THE NOTION OF MIXING AND RANK ONE EXAMPLES

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Abstract. This paper discusses the relationships among mixing, lightly mixing, weakly mixing and ergodicity. Besides proving the hierarchy of these notions, we construct examples of rank one transformations that exhibit each of these behaviors to demonstrate the differences.

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1. Preliminaries

We assume knowledge of measure theory (for example, chapters 1 and 2 of [5] would suffice). We use the same notation as in standard measure theory.

1.1. **Convergence.**

Definition 1.1. A bounded sequence $\{a_i\}$ may converge to a number a in at least three ways.
(1) Convergence:

$$\lim_{i\to\infty} a_i = a.$$

(2) Strong Cesàro convergence:

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} |a_i - a| = 0.$$

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(3) Cesàro convergence:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} a_i = a.$$

Lemma 1.2. *Let $\{a_i\}$ be a bounded sequence. Then convergence implies Strong Cesàro convergence, and Strong Cesàro convergence implies Cesàro convergence.*

Proof. The result follows easily by applying the ϵ - δ definition of limits. □

1.2. **Sets of Density Zero.**

Definition 1.3. A set D of nonnegative integers is said to be of **zero density** if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{I}_D(i) = 0,$$

where $\mathbb{I}_D$ is the characteristic function defined by $\mathbb{I}_D(x) = 1$ if $x \in D$ and $\mathbb{I}_D(x) = 0$ if $x \notin D$.

Definition 1.4. Let $a \in \mathbb{R}$. We say a sequence $\{a_i\}$ of real numbers **converges in density** to a if there exists a zero density set $D \subset \mathbb{N}$ such that for any $\epsilon > 0$, there is an integer N such that if $i > N$ and $i \notin D$, then $|a_i - a| < \epsilon$.

Lemma 1.5. *If $\{b_i\}$ is a bounded sequence of nonnegative real numbers, then $\{b_i\}$ converges Cesàro to 0 if and only if $\{b_i\}$ converges in density to 0.*

Proof. The result follows easily by applying the definitions of convergences. □

1.3. **Dynamical Systems.**

Definition 1.6 (Dynamical System)**.** A **measurable dynamical system** is a quadruple $(X, \mathcal{S}, \mu, T)$, where X is a space, $\mathcal{S}$ a σ-algebra of subsets of X, μ a measure on $\mathcal{S}$ and $T : X \to X$ a measurable transformation with respect to μ. Specifically, $(X, \mathcal{S}, \mu, T)$ is a **Lebesgue measure-preserving dynamical system** if $(X, \mathcal{S}, \mu)$ is a Lebesgue measurable space and T is a measure-preserving transformation $T : X \to X$. We will refer to such system simply as measure-preserving dynamical system in this paper.

Though we are not going to study the "sameness" of dynamical systems, we define the notion of *factor* for technical reasons. A factor of a dynamical system carries some of its dynamical properties. Later we show if a dynamical system has an ergodic transformation, then so does its factor.

Definition 1.7. Let $(X, \mathcal{S}, \mu, T)$ be a measurable dynamical system. A dynamical system $(X', \mathcal{S}', \mu', T')$ is a **factor** of $(X, \mathcal{S}, \mu, T)$ if there exist measurable sets $X_0 \subset X$ and $X_0' \subset X'$ of full measure with $T(X_0) \subset X$, $T'(X_0') \subset X'$, and a **factor map** $\phi : X_0 \to X_0'$ such that:

(1) ϕ is onto
(2) $\phi^{-1}(A) \in \mathcal{S}(X_0)$ for all $A \in X_0'$
(3) $\mu(\phi^{-1}(A)) = \mu'(A)$ for all $A \in \mathcal{S}'(X_0')$
(4) $\phi(T(x)) = T'(\phi(x))$.

2. Ergodicity

Definition 2.1 (Ergodicity)**.** A measure-preserving transformation T is **ergodic** if whenever A is a measurable set that is strictly T-invariant (i.e., $T^{-1}(A) = A$), then $\mu(A)$ is either 0 or 1.

Theorem 2.2 (Birkhoff's ergodic theorem)**.** *Let $(X, \mathcal{S}, \mu)$ be a probability space and let T be a measure-preserving transformation on $(X, \mathcal{S}, \mu)$. If $f : X \to \mathbb{R}$ is an integrable function, then*

(1) $\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i(x))$ exists for all $x \in X \setminus N$, for some null set N depending on f. Denote this limit by $\tilde{f}$.
(2) $\tilde{f}(Tx) = \tilde{f}(x)$ a.e.
(3) For any measurable set A that is T-invariant,

$$\int_A f d\mu = \int_A \tilde{f} d\mu.$$

In particular, if T is ergodic, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i(x)) = \int f d\mu \ \ a.e.$$

For the proof, see chapter 5 of [9].

Theorem 2.3. *Let T be a finite measure-preserving transformation on a probability space $(X, \mathcal{S}, \mu)$. T is ergodic if and only if for all measurable sets A and B,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mu(T^{-i}(A) \cap B) = \mu(A)\mu(B). \tag{2.4}$$

Proof. First we prove the forward direction. Suppose T is ergodic and A, B measurable. Apply the Birkhoff ergodic theorem on T, A, B, and the characteristic map $\mathbb{I}_A$ $\mathbb{I}_B$ to get

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{I}_A(T^i(x)) = \int \mathbb{I}_A d\mu = \mu(A) \quad \text{a.e.}$$

Thus,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{I}_A(T^i(x))\mathbb{I}_B(x) = \mu(A)\mathbb{I}_B(x) \quad \text{a.e.}$$

Observe that $|\mathbb{I}_A(T^i(x))\mathbb{I}_B(x)| \leq 1$ *a.e.*, which enables us to apply the dominated convergence theorem. This gives

$$\begin{aligned} \lim_{n\to\infty} \int \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{I}_A(T^i(x))\mathbb{I}_B(x) d\mu(x) &= \int \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{I}_A(T^i(x))\mathbb{I}_B(x) d\mu(x) \\ &= \int \mu(A)\mathbb{I}_B(x) d\mu(x) = \mu(A)\mu(B). \end{aligned}$$

Also, we know

$$\int \frac{1}{n}\sum_{i=0}^{n-1} \mathbb{I}_A(T^i(x))\mathbb{I}_B(x)d\mu(x) = \frac{1}{n}\sum_{i=0}^{n-1}\int \mathbb{I}_A(T^i(x))\mathbb{I}_B(x)d\mu(x)$$
$$= \frac{1}{n}\sum_{i=0}^{n-1}\mu(T^{-i}(A)\cap B).$$

We take the limit of both sides and obtain what we want to show.

Now we prove the reverse direction. Suppose (2.4) holds for all sets A and B in X. Then it also holds for $A = B$. Let A be T-invariant. By the definition of T-invariant, $\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\mu(T^{-i}(A)\cap A) = \mu(A)$. Let $A = B$. We get $\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\mu(T^{-i}(A)\cap A) = \mu(A)^2$. This gives $\mu(A) = \mu(A)^2$, which implies $\mu(A)$ is 0 or 1. Thus T is ergodic. □

Proposition 2.5. *Let T be defined on a probability space $(X, \mathcal{S}(X), \mu)$, $T \times T$ defined on $(X\times X, \mathcal{S}(X)\times\mathcal{S}(X), \mu\times\mu)$. If $T\times T$ is an ergodic transformation, then T is also ergodic.*

Proof. Let A be a strictly T-invariant measurable set. Then $(T\times T)^{-1}(A\times A) = T^{-1}(A)\times T^{-1}(A) = A\times A$, so $A\times A$ is $(T\times T)$-invariant. The property of product measure shows

$$(\mu\times\mu)(A\times A) = \mu(A)\mu(A).$$

Since $T\times T$ is ergodic, $(\mu\times\mu)(A\times A)$ is 1 or 0. Therefore $\mu(A)$ is either 1 or 0. Hence T is ergodic. □

Theorem 2.6. *Let $(Y, \mathcal{S}'(Y), \nu, S)$ be a factor of $(X, \mathcal{S}(X), \mu, T)$. If T is ergodic, so is S.*

Proof. Let $\phi : X \to Y$ be a factor map. If A is a strictly S-invariant set, then

$$T^{-1}(\phi^{-1}(A)) = \phi^{-1}(S^{-1}(A)) = \phi^{-1}(A).$$

(The first equality holds by Definition 1.7 and the second holds because A is S-invariant.) Thus $\phi^{-1}(A)$ is a T-invariant set. Also by Definition 1.7, $\mu(\phi^{-1}(A)) = \nu(A)$. Since T is ergodic, $\nu(A)$ is 0 or 1, i.e., S is ergodic. □

Definition 2.7 (Eigenvalue and Eigenfunction)**.** Let $(X, \mathcal{S}, \mu)$ be a probability space and let $T : X \to X$ be a measure-preserving transformation. We say a number $\mu \in \mathbb{C}$ is an **eigenvalue** of T if there exists a function $f \in L^2(X, \mathcal{S}, \mu)$ which is nonzero a.e. such that

$$f(T(x)) = \lambda f(x)\ \mu\text{-a.e.}$$

The function f is called an **eigenfunction**.

Lemma 2.8. *If λ is an eigenvalue for a measure-preserving transformation T, then $|\lambda| = 1$.*

Proof. Let T be a measure-preserving transformation and f be its eigenfunction. Let λ be its eigenvalue. Then $|\lambda|^2\int|f|^2d\mu = \int|\lambda|^2|f|^2d\mu = \int|f\circ T|^2d\mu = \int|f|^2\circ Td\mu = \int|f|^2d\mu$. □

Lemma 2.9. *If T is ergodic and f is an eigenfunction, then $|f|$ is constant a.e.*

Proof. Let f be an eigenfunction of T with eigenvalue λ, i.e., $f \circ T = \lambda f$ a.e. By Lemma 2.8, we know $|\lambda| = 1$. Then $|f| \circ T = |f \circ T| = |\lambda f| = |\lambda||f| = |f|$. Since T ergodic, $|f|$ is constant a.e. □

Corollary 2.10. *If T is ergodic and f is its eigenfunction, then $\frac{f}{|f|}$ is also an eigenfunction. Moreover, $\frac{f}{|f|}$ has absolute value 1. Thus we may choose an eigenfunction f of T with $|f| = 1$ a.e. (i.e., the values of f lie in the unit circle).*

Proof. Note that $|f| \neq 0$ a.e. since f is non-zero a.e. □

Definition 2.11 (Continuous Spectrum)**.** A measure-preserving transformation T is said to have **continuous spectrum** if $\lambda = 1$ is its only eigenvalue and it is simple; equivalently, T is ergodic and $\lambda = 1$ is its only eigenvalue.

