using Brouwer's fixed point theorem.

There is definitely further room to go in our analysis of Social Choice. In particular, much literature has devoted effort into redefining assumptions which permit the creation of a suitable aggregation map. The topological approach provides a much simpler path to test the validity of these approaches. The topological approach also affords another angle to the problem. Rather than assumptions on the map, an investigation into the type of choice space which admits a suitable aggregation map can also be performed.

Mathematically, this approach can also be developed further. The development of cellular homology, and the recognition that all simplicial complexes are cell complexes provides a natural point for further sophistication of the topological model. This is particularly interesting since almost all spaces can be approximated by a cell complex. Naturally, combinatorial topological methods can also be used to investigate the problem in a finite setting.

In conclusion, while we can gain valuable intuition about the problem of Social Choice through the topological approach, the problem of social choice also provides a unique example of the appeal of using a topological framework to investigate problems.

Acknowledgments. It is a pleasure to thank my mentor, Lei Chen for her help in clarifying the various topological proofs and mathematical modeling of preferences and profiles, as well as her constant advice in the structure of the paper. I would also like to thank Peter May, the director of the REU Program, without whom this would not be possible.

References

[1] A. Hatcher. Algebraic Topology. Cambridge University Press. 2001.
[2] J. P. May. A Concise Course in Algebraic Topology. University of Chicago Press. 1999.
[3] K. J. Arrow. Social Choice and Individual Values. Yale University Press. 1978.

[4] G. Chichilnisky. The Topological Equivalence of the Pareto Condition and the Existence of a Dictator. Journal of Mathematical Economics. 9. 1982. 223-233
[5] G. Chichilnisky, G. Heal. Necessary and Sufficient Conditions for the Resolution of the Social Choice Paradox. Journal of Economic Theory 31. 1983. Available at SSRN: http://ssrn.com/abstract=1370665
[6] Y. M. Baryshnikov Unifying Impossibility Theorems: A Topological Approach Advances in Applied Mathematics 14. 1993. 404-415
[7] Y. M. Baryshnikov. Topological and Discrete Social Choice: in search of a theory. Social Choice and Welfare 14. 1997. 199-209
[8] Y. M. Baryshnikov. On Isotopic Dictators and Homological Manipulators. Journal of Mathematical Economics. 33. 2000. 123-134
[9] S. Weinberger. On the Topological Social Choice Model. Journal of Economic Theory 115. 2004. 377-384