A Survey on Representation Theory

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Abstract

We present an overview on the principal facts and applications of unitary representations, and representations for finite and compact groups. While doing so, we use matrix coefficients rather than characters to deal with the proofs.

1 Introduction

We begin by introducing the following motivating example. It is a non-trivial fact that the Hilbert space $L^2(\mathbb{S}^2)$ has an orthonormal basis formed by eigenfunctions of the Laplacian on the sphere, which we will define in detail later. By now, we point out that proving this result is not easy because the space $L^2(\mathbb{S}^2)$ has infinite dimension. Indeed, if it were finite-dimensional, this would follow straightforward since the Laplacian is a self-adjoint operator.

This statement can be proven by using analysis, however we will present here an algebraic approach taking advantage of the sphere's symmetries. Indeed, any such symmetry R induces by the formula $(R \cdot f)(x) := f(R^{-1}x)$ a *linear* symmetry on $L^2(\mathbb{S}^2)$.

These symmetries play an important role because the Laplacian on the sphere commutes with them. From linear algebra, we know that if two operators commute, they keep invariant the eigenspaces of each other. Hence, it is natural trying to diagonalize the Laplacian on each of the eigenspaces of R. The problem is that these eigenspaces may be still too big.

To solve this problem, the main idea is considering those subspaces invariant by the whole group $\mathrm{SO}(3)$ of rotations of the sphere, rather than those invariant by a single rotation.

This idea motivates our study in the following exposition of *linear group actions* on vector spaces as well as the study of spaces invariant by the action. These considerations naturally come with the following basic definitions of *representation theory*.

Definition 1.1. Given a locally compact group G, a *complex representation* (V, π) is a homomorphism π of G into the group of linear automorphisms of the complex vector space V such that the resulting map $G \times V \to V$ given by $(g, v) \mapsto \pi(g) \cdot v$ is continuous.

Definition 1.2. A *morphism* between two representations (V_1, π_1) and (V_2, π_2), is a linear transformation $T : V_1 \to V_2$ such that $T(\pi_1(g) \cdot v) = \pi_2(g) \cdot T(v)$ for every g in G and v in V_1. The set of all morphisms from V_1 to V_2 is denoted by $\mathrm{Hom}_G(V_1, V_2)$.

The above defines the *category of representations* of the group G. The main objective of representation theory is to understand this category. This problem can be separated in two parts. First, we want to understand how representations can be built up from smaller ones, such that they are minimal. Second, we want to classify these minimal representations.

For instance, understanding the *direct sum* $V = V_1 \oplus V_2$ of two representations is entirely equivalent to understand each of the two smaller *invariant subspaces*. Hence, it turns out useful finding decompositions of this style for a given representation.

However, it may happen that V contains an invariant subspace V_1, with $V/V_1 =: V_2$, but there is no decomposition as before. The problem is that we cannot necessarily find a subrepresentation V_2, such that it provides a complement for V_1.

Before giving any concrete example, let us introduce adequate terminology in order to talk about these concepts in an accurate way.

Definition 1.3. A representation is called *irreducible*, if it does not have invariant subspaces under the action of the group. It is said *indecomposable*, if the vector space can not be written as the direct sum of two invariant subspaces under the action of G.

Observe that if a representation is irreducible, then by definition it is indecomposable. But, it is not true in general that an indecomposable representation is also an irreducible one.

Example 1.4. Suppose $G = \mathbb{Z}$. Observe that a representation of $\mathbb{Z}$ is completely defined by choosing an arbitrary transformation T which corresponds to the number 1.

Now, if the representation is finite-dimensional, we can consider the eigenvector associated to some eigenvalue for T. The span of this eigenvector is a subspace invariant by the representation. Therefore, all irreducible representations are one-dimensional.

However, it is not true that these irreducible invariant subspaces necessarily admit an invariant complement. If that were the situation, then clearly T would be diagonalizable, and we know this is not always true. Recall that the Jordan form encodes the structure of the invariant subspaces for a given transformation. Actually, the representation of $\mathbb{Z}$ associated to the operator T is indecomposable if and only if T is made up of only one Jordan block.