3. Mixing and Weakly Mixing

The notion of mixing is an abstract concept originated from the study of physics. We formalize this notion to describe the behavior of transformations in dynamical systems. In order to study a transformation T on a probability space, we study $T^n(A) \cap B$ for measurable sets A and B. Intuitively, the ideally "mixed" state is that the size of A in B is proportional to the measure of A and B, i.e., after applying T for some n times, $\mu(T^n(A) \cap B) = \mu(A)\mu(B)$. For example, if we mix 1 ounce of Vodka with 1 ounce of Gin, we want any subset of the new drink to be half-Vodka-half-Gin. Transformations that make the system converge to this state is defined as "mixing" in Definition 3.1.

However, we can also consider a weaker mode of convergence to the ideal state. We define it as weakly mixing. Next chapter we will discuss yet another notion of mixing that only requires the limit inferior of $T^n(A) \cap B$ to be positive, not caring whether is it proportional to the measure of A and B.

Now we explore the relationships between mixing, weakly mixing and ergodicity. For finite measure-preserving transformations, we see that

(1) mixing $\implies$ weakly mixing
(2) weakly mixing $\implies$ ergodic
(3) weakly mixing $\iff$ doubly ergodic.

Definition 3.1 (Mixing)**.** A measure-preserving transformation T on a probability space $(X, \mathcal{S}, \mu)$ is **mixing** if for all measurable sets A and B,

$$\lim_{n\to\infty} \mu(T^{-n}(A) \cap B) = \mu(A)\mu(B).$$

Definition 3.2 (Weakly mixing)**.** A measure-preserving transformation T on a probability space $(X, \mathcal{S}, \mu)$ is **weakly mixing** if for all measurable sets A and B,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} |\mu(T^{-i}(A) \cap B) - \mu(A)\mu(B)| = 0.$$

Definition 3.3 (Doubly ergodic)**.** A measure-preserving transformation T on a probability space $(X, \mathcal{S}, \mu)$ is **doubly ergodic** if for all measurable sets A and B, there exists an integer $n > 0$ such that

$$\mu(T^{-n}(A) \cap A) > 0 \text{ and } \mu(T^{-n}(A) \cap B) > 0.$$

Remark 3.4. Construct a sequence $\{a_i\}$ with $a_i(A,B) = \mu(T^{-i}(A) \cap B)$. Notice that (2.4) shows that ergodicity is equivalent to Cesàro convergence of $\{a_i\}$ to $\mu(A)\mu(B)$. From the definitions above, we also know weakly mixing is equivalent to strong Cesàro convergence of $\{a_i\}$ to $\mu(A)\mu(B)$, and mixing is equivalent to convergence of $\{a_i\}$ to $\mu(A)\mu(B)$.

Theorem 3.5. *Let T be a probability-preserving transformation.*

1. *If T is weakly mixing, then it is ergodic.*
2. *If T is mixing, then it is weakly mixing.*

Proof. This result follows from Remark 3.4 and Lemma 1.2. □

The next lemma provides sufficient conditions for ergodicity, mixing, and weakly mixing that will be helpful in later proofs. The conditions rely on the notion of a sufficient semi-ring.

Definition 3.6 (Semi-Ring)**.** A **semi-ring** on a nonempty set X is a collection $\mathcal{R}$ of subsets of X such that

1. $\mathcal{R}$ is nonempty;
2. if $A, B \in \mathcal{R}$, then $A \cap B \in \mathcal{R}$;
3. if $A, B \in \mathcal{R}$, then
$$A \setminus B = \bigsqcup_{j=1}^{n} E_j,$$
where $E_j \in \mathcal{R}$ are disjoint.

Definition 3.7. Let $(X, \mathcal{S}, \mu)$ be a measure space. A semi-ring $\mathcal{C}$ of measurable subsets of X of finite measure is said to be a **sufficient semi-ring** for $(X, \mathcal{S}, \mu)$ if it satisfies: for every $A \subset \mathcal{S}$,
$$\mu(A) = \inf \left\{ \sum_{j=1}^{\infty} \mu(I_j) : A \subset \bigcup_{j=1}^{\infty} I_j \text{ and } I_j \in \mathcal{C} \text{ for } j \geq 1 \right\}.$$

Lemma 3.8. *Let T be a measure-preserving transformation on a probability space $(X, \mathcal{S}, \mu)$ with a sufficient semi-ring $\mathcal{C}$.*

1. *If for all $I, J \in \mathcal{C}$, $\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mu(T^{-i}(I) \cap J) = \mu(I)\mu(J)$, then T is ergodic.*
2. *If for all $I, J \in \mathcal{C}$, $\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} |\mu(T^{-i}(I) \cap J) - \mu(I)\mu(J)| = 0$, then T is weakly mixing.*
3. *If for all $I, J \in \mathcal{C}$, $\lim_{n\to\infty} \mu(T^{-i}(I) \cap J) = \mu(I)\mu(J)$, then T is mixing.*

Proof. We provide a brief sketch of the proof. For a complete proof, see chapter 6.3 of [9] for details. Let A, B be two sets of positive measures in $\mathcal{S}$. The proof is done by constructing E and F which are both finite disjoint unions of finite sequences of sets in $\mathcal{C}$ such that the measure of set difference between A and E, B and F are within ϵ. Then we can show by computation that if the limits in concern are bounded by ϵ for such E and F, then they are also bounded for A and B. □

Remark 3.9. We can now study a sufficient semi-ring of a measure space and extend the result to all measurable sets in the space. One typical sufficient semi-rings we utilize is intervals with dyadic rational endpoints. Another example: for $(X, \mathcal{S}(X), \mu)$ and $(Y, \mathcal{S}(Y), \nu)$, define $\mu \times \nu$ on $\mathcal{S}(X) \times \mathcal{S}(Y)$ by $(\mu \times \nu)(A \times B) =$

$\mu(A)\nu(B)$. Then the semi-ring of measurable rectangles is a sufficient semi-ring for the extension measure $\mu \times \nu$.

Now we explore the relations between ergodicity, mixing, and weakly mixing.

Proposition 3.10. *Let T be a measure-preserving transformation on a probability space $(X, \mathcal{S}, \mu)$. Then the following are equivalent:*

(1) T is weakly mixing.
(2) For each pair of measurable sets A and B, there is a zero density set $D = D(A,B)$ such that

$$\lim_{i\to\infty, i\notin D} \mu(T^{-i}(A)\cap B) = \mu(A)\mu(B).$$

(3) For each pair of measurable sets A and B,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1}(\mu(T^{-i}(A)\cap B) - \mu(A)\mu(B))^2 = 0.$$

Proof. (1) $\Leftrightarrow$ (2): Let

$$b_i = |\mu(T^{-i}(A)\cap B) - \mu(A)\mu(B)|.$$

Then $\{b_i\}$ is a sequence of bounded, non-negative real numbers. Notice that (1) is the condition that $\{b_i\}$ converges Cesàro to 0 and (2) is the condition that $\{b_i\}$ converges in density to 0. Hence, by Lemma 1.2, (1) and (2) are equivalent.

(2) $\Leftrightarrow$ (3): By the fact that $\{b_i\}$ converges to 0 if and only if $\{b_i^2\}$ converges to 0, (2) is equivalent to

$$\lim_{i\to\infty, i\notin D}(\mu(T^{-i}(A)\cap B) - \mu(A)\mu(B))^2 = 0. \tag{3.11}$$

By Lemma 1.5, (3.11) is equivalent to (3). This completes the proof. □

Theorem 3.12. *Let T be a measure-preserving transformation on a probability space $(X, \mathcal{S}, \mu)$. Then the following are equivalent.*

(1) T is weakly mixing.
(2) $T \times T$ is weakly mixing.
(3) $T \times T$ is ergodic.

Proof. (1)$\Rightarrow$(2): Assume T is weakly mixing. Let A, B, C, D be measurable subsets of X. By Proposition 3.10, there exist $D_1 = D_1(A,B)$ and $D_2 = D_2(C,D)$ such that for $n \notin D_1 \cup D_2$,

$$\lim_{n\to\infty} \mu(T^{-n}(A)\cap B) = \mu(A)\mu(B)$$
$$\lim_{n\to\infty} \mu(T^{-n}(C)\cap D) = \mu(C)\mu(D).$$

Then for all $n \notin D_1 \cup D_2$,

$$\lim_{n\to\infty}(\mu\times\mu)[(T\times T)^{-n}(A\times C)\cap(B\times D)] = (\mu\times\mu)(A\times C)(\mu\times\mu)(B\times D). \tag{3.13}$$

By the observation in Remark 3.9, (3.13) holds on a sufficient semi-ring. Then by Lemma 3.8, it holds for all measurable sets of X. Then by Proposition 3.10, (3.13) implies $T \times T$ is weakly mixing.

(2)$\Rightarrow$(3): This follows from Theorem 3.5.

(3)⇒(1): Assume $T \times T$ is ergodic. We will show the condition in Proposition 3.10(3) holds. Expand the coefficients in that condition:

$$\begin{aligned}(3.14)\qquad &\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}(\mu(T^{-i}(A)\cap B)-\mu(A)\mu(B))^2\\ &=\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}(\mu(T^{-i}(A)\cap B))^2\\ &\quad-2\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}(\mu(T^{-i}(A)\cap B))\mu(A)\mu(B)+(\mu(A)\mu(B))^2.\end{aligned}$$

We compute each of the three terms on the right hand side. Since $T\times T$ is ergodic, T is ergodic by Proposition 2.5. Thus apply Theorem 2.3 on T, A, B to get

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\mu(T^{-i}(A)\cap B)=\mu(A)\mu(B).$$

Apply Theorem 2.3 on $T\times T$, $A\times A$, $B\times B$ to get

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\mu^2[(T\times T)^{-i}(A\times A)\cap(B\times B)]=\mu^2(A\times A)\mu^2(B\times B)$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}(\mu(T^{-i}(A)\cap B))^2=(\mu(A))^2(\mu(B))^2.$$

Thus, (3.14) equals $\mu(A)^2\mu(B)^2-2\mu(A)^2\mu(B)^2+\mu(A)^2\mu(B)^2=0$. By Proposition 3.10, T is weakly mixing. □

The next theorem gives the equivalent condition for weakly mixing and doubly ergodic transformations.

Theorem 3.15. *Let T be an invertible measure-preserving transformation on a Lebesgue probability space $(X,\mathcal{S},\mu)$. Then the following are equivalent.*

1. *T is weakly mixing.*
2. *T has continuous spectrum. (Recall Definition 2.11.)*
3. *T is doubly ergodic.*
4. *$T\times S$ is ergodic for any ergodic, finite meausure-preserving transformation S.*

Proof. (1)⇒(2): Assume T is weakly mixing. By Theorem 3.12, T and $T\times T$ are ergodic. By Lemma 2.8 and Corollary 2.10, there exist an eigenvalue λ and an eigenfunction $f\in L^2$, with $|\lambda|=1$ and $|f|=1$ such that $f(T(x))=\lambda f(x)$ a.e. To show T has continuous spectrum, we need to show $\lambda=1$ is the only eigenvalue. Define $g: X\times X\to\mathbb{C}$ by $g(x,y)=f(x)\bar{f}(y)$. Since $\lambda\bar{\lambda}=|\lambda|^2=1$,

$$g(T(x),T(y))=f(T(x))\bar{f}(T(y))=\lambda f(x)\bar{\lambda}\bar{f}(y)=f(x)\bar{f}(y)=g(x,y).$$

This shows g is the eigenfunction for $T\times T$. Since $T\times T$ is ergodic, g is constant a.e.; thus f is constant a.e.

(1)⇒(3): Assume T is weakly mixing, and A and B are measurable sets of positive measures. Then by Proposition 3.10 there exist $D_1=D_1(A,B), D_2=$

$D_2(A, A)$ such that for $n \notin (D_1 \cup D_2)$,

$$\lim_{n\to\infty} \mu(T^{-n}(A) \cap B) = \mu(A)\mu(B),$$
$$\lim_{n\to\infty} \mu(T^{-n}(A) \cap A) = \mu(A)\mu(A).$$

Since $\mathbb{Z}_{\geq 0} - (D_1 \cup D_2)$ has density 1, there exists a nonnegative integer n such that $\mu(T^{-n}(A) \cap A) > 0$ and $\mu(T^{-n}(A) \cap B) > 0$, as required.

(2)⇒(4): This proof involves a lot of functional analysis, which is not a focus of this paper, so we only present a outline. For the complete proof of this theorem, see [8, page 67-71]. We define *fiber sets* $A_x = \{y \in Y : (x, y) \in A\}$. If $T \times S$ is not ergodic, then there would be an invariant set $(T \times S)(A) = A$. We then know $S(A_x) = A_{T(x)}$ for a.e. x. So S takes the fiber at x to the fiber at $T(x)$. Define the metric $d(x, x') = \mu(A_x \triangle A_{x'})$. Identify $x \sim x'$ if and only if $A_x = A_{x'}$ a.e. Then $X/\sim$ with d is a metric space. We claim that T acts on $X/\sim$ as an isometry and this space is compact. Then T on $X/\sim$ is isomorphic to a rotation on a compact map, which can be shown to admit a non-constant eigenfunction. This is a contradiction.

(3)⇒(2): Assume T is doubly ergodic. By definition, T is also ergodic. Similar to the previous proof, there exist an eigenvalue λ and an eigenfunction $f \in L^2$ with $|\lambda| = 1$, $|f| = 1$ such that $f(T(x)) = \lambda f(x)$ a.e.

We aim to show $\lambda = 1$ is the only eigenvalue. We will proceed by contradiction. Suppose there are multiple eigenvalues on unit circle. Parametrize this unit circle. Write $\lambda = e^{2\pi i\alpha}$ and $f(x) = e^{2\pi i g(x)}$ for some $\alpha \in [0, 1)$ and measurable $g : X \to [0, 1)$. Define $R : [0, 1) \to [0, 1)$ by $R(t) = \alpha + t \pmod 1$. Then $g \circ T = R \circ g$.

Now we construct a factor of the probability space as below: Let ν be a measure on $[0, 1)$ with $\nu(A) = \mu(g^{-1}(A))$. Then g is a factor map from T to R. By Theorem 2.6, R is ergodic because T is. There are two cases for R: (a) If α is rational, then ν is atomic and concentrated on finitely many points, thus not doubly ergodic. (This can be shown by taking A and B to be sets with singleton points.) (b) If α is irrational, then ν is a Lebesgue measure. Lebesgue measure is not doubly ergodic for irrational rotation by the following argument: Let $A = [0, \frac{1}{8})$ and $B = [\frac{1}{2}, \frac{3}{4})$. Then for any integer n such that $R^n(A) \cap B \neq 0$, $R^n(A) \cap A = 0$. Since in both cases R is not doubly ergodic, so we get a contradiction. Thus $\lambda = 1$ is the only eigenvalue, i.e., T has continuous spectrum.

(4)⇒(1): Let $S = T$; the result follows directly from Theorem 3.12. □

Remark 3.16. (1) and (3) are proved to be equivalent for finite measure-preserving transformations, but the result may not always hold for infinite measures. In fact, [2] shows that for infinite measure-preserving transformations, doubly ergodic implies weakly mixing but there exists a weakly mixing map that is not doubly ergodic.

4. Mixing and Lightly Mixing

As we introduced in the previous chapter, lightly mixing is a notion that is stronger than weakly mixing but weaker than mixing. Now we study it's relationship with the other notions.

Definition 4.1 (Lightly mixing)**.** A finite measure-preserving transformation T on a probability space $(X, \mathcal{S}, \mu)$ is **lightly mixing** if for all measurable sets A,B of positive measure,

$$\liminf_{n\to\infty} \mu(T^{-n}(A) \cap B) > 0.$$

Lemma 4.2. *T is lightly mixing if and only if for each set A of positive measure, there exists N such that $\mu(T^{-n}(A) \cap A) > 0$ for all $n \geq N$*

Proof. The forward direction follows by letting $B = A$ in the definition of lightly mixing. For the other direction, we prove the contrapositive. Suppose T is not lightly mixing. Then there exists a set E of positive measure such that $\liminf_{n\to\infty}$ $\mu(T^{-n}(E) \cap E) = 0$. Then we can always pick n_k large enough such that $E_k = T^{-n_k}(E) \cap E$ satisfies $\mu(E_k) < \frac{\mu(E)}{3^k}$

Now let $F = E - \bigcup_{k=1}^{\infty} E_k$ so that F is the part of the set that "does not come back." Then we have

$$\begin{aligned} \mu(F) &\geq \mu(E) - \sum_{k=1}^{\infty} \mu(E_k) && \text{by countable additivity of } \mu \\ &> \mu(E) - \frac{\mu(E)}{2} = \frac{\mu(E)}{2} && \text{by sum of geometric series.} \end{aligned}$$

Then this shows F has positive measure. Notice $T^{-n_k}(F) \cap F$ is empty for all k, since $T^{-n_k}(A) \subset T^{-n_k}(E)$ but $T^{-n_k}(E) \cap A = \emptyset$. □

Proposition 4.3. *Let T be a finite measure-preserving transformation on a probability space $(X, \mathcal{S}, \mu)$. If T is lightly mixing, then T is weakly mixing.*

Proof. Apply Lemma 4.2: if T is lightly mixing, then for each set A of positive measure, there exists N such that $\mu(T^{-n}(A) \cap A) > 0$ for all $n \geq N$. Fix this N. There exists an integer $m > N$ such that $\mu(T^{-m}(A) \cap B) > 0$ by property of limit inferior. We have $\mu(T^{-m}(A) \cap A) > 0$ and $\mu(T^{-m}(A) \cap B) > 0$, so T is doubly ergodic. For finite measure this means T is weakly mixing by Theorem 3.15. □

5. Rank One Maps

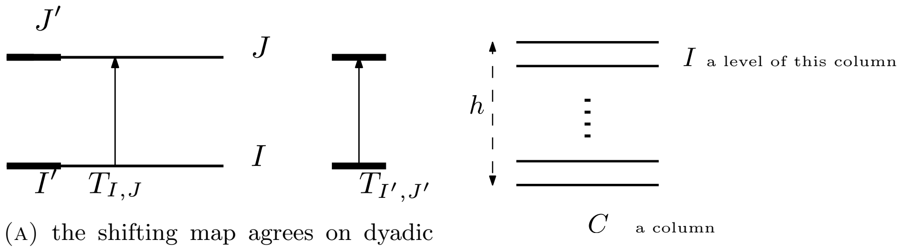
In Theorem 3.5 we showed that mixing implies weakly mixing, and that weakly mixing implies ergodic. In this section we show by constructing examples that the converses do not hold. To do this, we construct a map that is ergodic but not weakly mixing in Section 5.1, a map that is weakly mixing but not lightly mixing in Section 5.2 and a map that is lightly mixing but not mixing in Section 5.3. These constructions employ a technique called "cutting and stacking."

These transformations are also called "rank one transformations" because they are constructed in a particularly simple way. (See Remark 5.19.) We will give a rigorous definition of rank one after we show the process of construction.

For this chapter, μ denotes the Lebesgue measure.

Construction 5.1 (Shifting Map for Two Intervals)**.** Given two intervals $I = [a, b)$ and $J = [c, d)$ of the same length, define $T_{I,J} : I \to J$ by $T_{I,J}(x) = x + c - a$. It is easy to check that:

(1) $T_{I,J}$ is determined by I and J and is one-to-one.
(2) for any measurable set $A \subset I$, $T_{I,J}(A) \subset J$ is measurable and $\mu(T_{I,J}(A)) = \mu(A)$.
(3) if I' and J' are both dyadic subintervals of I and J of the same order (i.e., the end points of I and J are dyadic rationals of the same order), then $T_{I,J}$ agrees with $T_{I',J'}$ on I'. (See Figure 1a.)



(A) the shifting map agrees on dyadic subintervals

(B) visualization of a column

FIGURE 1. shifting map and columns

Definition 5.2 (Column)**.** Let a **column** be a finite sequence of disjoint intervals of the same length. Each interval is called a **level** and the number of intervals in a column is called the **height** of the column. (See Figure 1b.)

Definition 5.3 (Shifting Map for a Column)**.** If C is a column, we define T_C, the shifting map for C, as follows: for each level I of C except for the top level, T_C maps I to the level directly above it, via the shifting map for two intervals. (See Construction 5.1.)

For example, suppose C has levels $I_1, \dots, I_n$ with I_1 the bottom and I_n the top. Then T_C is defined to be T_{I_1,I_2} on I_1, T_{I_2,I_3} on I_2, …, and T_{I_{n-1},I_n} on I_{n-1}, and not defined on I_n. (Notice that the domain of T_C is all levels in C except for the top one.)