Above, we relied on the existence of eigenvalues and eigenvectors for finite-dimensional vector spaces. In arbitrary Hilbert spaces, we can use instead the following version of the spectral theorem: every normal operator in a Hilbert space is isomorphic to a multiplication operator $T_f : L^2(X) \to L^2(X)$, given by $T_f(g) = fg$, where f belongs to $L^\infty(X)$. Observe that in particular this completely classifies unitary representations of $\mathbb{Z}$.

Throughout this exposition, we will show how representation theory can be used to solve the following problems.

1. Show that any continuous map φ from $\mathbb{R}$ into $\mathrm{GL}_n(\mathbb{C})$ such that $\varphi(x+y) = \varphi(x) \cdot \varphi(y)$ is given by $\varphi(t) = e^{tA}$, where A is a matrix.

2. Up to conjugation, determine how many pairs of matrices A, B belonging to $\mathrm{GL}_6(\mathbb{C})$ are there such that they satisfy the following: $A^2 = B^2 = (AB)^3 = Id$.

3. Compute the spectrum of the Laplacian operator acting on the two dimensional sphere.

2 Unitary representations

In the following, G will always denote a locally compact group with Haar measure μ that is invariant under left translations, but not necessarily under right translations.

It will be useful to restrict our study to *unitary* representations, that is, representations where the group action is also required to respect the inner product of the vector space.

The objective of this section is to explain generalities for this kind of representations. In particular we will present two of the most fundamental theorems in representation theory, which are Schur's lemma and Frobenius' Reciprocity.

Definition 2.1. Given a locally compact group G, an *unitary representation* (V, π) is a homomorphism π of G into the group of linear unitary automorphisms of a complex Hilbert space V such that the resulting map $G \times V \to V$ given by $(g, v) \mapsto \pi(g) \cdot v$ is continuous.

The following proposition presents a couple of facts that are not true for general representations, but they do work when the representation is unitary.

Proposition 2.2. *If (V, π) is an unitary representation, then*

1. *The orthogonal complement of any invariant subspace of V is invariant. In particular, indecomposable representations are irreducible.*

2. *If T is an operator on V that commutes with the action of G, then its adjoint T^* also commutes with the action of G.*

Proof. For the first part, it suffices to observe that if a subspace is invariant under some linear transformations, its orthogonal complement is invariant under the adjoint operators. Since operators associated to elements of the group are unitary, their adjoint is the inverse, and therefore the orthogonal complement is also an invariant subspace.

For the second it is enough to observe the following equality

$$\langle u, T^*(\pi(g) \cdot v)\rangle = \langle \pi(g^{-1}) \cdot T(u), v\rangle = \langle T(\pi(g^{-1}) \cdot u), v\rangle = \langle u, \pi(g) \cdot T^*(v)\rangle$$

which implies the desired result. □

Observe that, any finite-dimensional representation can be written as the direct sum of irreducible representations. Indeed, if the representation is irreducible, we are done. If it is not, it can be written as the direct sum of an invariant subspace, and its (also invariant) orthogonal complement. Proceed recursively on each non-irreducible component and by the finite-dimensional assumption, the process has to finish.

Definition 2.3. Let G be a group and (W, ϕ) be a representation of a closed subgroup H. We define the *induction* of W from H to G as the representation

$$\operatorname{Ind}_H^G W := \{f : G \to W \text{ such that } f(gh) = \phi(h)^{-1} \cdot f(g) \text{ if } h \in H,\ \|f\|_2 < \infty\}$$

endowed with the action of G given by $(g \cdot f)(x) = f(g^{-1}x)$.

Example 2.4. Take $W = \mathbb{C}$ with the trivial action of H. The vector space will be given by all functions $f : G \to \mathbb{C}$ such that $f(gh) = h^{-1} \cdot f(g) = f(g)$ and $\|f\|_2 < \infty$. Since the values of the function only depend on the coset of the argument, then the vector space is $L^2(G/H)$. In particular, if H is the trivial group, then $\operatorname{Ind}_H^G \mathbb{C} = L^2(G)$.