The three maps in this chapter are all constructed by applying this shifting map on different types of columns and taking the "limit map."

5.1. **The Dyadic Odometer.**

Construction 5.4. The construction is basically "cutting column into two, and stacking the right above the left". We inductively construct the columns C_0, C_1, $C_2, \dots$.

Base construction: Consider the interval $[0, 1)$. Let C_0 denote the first column. Then,

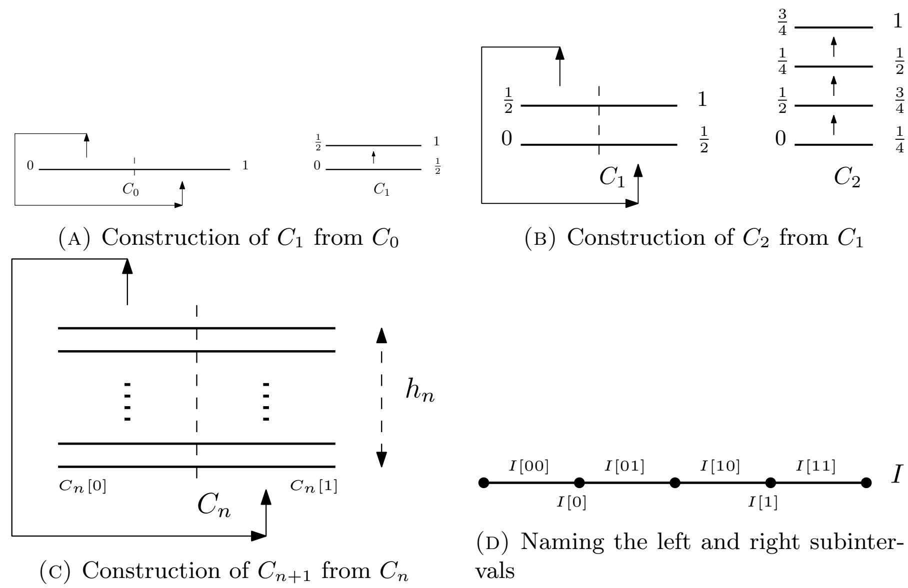
$$C_0 = ([0, 1)),\ h_0 = 1.$$

Construction of C_1: Divide C_0 into two disjoint subintervals $[0, \frac{1}{2})$ and $[\frac{1}{2}, 1)$. Stack the right above the left. (See Figure 2a.) Then,

$$C_1 = ([0, \tfrac{1}{2}), [\tfrac{1}{2}, 1)),\ h_1 = 2.$$

Induction: To obtain C_{n+1} from $C_n = (I_{n,0}, I_{n,1}, \dots, I_{n,h_n-1})$, divide each level into two disjoint subintervals of the same length as above. In this way the column C_n is divided into two subcolumns $C_n[0]$ and $C_n[1]$, repersenting the left and the right subcolumns. Stack $C_n[1]$ above $C_n[0]$. (See Figure 2c.)

Since each level is cut into two, we have $h_{n+1} = 2h_n$ and $h_n = 2^n$.



(A) Construction of C_1 from C_0

(B) Construction of C_2 from C_1

(C) Construction of C_{n+1} from C_n

(D) Naming the left and right subintervals

FIGURE 2. the dyadic odometer

Notation 5.5 (Naming the subintervals)**.** Let C_n denote the nth column, h_n denote the height of C_n, and $I_{n,m}$ denote the mth level in C_n, counting from the bottom $(m = 0, 1, \dots h_n - 1)$.

For dyadic columns, name the leftmost sublevel of a level I in C_n as $I[0]$ and the rightmost $I[1]$. For $a_0, \dots, a_k \in \{0, 1\}$, define $I[a_0 a_1 \dots a_k]$ recursively to represent each level in C_n in this way: if $I[a_0 a_1 \dots a_{k-1}]$ is defined, then $I[a_0 a_1 \dots a_{k-1} 0]$ is the left sublevel of $I[a_0 a_1 \dots a_{k-1}]$ and $I[a_0 a_1 \dots a_{k-1} 1]$ is the right sublevel.

Construction 5.6 (Sequence of Partial Maps T_{C_n})**.** Now we apply the shifting map on the sequence of columns we constructed. Recall Definition 5.3. For each n, define T_{C_n}. It is easy to see that $T_{C_{n+1}}$ agrees with T_{C_n} whenever T_{C_n} is defined. Notice that T_{C_n} is undefined on only the top level of C_n, of measure $\frac{1}{2^n}$.

Remark 5.7. Figure 2c is an intuitive way of representing the columns. In C_n, each level is mapped into the level directly above it. Another way of writing the column is to consider C_n as an ordered tuple. Then $C_{n+1} = (I_{n,0}[0], \dots, I_{n,h_n-1}[0], I_{n,0}[1], \dots, I_{n,h_n-1}[1])$. Rename the subintervals as $C_{n+1} = (I_{n+1,0}, \dots, I_{n+1,h_{n+1}-1})$. Each of the elements in the set is mapped into the element immediately after it. It is easy to see that T_{C_n} is not defined only on the last element of this set. Also notice that by our construction, $I_{n,i}[0]$ is in fact $I_{n+1,i}$ and $I_{n,i}[1]$ is I_{n+1,h_n+i}.

Definition 5.8 (Dyadic Odometer)**.** Let $T : [0, 1) \to (0, 1]$ be the map $T(x) = \lim_{n\to\infty} T_{C_n}(x)$. This map is called the dyadic odometer.

Remark 5.9 (Invertibility). This map is well defined since for each $x \in [0, 1)$, there is some $n > 0$ such that x is in some level of C_n that is not the top. T is one-to-one with a similar argument. Also it is easy to check that T^{-1} is defined for all $x \in (0, 1]$, i.e., T is invertible. The invertibility is very important because it allows us to apply Theorem 3.15.

Remark 5.10. In this chapter, each of the maps we construct is invertible. Also, the columns we construct are all T-invariant for T the limit of shifting map. Since sets that are positive invariant are strictly T-invariant mod μ for T invertible and measure-preserving, we can study the forward image of T and conclude the same result as studying the pre-image of T. That is, instead of studying the behavior of $\mu(T^{-i}(A) \cap B)$, we can just study the behavior of $\mu(T^{i}(A) \cap B)$, which is more intuitive in specific examples.

Now we show some properties of this map.

Lemma 5.11. *Let T be the dyadic odometer. Then:*
(1) for all $n > 0$, $T(I_{n,h_n-1}) = I_{n,0}$.
(2) for all $n > 0$, $i = 0, \dots, h_n - 1$, $T^{h_k}(I_{n,i}) = I_{n,i}$ for all $k \geq n$.

Proof. (1): Let I and J be the top and bottom level of C_n. That is, $I = I_{n,h_n-1}$ and $J = I_{n,0}$. We prove by induction. Base step: $T(I[0]) = J[1]$ by definition of T_{C_n}. Induction step: if $T(I[\underbrace{1\dots1}_{k}0]) = J[\underbrace{0\dots0}_{k}1]$, then $T(I[\underbrace{1\dots1}_{k}10]) = J[\underbrace{0\dots0}_{k}01]$ by $T_{C_{k+1}}$. So $T(I[\underbrace{1\dots1}_{k}0]) = J[\underbrace{0\dots0}_{k}1]$ holds for all k. Thus $I[1] = \bigsqcup_{k>0} I[\underbrace{1\dots1}_{k}0]$, so $T(I) = T(I[0]) + T(I[1]) = J[1] + \sum_{k>0} T(I[\underbrace{1\dots1}_{k}0]) = J$ as required.

(2): Since both (1) and $T_{C_n}(I_{n,i}) = I_{n,i+1}$ hold and T agrees with T_{C_n}, we have $T^{h_n}(I_{n,i}) = I_{n,i}$ for all $0 \leq i < h_n$. Thus, also $T^{h_n}(T^{h_n}(I_{n,i})) = I_{n,i}$. (2) can be shown by induction. □

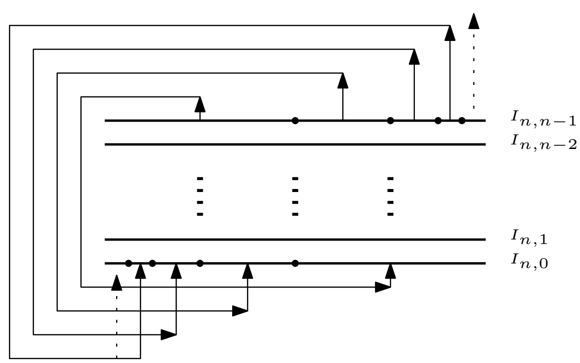


FIGURE 3. $T(I_{n,h_n-1}) = I_{n,0}$

Definition 5.12. Given measurable sets A and I and $0 < \alpha \leq 1$, we say I is **α-full of** A if $\mu(A \cap I) > \alpha\mu(I)$.

Lemma 5.13. *Let $(X, \mathfrak{L}, \mu)$ be a nonatomic measure space with a sufficient semi-ring $\mathcal{C}$. If $A \in \mathfrak{L}$ is of finite positive measure, then for any $1 > \alpha > 0$, there exists $I \in \mathcal{C}$ such that I is α-full of A.*

Proof. The proof is easily done in measure theory by applying the properties of semi-rings. □

The notion of "α-full of" is very important in showing the ergodicity of cutting and stacking constructions. It provides a notion of "large proportion" of sets. Lemma 5.13 makes it possible for any measurable set in the σ-algebra to have a set in the semi-ring that is a "large proportion" of it. Thus, this lemma enables us to pick levels in the dyadic odometers to approximate any set in $[0, 1)$.

Lemma 5.14. *Let A be a set of positive measure and I be a dyadic interval that is $\frac{3}{4}$-full of A. Let I_0 and I_1 be the left and right half of I. Then one of I_0 and I_1 is $\frac{3}{4}$-full of A and both I_0 and I_1 are $\frac{1}{2}$-full of A.*

Proof. Prove both statements by their contrapositives: if none of the two intervals is $\frac{3}{4}$-full of A, then their union is not. If either is not $\frac{1}{2}$-full of A, the union cannot be $\frac{1}{2}$-full of A even if the other interval is full of A. □

Theorem 5.15. *If T is the dyadic odometer, then T is invertible mod μ on $[0, 1)$. Also T is measure-preserving and ergodic.*

Proof. Recall that the dyadic intervals forms a sufficient semi-ring. We know for a measure space $(X, \mathcal{S}, \mu)$ with a sufficient semi-ring $\mathcal{C}$, if $T^{-1}(I)$ is measurable and $\mu(T^{-1}(I)) = \mu(I)$ for all I in $\mathcal{C}$, then T is measure-preserving. (For the proof, see chapter 3.4 of [9].) Then the dyadic odometer is measurable and measure-preserving. The argument in Remark 5.9 shows that T is invertible mod μ.

Now we show T is ergodic: Let A_1, B_1 be two sets of positive measures in $[0, 1)$. There exist dyadic intervals I and J that are $\frac{3}{4}$-full of A_1 and B_1. If I and J have different measures, say, $\mu(I) < \mu(J)$, then since Lemma 5.14 implies at least half of J is $\frac{3}{4}$-full of B_1, we can divide J and rename the half that is $\frac{3}{4}$-full of B_1 as J. Continue this process until I and J have the same measure (which will eventually happen since they are dyadic). Then they are both levels of the same column (which follows from construction). Name this column C_{n-1} . By Lemma 5.14, each half of J (in C_n) is $\frac{1}{2}$-full of B_1, and the same for I and A_1. This means we are able to find I and J each $\frac{1}{2}$-full of A_1 and B_1 and I above J (i.e., $T^\ell(I) = J$ for some ℓ). Let $A = A_1 \cap I$ and $B = B_1 \cap J$. We have

$$\begin{aligned}\mu(T^\ell(A_1) \cap B_1) &\geq \mu(T^\ell(A) \cap B)\\ &\geq \mu(T^\ell(I) \cap J) - \mu(I \setminus A) - \mu(J \setminus B) \qquad \text{by set theory}\\ &> \mu(J) - \tfrac{1}{2}\mu(I) - \tfrac{1}{2}\mu(J) = 0.\end{aligned}$$

Pick any set E and its complement E^c for A and B. Then $\mu(T^\ell(E) \cap E^c) > 0$, a contradiction. So E is cannot have positive measure, i.e., $\mu(E)$ is either 0 or 1. □

Definition 5.16. Let $n > \ell$. Suppose I is an interval in C_ℓ. Then there exist intervals I_k in C_n such that $I = \bigcup I_k$. We say that the **copies of I in C_n** are the intervals I_k. (Each I_k is a **copy** of I.)

We also define copies of columns. For $n > \ell$, a **C_ℓ-copy in C_n** is a group of consecutive intervals $J_0, \ldots, J_{h_n-1}$ in C_n such that J_k is a copy of I_k for each k.

For this dyadic odometer example, any level I in C_ℓ has two copies in $C_{\ell+1}$, each of length $\frac{1}{2}|I|$, and has 4 copies in $C_{\ell+2}$, each of length $\frac{1}{4}|I|$. Each C_{n+1} has exactly two copies of C_n. (See Figure 4.)

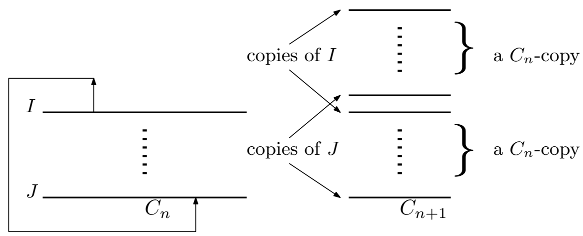


FIGURE 4. each level in C_n has two copies in C_{n+1}

Definition 5.17. We say levels I and J in C_n are $|\mathbf{i}-\mathbf{j}|$ **apart** if J is the jth level in C_n and I is the ith level of C_n.

Proposition 5.18. *T is not weakly mixing.*

Proof. Since the dyadic odometer is finite-measurable, to show T is not weakly mixing we only need to show it is not doubly ergodic (by Theorem 3.15). To test doubly ergodicity, simply pick A and B to be the top and bottom levels of C_n. Then A is $h_n - 1$ levels above B. The copies of A are h_n levels apart, and the copies of B are also h_n levels apart. So if $\mu(T^m(A)\cap B) > 0$, then $m = rh_n + 1$ for some $r \in \mathbb{N}$. If $\mu(T^m(A)\cap A) > 0$, then $m = rh_n$ for some $r \in \mathbb{N}$. But there are no such r and s that satisfy $rh_n = sh_n + 1$, so $\mu(T^m(A)\cap A)$ and $\mu(T^m(A)\cap B)$ cannot be both positive for any m. □

Remark 5.19 (rank one transformation). The dyadic odometer and the two maps we construct next are all rank one transformation. However, though it is easy to give examples of rank one, it is hard to define the notion concisely. Seven mainstream definitions are given in section 1 of [4]. In this paper, we use "rank one" to describe transformations constructed with a single cutting and stacking at each step.

Since the dyadic odometer is ergodic but not weakly mixing, we have shown that ergodicity and weakly mixing are not equivalent.

5.2. **Chacón's Transformation.** This transformation from R. V. Chacón's 1969 Paper [3] is an example that is weakly mixing but not mixing. Moreover, we show that it is not lightly mixing.

Construction 5.20 (Chacón's Transformation)**.** We use the same notations of column, level and height. The construction is basically "cutting the column into three, putting a spacer above the middle column, and stacking from left to right." We inductively construct the columns C_0, C_1, $C_2, \ldots$.

Base construction: Let C_0 denote the column consisting of a single interval $I_{0,0} = [0, \frac{2}{3})$. Then $h_0 = 1$.

Construction of C_1: Divide C_0 into three disjoint subintervals $[0, \frac{2}{9})$, $[\frac{2}{9}, \frac{4}{9})$ and $[\frac{4}{9}, \frac{2}{3})$. Put a *spacer* S_0 (a new interval) above the middle subinterval and stack from the right on top to the left on bottom. S_0 is chosen to abut the current column and be of the same length of the middle subinterval. In this case, the spacer is $[\frac{2}{3}, \frac{8}{9})$. The union of all the intervals is $[0, \frac{8}{9})$. Notice C_1 has four levels (three from dividing C_0 and one spacer), so $h_1 = 4$. (See Figure 5a.)

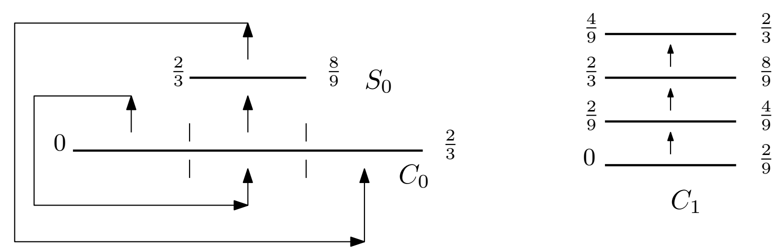
Induction: We obtain C_{n+1} from C_n. (This is a generalization of the construction of C_1 from C_0 in the previous paragraph.) First, note that the column C_n has

$\sum_{i=0}^{n} 3^i = \frac{1}{2}(3^{n+1} - 1)$ levels, each of length $2 \cdot 3^{-n-1}$ and that the union of these levels is $[0, 1 - 3^{-n-1})$. To obtain C_{n+1} from C_n, we divide each level into three disjoint subintervals of equal length, i.e., of length $2 \cdot 3^{-n-2}$. We define the spacer S_n to also have length $2 \cdot 3^{-n-2}$ and to have left endpoint at $1 - 3^{-n-1}$. (In other words, the new level is chosen to abut the current interval.) We stack the middle subcolumn on top of the left subcolumn, then we stack the spacer on top of the middle subcolumn, and then we stack the right subcolumn on top of the spacer. (See Figure 5b.) A level I in C_ℓ has three copies in $C_{\ell+1}$, each of length $\frac{1}{3}|I|$. Notice that C_{n+1} has height $h_{n+1} = 3h_n + 1$.

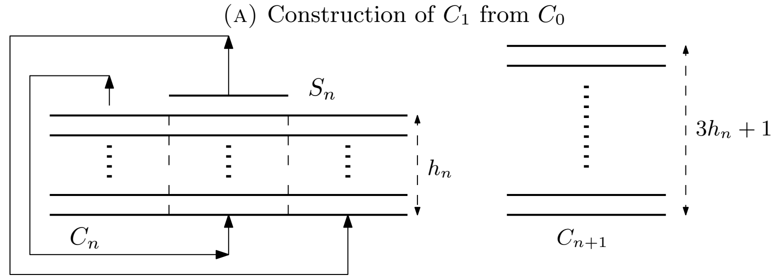
The lengths of spacers added is a geometric sequence with total length

$$\tfrac{2}{9} + \tfrac{2}{27} + ... = \tfrac{1}{3}$$

Let T_{C_n} be defined on the column C_n as in Definition 5.3. Taking n to infinity gives a measure-preserving transformation on [0, 1). The resulting transformation T agrees to each transformation T_n, as in the case of dyadic odometer. This new map is called the **canonical Chacón's Transformation**.



(A) Construction of C_1 from C_0



(B) Construction of C_{n+1} from C_n

FIGURE 5. the canonical Chacón's Transformation

We observe that each level in C_n has 3 copies in C_{n+1}. Pick a level, say level I_m, of C_n. We can see from Figure 6 that:

(1) T^{h_n} maps the leftmost copy of I_m back to level I_m itself. (Recall Definition 5.16.)
(2) T^{h_n} maps the middle copy of I_m level to $T^{-1}(I_m)$ for $m > 0$. For $m = 0$, it simply maps I_0 to the spacer S_n and we know $S_n \subset T^{-1}(I_0)$.

The next lemma is based on this observation.

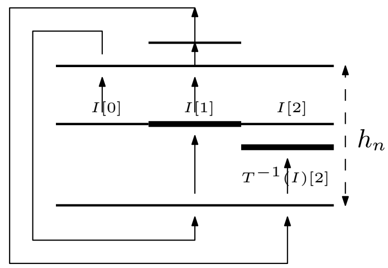


FIGURE 6. $T^{h_n}(I)$ intersects with itself and the level below it for at least $\frac{1}{3}\mu(I)$

Lemma 5.21. *Let $n > 0$. Then:*

(1) For all $k \geq n$, $\mu(T^{h_k}(I) \cap I) \geq \frac{1}{3}\mu(I)$.
(2) For each $\ell \geq 0$, there exists an integer $H = H(n, \ell)$ such that if I and J are at most $\ell \geq 0$ apart, with I above J, then $\mu(T^H(I) \cap J) \geq (\frac{1}{3})^\ell \mu(J)$.
(3) If I is the top level of C_n, then $T^{h_n} I \subset I \cup T^{-1} I$.