Theorem 2.5. *(Frobenius reciprocity) Let G be a group with a closed subgroup H. For every unitary representation (V, π) of G and for every unitary representation (W, ϕ) of H, we have:*

$$Hom_G(V, Ind_H^G\, W) = Hom_H(Res\ V, W)$$

Where Res V denotes the representation of H given by restricting the representation of G.

Proof. First, suppose that $\Phi \in \operatorname{Hom}_G(V, \operatorname{Ind}_H^G W)$. Consider $\Psi : V \to W$ given by

$$\Psi(v) := \Phi(v)(e)$$

Where e is the identity element in G. It is straightforward to check that $\Psi \in \operatorname{Hom}_H(\operatorname{Res} V, W)$.

On the other hand, given $\Psi \in \operatorname{Hom}_H(\operatorname{Res} V, W)$, consider $\Phi : V \to \operatorname{Ind}_H^G W$ by

$$\Phi(v)(g) := \Psi(g^{-1} \cdot v)$$

It is also easy to check that $\Phi(v) \in \operatorname{Ind}_H^G W$ and that $\Phi \in \operatorname{Hom}_G(V, \operatorname{Ind}_H^G W)$.

Finally, since clearly $\Phi(v)(g) = \Phi(g^{-1} \cdot v)(e)$, we obtain that these processes are inverse to each other, and we conclude the proof. □

Let us point out an interesting interpretation of Frobenius Reciprocity. Note that Res is a functor from the category of representations of G into the category of representations of H. What Frobenius Reciprocity actually states is that Ind is the right adjoint of Res.

Now, we define the functor ind, which under some conditions, is the left adjoint of Res. In general, there exists another definition for ind such that the analogous reciprocity holds.

Definition 2.6. Let G be a group and (W, ϕ) be a representation of a closed subgroup H. We define the *compact induction* of W from H to G as the representation

$$\operatorname{ind}_H^G W := \{f : G \to W \text{ such that } f \in \operatorname{Ind}_H^G W \text{ and } \operatorname{supp}(f) \text{ is compact mod } H\}$$

endowed with the action of G given by $(g \cdot f)(x) = f(g^{-1}x)$.

The following theorem can be stated more generally, however, we present a proof when G/H is compact. Also, observe that if G/H is compact, then ind and Ind coincide

Theorem 2.7. *Let G be a group with a closed subgroup H such that G/H is compact. For every unitary representation (V, π) of G and for every unitary representation (W, ϕ) of H:*

$$Hom_G(ind_H^G\, W, V) = Hom_H(W, Res\ V).$$

Proof. First, let us point out that there exist a copy of W inside $\operatorname{ind}_H^G W$. For every $w \in W$, consider $e_w \in \operatorname{ind}_H^G W$ given by $e_w(h) = \phi(h^{-1})w$ if $h \in H$ and $e_w(g) = 0$ otherwise.

This identification leads to a correspondence between the sets. The idea is that maps from $\operatorname{ind}_H^G W$ to V can be restricted to maps from the copy of W into V. For the other direction, the idea is to extend maps such that these processes are inverse to each other.

If $\Phi \in \mathrm{Hom}_G(\mathrm{ind}_H^G\, W, V)$, define $\Psi : W \to V$ by

$$\Psi(w) = \Phi(e_w).$$

Straight calculations show that $h \cdot \Psi(w) = \Psi(h \cdot w)$, and therefore $\Psi \in \mathrm{Hom}_H(W, \mathrm{Res}_H V)$.

For the other direction let $\Psi \in \mathrm{Hom}_H(W, \mathrm{Res}_H V)$ and take $f \in \mathrm{ind}_H^G W$. Consider the linear map $\Phi : ind_H^G W \to V$ given by

$$\Phi(f) = \int_{G/H} \pi(g)\Psi(f(g))\, dg.$$

It is straightforward to check that Φ is a morphism. From the construction, it follows that the processes are inverse to each other and the result follows. □

The next result is Schur's lemma which is very important in order to study the possible maps between irreducible unitary representations. The first part is true for any representation. The second part of this result is also true if the representation is not unitary, but the irreducible representations are finite-dimensional.