Proof. (1): First consider $k = n$. For any level K in C_n, name the three copies of K in C_{n+1} by $K[0], K[1]$ and $K[2]$, from left to right respectively. We have

$$T^{h_n}(K[0]) = K[1] \text{ and } T^{h_n}(K[1]) = (T^{-1}(K))[2]. \tag{5.22}$$

Thus, $T^{h_n}(I)$ intersects both I and $T^{-1}(I)$ in at least $\frac{1}{3}$ of the measure of I. This means

$$\mu(T^{h_n}(I) \cap I) \geq \frac{1}{3}\mu(I) \tag{5.23}$$

$$\mu(T^{h_n}(I) \cap T^{-1}(I)) \geq \frac{1}{3}\mu(I). \tag{5.24}$$

For $k \geq n$, each I in C_n has 3^{k-n} copies in C_k. Applying (5.23) and (5.24) to each copy I' of I, we get $\mu(T^{h_n}(I') \cap I') \geq \frac{1}{3}\mu(I')$. Then we add the copies together to get (1).

(2): We claim that we can take $H(n, \ell) = \sum_{i=0}^{\ell-1} h_{n+i}$. (In particular, $H(n, 0) = 0$.) We prove this result via induction on ℓ. Fix n and let $H_\ell = H(n, \ell)$

The base case $\ell = 0$ is trivial. For the induction step, suppose the result works for $\ell \leq k$. This means T^{H_k} intersects with $T^{-i}(I)$ in measure at least $(\frac{1}{3})^k$ times for all $0 \leq i \leq k$. By (5.22), we know that if K contains a full level in C_n, then T^{h_n} contains two full levels in C_{n+1}. Our choice of H_k makes it so that T^{H_m} has two full levels in C_{n+m} for all $m > 0$. Then we know $T^{H_k + h_{n+k}}(I)$ intersects with $T^{-i}(I)$ in measure at least $(\frac{1}{3})^{k+1}$ for all $0 \leq i \leq k$ by (5.23). The case $i = k+1$ holds by (5.24). So by induction, $H(n, \ell) = \sum_{i=0}^{\ell-1} h_{n+i}$ for all ℓ.

(3): Let I be the top level of C_n. We already know $T^{h_n}(I[0]) = I[1] \subset I$ and $T^{h_n}(I[1]) = T^{-1}(I[2]) \subset T^{-1}I$. Now consider $T^{h_n}(I[2])$. Observe that $T^{h_n}(I[20]) = I[11] \subset I$ and $T^{h_n}(I[21]) = T^{-1}(I[12]) \subset T^{-1}I$. Induction shows that $T^{h_n}(I[2 \ldots 20]) \subset I$ and $T^{h_n}(I[2 \ldots 21]) \subset T^{-1}I$. Since $I[2 \ldots 2]$ converges to the single point $\{1\}$ and T is defined on $[0, 1)$, the induction shows that (3) holds for all n. □

Theorem 5.25. *The canonical Chacón's transformation is a measure-preserving transformation on a probability Lebesgue space that has continuous spectrum.*

Proof. By an argument similar to Remark 5.9 and Theorem 5.15, it is easy to check that T is measure-preserving and ergodic. (We only change dyadic levels into triadic levels.) To show T has continuous spectrum, let $f \in L^2, |f| = 1$ be the eigenfunction of T with eigenvalue $|\lambda| = 1$ such that $f(T(x)) = \lambda f(x)$ a.e. We show $\lambda = 1$ is the only eigenvalue. Since f is a measurable function that is nonzero on sets of positive measure, there exists a constant c such that the set

$$A = \{x : |f(x) - c| < \epsilon\}$$

has positive measure. (If not, we would contradict countable additivity.) Now by Lemma 5.13, there exists a level I in some column C_n such that $\mu(I \cap A) > \frac{5}{6}\mu(I)$. (5.22) gives $T^{h_n}(I[0]) = I[1]$ and $T^{h_n+1}(I[1]) = I[2]$. Since I is $\frac{5}{6}$-full of A and T is measure-preserving, there must be a point $x \in A \cap I$ such that $T^{h_n}(x) \in A \cap I$ and $T^{h_n+1}(T^{h_n}(x)) \in A \cap I$. This gives

$$|f(x) - c| < \epsilon, \quad |\lambda^{h_n} f(x) - c| < \epsilon, \quad |\lambda^{2h_n+1} f(x) - c| < \epsilon.$$

By triangle inequalities, $|f(x)||\lambda^{h_n} - 1| < 2\epsilon$ and $|f(x)||\lambda^{2h_n+1} - 1| < 2\epsilon$. Since $|f| = 1$, we get $\lambda = 1$. Thus, T has continuous spectrum. □

Remark 5.26. The proof above is as presented in Chacón's original paper. Another way to show the weakly mixing property, in the next theorem, follows the same track of Lemma 5.14.

Theorem 5.27. *The canonical Chacón's transformation T is not doubly ergodic.*

Proof. Pick any two sets A_1 and B_1 of positive measure. We are able to choose levels I_1 and J_1 in some column C_n that are $\frac{2}{3}$-full of A_1 and B_1 respectively. We can choose I_1 and J_1 so that I_1 is above J_1 and they are ℓ apart with $0 \le \ell \le h_n$. Let $\delta = (\frac{1}{3})^\ell$. Following the same approximation method in Theorem 5.15, we are able to pick I and J subintervals of I_1 and J_1 that are $(1 - \frac{\delta}{3})$-full of I_1 and J_1 respectively. Let $A = A_1 \cap I$, $B = B_1 \cap J$.

Let $H = \sum_{i=0}^{\ell-1} h_{n+i}$. Lemma 5.21(2) gives $\mu(T^H(I) \cap J) \ge (\frac{1}{3})^\ell \mu(J)$ and $\mu(T^H(I) \cap I) \ge (\frac{1}{3})^\ell \mu(I)$. Then

$$\begin{aligned}\mu(T^H(A) \cap B) &\ge \mu(T^H(I) \cap J) - \mu(I \setminus A) - \mu(J \setminus B) \\ &\ge \delta\mu(J) - \frac{\delta}{3}\mu(I) - \frac{\delta}{3}\mu(J) > 0\end{aligned}$$

and

$$\mu(T^H(A) \cap A) \ge \mu(T^H(I) \cap I) - \mu(I \setminus A) - \mu(I \setminus A) > 0.$$ □

Proposition 5.28. *The canonical Chacón's transformation T is weakly mixing.*

Proof. It follows directly from Theorem 3.15. □

Theorem 5.29. *The canonical Chacón's transformation is not mixing.*

Proof. Let $n > 0$ be such that if I is a level in C_n, then $\mu(I) < \frac{1}{3}$. Then by Lemma 5.21, for all $k \ge n$,

$$\mu(T^{h_k}(I) \cap (I)) \ge \frac{1}{3}\mu(I) > \mu(I)\mu(I) + \frac{1}{9}\mu(I) > \mu(I)\mu(I) + \frac{1}{27}.$$

Thus, T is not mixing. □

Theorem 5.30. *The canonical Chacón's transformation is not lightly mixing.*

Proof. Let I be the top level in C_n for some n. Lemma 5.21(3) gives us that $T^{h_n}I \subset (I \cup T^{-1}I)$, which means $T^{h_n}I \cap (I \cup T^{-1}I)^c = \emptyset$. Thus, for any natural number N, pick I the top level of C_N and J a level in $(I \cup T^{-1}I)^c$. There exists $h_N > N$ such that $T^{h_N}I \cap J = \emptyset$. By Lemma 4.2 , T is not lightly mixing. □

5.3. **Just Lightly Mixing Map.** In N. A. Friedman and J. L. King's paper [6], a map that is lightly mixing but not strongly mixing is constructed with the cutting and stacking strategy. We will call this map **FK's map**.

Construction 5.31. This construction is basically "cutting the column into two, adding a spacer on the right, and stacking from left to right." We inductively construct the columns C_0, C_1, C_2, ...

Base construction: For convenience, let C_0 be empty. Then $h_0 = 0$.

Construction of C_1: Stack spacer $S_1 = [0, \frac{1}{2})$. So $C_1 = ([0, \frac{1}{2}))$ and $h_1 = 1$.

Construction of C_2: Cut C_1 into two subcolumns and stack $S_2 = [\frac{1}{2}, \frac{3}{4})$ above $C_{1,1}$. So $C_2 = ([0, \frac{1}{4}), [\frac{1}{4}, \frac{1}{2}), [\frac{1}{2}, \frac{3}{4}))$ and $h_2 = 3$.

Induction: To obtain C_{n+1} from C_n, divide C_n into two subcolumns, $C_{n,0}$ and $C_{n,1}$, representing the left half and the right half respectively. Let $S_{n+1} = [1-\frac{1}{2^n}, 1-\frac{1}{2^{n+1}})$. We stack $C_{n,1}$ above $C_{n,0}$ and S_{n+1} above $C_{n,1}$. Notice that $h_{n+1} = 2h_n+1$.

Let T_{C_n} be defined on the column C_n as in Definition 5.3. Taking n to infinity gives a measure-preserving transformation on [0, 1). The resulting transformation T agrees with each transformation T_n. Call it **FK's map**.

Remark 5.32. It is easy to check that T is invertible mod μ on $[0, 1)$. Also, T is measure-preserving as in Remark 5.9.

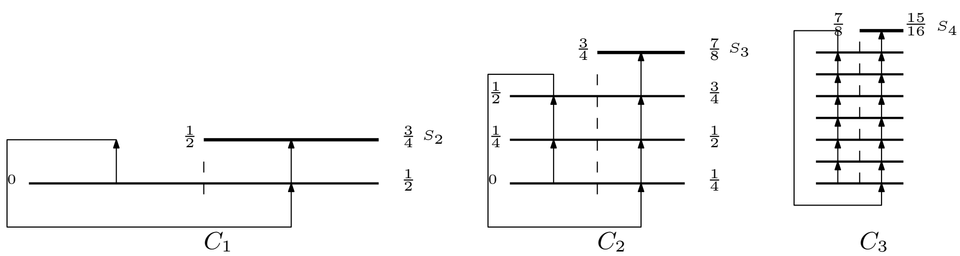


FIGURE 7. construction of C_2 and C_3

Notation 5.33. Similar to Definition 5.2, denote left sublevel of I_n as $I_n[0]$ and right of I_n as $I_n[1]$. For $1 \le i \le h_n$, let $I_{n,i}$ denote the ith level in column C_n, starting from the top (as in Figure 8). For example, $I_{n,i}[0]$ is the left sublevel of ith level in C_n, $I_{n,i}[0] = I_{n+1,h_n+1+i}$ and $I_{n,i}[1] = I_{n+1,i+1}$.

A lemma similar to Lemma 5.11 can be proved:

Lemma 5.34. *For a spacer S_n, $T^{h_n}(S_n) = S_{n+h_n} \cup (\bigcup_{i=1}^{h_n}(T^{h_n}(S_n) \cap I_{n,i}))$*

Proof. By the construction of $T_{C_{n+1}}$, we get $T^{h_n}(S_n[0]) = S_n[1] = I_{n,1}[1]$. Similarly, $T_{C_{n+2}}$ gives us $T^{h_n}(S_n[10]) = I_{n,2}[01]$. Induction shows us $T^{h_n}(S_n[\underbrace{1...1}_{k}0]) = I_{n,k+1}[\underbrace{0...0}_{k}1]$ for $0 \le k \le h_n - 1$. The union of all such intervals is $\bigcup_{i=1}^{h_n}(T^{h_n}(S_n) \cap$

$I_{n,i}$, as shown in the bold lines in Figure 8. Also notice $S_n[\underbrace{1...1}_{h_n}]$, the rightmost sublevel of S_n, is sent to S_{n+h_n}. □

This lemma shows that each spacer S_n is sent to the union of a new spacer S_{n+h_n} and h_n intervals, one on each level of C_n, with length decreasing by a factor of $\frac{1}{2}$. Observe that $\mu(T^{h_n}(S_n) \cap I_{n,i}) = \frac{\mu(S_n)}{2^i}$. Call the configuration of these intervals a *crescent*.

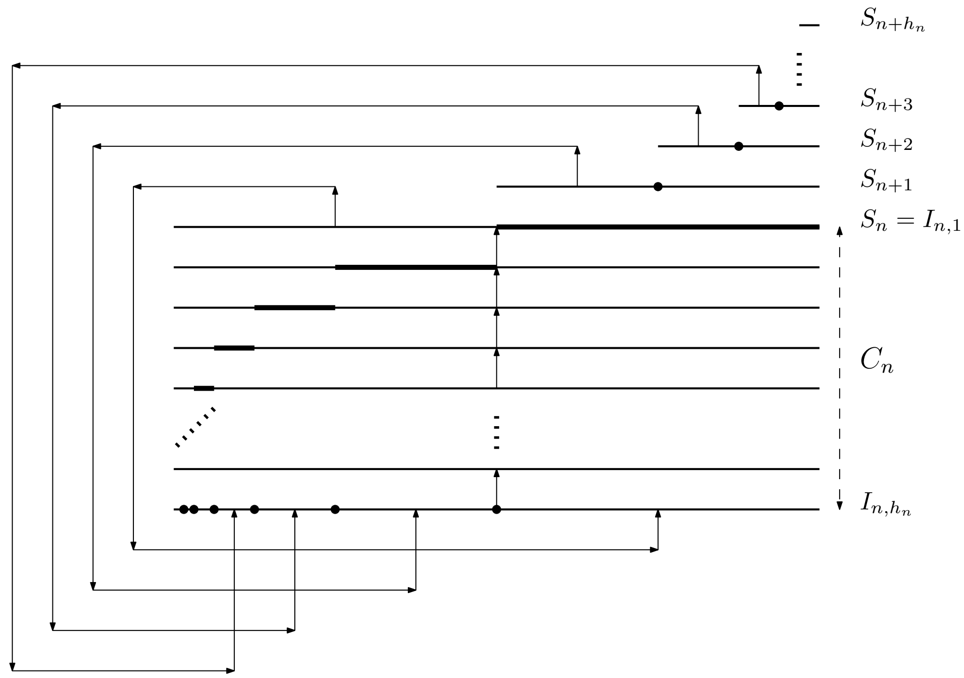


FIGURE 8. $T^{h_n}(S_n)$ is the union of the spacer S_{n+h_n} with the bolded lines with decreasing lengths. Call the configuration of the latter a crescent.

Remark 5.35. Recall the notion of copy in Definition 5.16. For each level in C_n, there are 2 copies in C_{n+1} and C_{n+1} has 2 C_n-copies. For $m \geq k$, C_m appears in C_k as 2^{m-k} disjoint C_k-copies.

Notation 5.36. A C_k-copy in C_n is **preceded by** u **spacers** if there are u spacers between this copy and the closest copy below it. For example, if $n = k + 1$, one of the two C_k-copies is preceded by $u = 0$ spacers. If $n = k + 2$, two of the four C_k-copies are preceded by $u = 0$ spacers, and one is preceded by $u = 1$ spacer.

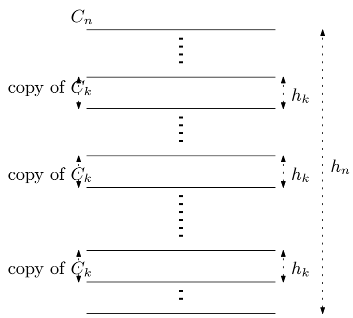


FIGURE 9. the C_k-copies in C_n

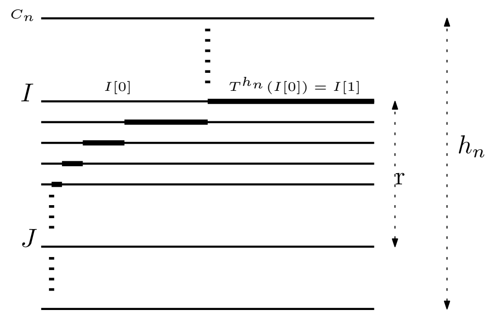


FIGURE 10. $T^{h_n}(I) \cap J$ is a subset of the right-most subinterval of J

Remark 5.37. Induction shows that of the 2^{n-k} C_k-copies in C_n, $2^{n-k-u-1}$ copies are preceded by u spacers, for $0 \leq u < n-k$. Thus, the the fraction of C_k-copies in C_n that is **preceded by at most u spacers** is $\frac{1}{2} + \frac{1}{4} + ... + (\frac{1}{2})^{u+1} = 1 - (\frac{1}{2})^{u+1}$.

Remarks 5.38. Consider any level $I_{n,i}$ in C_n, similar to Lemma 5.34. The copies of $T^{h_n}(I_{n,i})$ form a crescent in C_n from $I_{n,i}$ to bottom, and the remainder is in S_{n+h_n-i+1}

Lemma 5.39. *If I and J are both levels in C_n and J is $r-1$ levels below I, then*

$$T^{h_n}(I) \cap J \subset J[\underbrace{0...0}_{r}] \tag{5.40}$$

Proof. Similar to Lemma 5.34, this proof is also done by induction. □

This lemma shows that the intersection $T^{h_n}(I) \cap J$ is always subset of the leftmost subinterval of J with length $\frac{\mu(J)}{2^r}$.

Theorem 5.41. *T is not mixing.*

This proof is similar to Proposition 5.18 in that we keep track of the copies of the top level of some C_n. The difference is, however, that in the dyadic odometer case, the distance between copies of top levels and copies of bottom levels is always $rh_n + 1$ for some $r \in \mathbb{N}$. Here, because of the existence of spacers, the copies of the top level "mixes" into the levels below in the form of a crescent. (For this reason, FK's map *is* doubly ergodic, as opposed to dyadic odometer).

Proof. Fix k so large that $\frac{h_k}{2^{h_k}} < \frac{1}{4}$. Let A be the top level of C_k and B be the bottom of C_k. Then B is $h_k - 1$ levels below A. For $n > k$, by the observation in Remark 5.35, A appears in 2^{n-k} copies of C_k in C_n. Name them I_j for $1 \leq j \leq 2^{n-k}$. We have $I_j \subset A$ for all j. Now we do the same for B: there exist 2^{n-k} copies J_j for $1 \leq j \leq 2^{n-k}$. We have $J_j \subset B$ for all j. For each j, the level J_j is $h_k - 1$ levels below I_j. For those I_p that are below J_j, $T^{h_n}(I_p) \cap J_j = \emptyset$ by the same argument as in Proposition 5.18. For those I_p that are above J_j, we have $T^{h_n}(I_p) \cap J_j \subset J_j[\underbrace{0...0}_{h_k}]$.

We know the leftmost subinterval of J_i has measure less than $\frac{\mu(J_i)}{2^{h_k}}$, by Lemma 5.39.

Thus,

$$\mu(T^{h_n}(A)\cap B)=\sum_{j=1}^{2^{n-k}}\mu(T^{h_n}(A)\cap J_j)\le\sum_{j=1}^{2^{n-k}}2^{-h_k}\mu(J_j)=2^{-h_k}\mu(B).$$

We know $\mu(A)=\mu(S_k)=\frac{1}{2^{k+1}}$ and $h_k=2^k-1$, so $2h_k\mu(A)<1$. Then it is easy to see that $\mu(T^{h_n}(A)\cap B)<2^{-h_k}(2h_k\mu(A))\mu(B)<\frac{1}{2}\mu(A)\mu(B)$. □

Remark 5.42. A transformation is called **partially mixing** if $\liminf\mu(T^n(A)\cap B)\ge\alpha\mu(A)\mu(B)$ for some positive α. It is easy to check that mixing implies partially mixing. Instead of requiring $\frac{h_k}{2^{h_k}}<\frac{1}{4}$, we can require $\frac{h_k}{2^{h_k}}<\frac{\alpha}{4}$ in the proof above and conclude a stronger result that $\liminf\mu(T^n(A)\cap B)<\frac{1}{2}\alpha\mu(A)\mu(B)$ for all positive α. So T is not partially mixing.

Theorem 5.43. *T is lightly mixing.*

Remark 5.44. The following proof is long, but the idea is simple. We want to show $\mu(T^m(A)\cap A)>0$ for large m. Fix a set A, say, the top level of C_k. For any large m, choose n such that $h_n\le m<h_{n+1}$.

We consider two sets:

(1) the union of all "good" copies of C_k in C_n that are not far from each other.
(2) the union of all crescents created by applying T m times on A_i, the copies of A in C_n.

Our choice of k would make the two sets intersect in a set of positive measure, and our choice of n would make most of A_i have the property that $T^m(A_i)$ is a crescent or a pair of crescents for all m. So there would have to be a copy of A in C_n whose mth image, $T^m(A_i)$, intersects with the top level of C_k-copy, for all m.

Proof. This is a proof by computation. Let A be a set of positive measure. By Lemma 4.2, we need to find an integer M such that

$$\mu(T^m(A)\cap A)>0 \text{ for all } m\ge M. \tag{5.45}$$

We know T is defined on $[0,1)$. We define the measure of a column as the measure of the union of the levels in the column. Then $\lim_{n\to\infty}\mu(C_n)=1$.

Choose k large enough so that $\mu(C_k)>0.9$ and some level I in C_k satisfies $\mu(I\cap A)>0.99\mu(I)$ (i.e., I is 0.99-full of A). Without loss of generality, apply T on A several times to send A to the top level of C_k and then shrink A.[1] So now we assume $A\subset S_k$ where S_k is the top level of C_k, and also $\frac{\mu(A)}{\mu(S_k)}>0.99$.

For $n\ge k$, the level S_k appears as 2^{n-k} C_n-copies I_i for $1\le i\le 2^{n-k}$ by Remark 5.35. Also observe that each I_i is the top level of a C_k-copy in C_n.

Fix $1\ge\alpha>0$. We say a level n in C_n is *accurate* if it is α-full of A. Say a C_k-copy in C_n is *good* if its top level is *accurate* and the top level of the C_k-copy below it is *accurate*, and is preceded by at most $u=10$ spacers. Now fix N sufficiently large so that for all $n\ge N$,

$$\#[\text{accurate } C_n\text{-levels } I]>\left(\frac{\mu(A)}{\mu(S_k)}-0.01\right)\cdot 2^{n-k}>0.98\cdot 2^{n-k}. \tag{5.46}$$

[1]This manipulation is without loosing of generality because if (5.45) holds on a subset of A, it also holds on A.

The second inequality holds for sufficiently large n because $\frac{\mu(A)}{\mu(S_k)}$ increases as n increases.

By this inequality we know the fraction of C_k-copies whose top level is *not accurate* is dominated by 0.02. Also, Remark 5.37 shows that the proportion of C_k copies preceded by more than 10 spacers is $1-(\frac{1}{2})^{10+1}$. Let G_n be the union of all *good* C_k-copies in C_n. We then have

$$\mu(G_n) > (1-(\tfrac{1}{2})^{10+1} - 0.02\cdot 2)\mu(C_k) > 0. \tag{5.47}$$

Now we do the computation. Set the M in (5.45) to be h_N. For an $m \geq M$, let $n \geq N$ denote the value for which

$$h_n \leq m < h_{n+1} = 2h_n + 1.$$

Though the choice of n depends on m, it suffices to show $\mu(T^m A \cap A) > 0$ for all m in $[h_n, h_{n+1})$.

Let $S_{n,k} = \bigcup_{i=1}^{h_k} S_{n+i}$, as in Figure 8. Suppose I is an accurate interval in C_n whose image $T^m I$ is disjoint from $S_{n,k}$. Then $T^m I$ appears in C_n as either a single or double crescents, as in Figure 11.

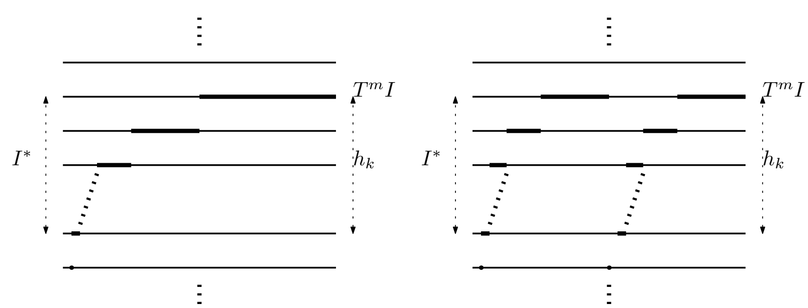
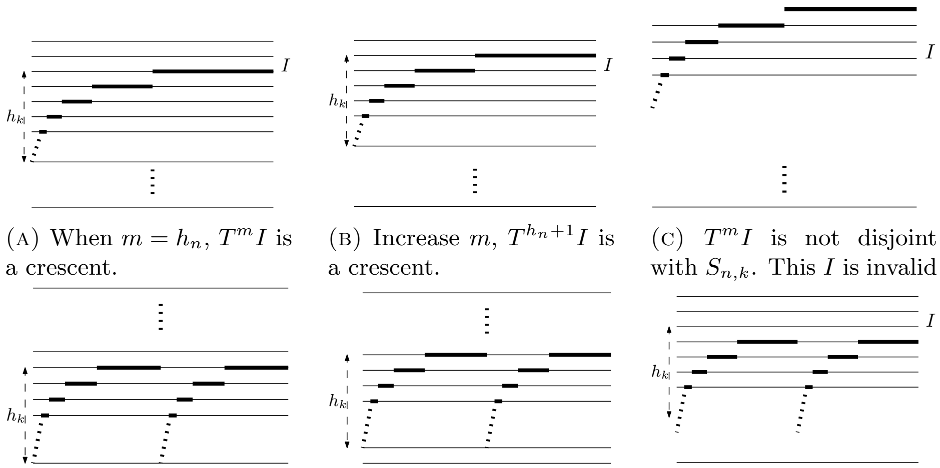


FIGURE 11. $T^m I$ is either single or a pair of crescents

Recall Remarks 5.38. We check the configuration of $T^m I$ for all values of m in $[h_n, h_{n+1})$. When $m = h_n$, $T^m I$ is a single crescent as in Figure 10. When m increases, $T^m I$ moves up. After the top level of $T^m I$ intersects with $S_{n,k}$, say, when $m = p$, this I becomes invalid. But after $m = p + h_k$, I becomes disjoint with $S_{n,k}$ again. Now the set $T^m I$ becomes a pair of crescents.



(A) When $m = h_n$, $T^m I$ is a crescent.

(B) Increase m, $T^{h_n+1} I$ is a crescent.

(C) $T^m I$ is not disjoint with $S_{n,k}$. This I is invalid

(D) $T^m I$ filters through $S_{n,k}$, I become valid again

(E) $T^m I$ is pair of crescents.

(F) T^{2h_n} is pair of crescents.

FIGURE 12. the pattern of $T^m I$ as m increases.

Notice that the accurate levels are at least h_k from each other and $S_{n,k}$ has h_k levels. So for any m, there can only be at most one accurate level that is invalid (as in Figure 12c).

Let I^* be the union of h_k many C_n-levels that contain the top h_k levels of $T^m I$ as in Figure 11. Thus I^* has the measure of a C_k-copy in C_n, which gives $\mu(I^*) = \frac{\mu(C_k)}{2^{n-k}}$. Let G_n^* be the union of all I^*. There is at most one I whose image fails to be disjoint from $S_{n,k}$. Thus,

$$\mu(G_n^*) > (\#[\text{accurate } C_n\text{-levels } I] \text{ - } 1)\frac{\mu(C_k)}{2^{n-k}} > 0.8 \quad \text{by (5.46)}.$$

This inequality, together with (5.47), shows $\mu(G_n \cap G_n^*) > 0$. Since $\mu(G_n \cap G_n^*) > 0$, there is an I^* and a *good* C_k-copy, call it D, such that the two intersect in a set of positive measure. There are two cases: either (i) the top level of I^* is below that of D, or (ii) the top level of I^* is the same or above that of D.

For case (i), let J denote the top level of the C_k-copy below D. For case (ii), let J denote the top level of D. In both cases, since I^* has height h_k and we required D to be preceded by at most 10 spacers, we know J is fewer than $h_k + 10$ levels away from the top level of I^*. Since I and J are both accurate,

$$\begin{aligned}
\mu(T^m(A) \cap A) &\geq \mu(T^m(A \cap I) \cap (A \cap J)) \\
&\geq \mu(T^m I \cap J) - (1-\alpha)\mu(I) - (1-\alpha)\mu(J) \\
&\geq \mu(J) \cdot ((\tfrac{1}{2})^{h_k+10} - 2(1-\alpha)) \qquad \text{by Lemma 5.39.}
\end{aligned}$$

This shows such m satisfies (5.45) □

6. DISCUSSION

We have discussed three different constructions of rank one maps. In fact, we see that as spacers take more importance in the construction of the columns, the map tends to mix "better."

First, we notice the hierarchy

(1) weakly mixing
(2) mildly mixing
(3) lightly mixing
(4) partially mixing
(5) mixing.

There are rank one constructions of maps that satisfy i but not $i+1$ for $i = 1, 2, 3, 4$.

Chacón's map is actually (2) but not (3). In fact, there are even weaker maps than Chacón's map, that is (2) but not (1), constructed by cutting each column into many subcolumns and stacking together without adding spacers. The concept of *mildly mixing* involves the definition of rigidity, thus is not mentioned in this paper.

The main idea of Chacón's map is that (i) T^{h_n} maps a fixed proportion of any level in C_n both to itself and to one level lower; (ii) the rightmost subinterval of the top-right level of C_n stays on the top-right after applying T^{h_m} for all $m > n$ because no spacer is added above the top-right. So there is no crescent created. In fact, it is easy to show that there are maps similar to Chacón's transformation that are only weakly mixing; for example, if we cut each column into 5 instead of 3 and put one spacer in the middle.

A crescent is created when a spacer is added above the top-right. Maps similar to FK's map would be expected if we cut each column into multiple subcolumns and add spacers on the top-right.

FK's map is (3) but not (4). We can construct even stronger rank one maps, but with more spacers. It is even possible to construct a rank one mixing map. In fact, Ornstein proved in [7] that a rank one map with a very large amount of spacers allocated randomly can be mixing. Apart from this stochastic construction, there is also the Smorodinsky Conjecture that states the existence of such a mixing map. M. Smorodinsky conjectured that by adding staircases whose heights increase consecutively by one, the resulting transformation (see Figure 13.) is mixing if $\lim_{n\to\infty} \frac{r_n^2}{h_n} = 0$. This conjecture is proved by Terrence Adams with a rank one example called the "staircases map," in [1].

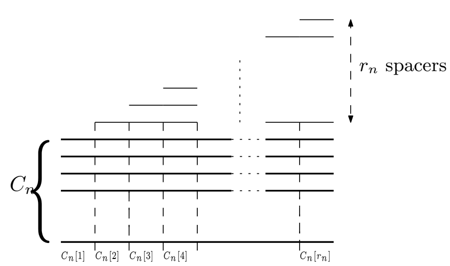


FIGURE 13. the staircase map

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