Theorem 2.8. *(Schur's lemma): Suppose (V_1, π_1) and (V_2, π_2) are two irreducible unitary representations of the group G. Let T be a operator between these two representations that commutes with the action of the group G.*

1. *If T is not null, then it is an isomorphism.*
2. *If $(V_1, \pi_1) = (V_2, \pi_2)$, T is a scalar map.*

Proof. For the first part, it suffices to observe that $\ker T$ and $\mathrm{im}\ T$ are invariant subspaces inside V_1 and V_2, therefore if T is either not injective or not surjective, it is the null operator.

For the second part define $(V, \pi) := (V_1, \pi_1)$. First, we prove the result when V is a finite-dimensional vector space. Since the underlying field is $\mathbb{C}$, T must have an eigenvalue λ. The operator $T - \lambda Id$ clearly commutes with the action of the group and has non-trivial kernel, then from the first part we obtain that it is null, that is: $T = \lambda Id$.

Generalizing this to infinite dimensional vector spaces is more difficult since there is no way to obtain eigenvalues. However, we can do the following. First we know that T^* commutes with the action of G, and since it is not null, by the first part is an isomorphism on (V, π). Now, consider the self adjoint operators $M = (T + T^*)/2$ and $N = (T - T^*)/2i$, which also commute with the action of G. If M and N were compact operators, then by the Spectral Theorem, we can find eigenvalues and therefore apply the same proof as before.

The general case can be proven using a more general version of the Spectral Theorem. Actually, it does not allow to take eigenvalues for M and N, but, assuming that one of them is not scalar, it guarantees the existence of a non-scalar orthogonal projection that commutes with the action of the group, and this contradicts the first part of the statement. □

Notice that this result is a classification theorem of morphisms between representations. Indeed, what it actually states is that, if V_1 and V_2 are irreducible representations of G, then the set $\mathrm{Hom}_G(V_1, V_2)$ is $\mathbb{C}$ or null depending on whether or not V_1 and V_2 are isomorphic.

Example 2.9. One of the first immediate consequences from Schur's lemma is that any irreducible representation of an abelian group is one-dimensional. Indeed, observe that if a group is abelian, then every element commutes with the action of the group, and by Schur's lemma it must be a scalar automorphism of the vector space. In particular every single one-dimensional subspace is invariant under the action of every element of the group, which implies that irreducible representations can only be one-dimensional.

In particular, this explains the following classical result. If there is a collection of linear transformations on a finite-dimensional vector space V which commute with each other, then there exists a basis of the vector space where all transformations are upper triangular matrices.

In fact, consider the group generated by these matrices. This group is abelian, and therefore every irreducible representation is one-dimensional. Then there must be a vector invariant by the action of all these transformations. Let us call this vector v_1. Since the group also acts on $V/\langle v_1\rangle$, we may iterate this argument so that we finish with a collection of vectors v_1, v_2,...; which constitute a basis for the space V. Furthermore if we represent these transformations as matrices with respect to this basis, they all result being upper triangular.

Example 2.10. Let us finish by considering the problem of the introduction: Show that any continuous map φ from $\mathbb{R}$ into $\mathrm{GL}_n(\mathbb{C})$ such that $\varphi(x+y) = \varphi(x)\cdot\varphi(y)$ is given by $\varphi(t) = e^{tA}$.

Such φ is, by definition, a representation of $\mathbb{R}$. Since $\mathbb{R}$ is an abelian group, the previous example implies the existence of a basis of $\mathbb{C}^n$ such that the image of φ consists of lower triangular matrices under this basis. On the other hand, since exponentiating a matrix made up of blocks is equivalent to exponentiate each block, it suffices to show the problem for indecomposable representations of $\mathbb{R}$.

Since these matrices commute, they all preserve the generalized eigenspaces of the others. In particular, if one matrix has two distinct eigenvalues, then the whole representation is decomposable. We conclude that, every single matrix has only one eigenvalue, and that is the only value that appears in its diagonal entries.

Now, we claim that we can assume without loss of generality that all these matrices have 1s in the diagonal. Sending each matrix to its eigenvalue gives us a continuous homomorphism from $\mathbb{R}$ into $\mathbb{R}^*$, therefore the eigenvalue of the matrix associated to r is $e^{r\lambda}$. If we multiply each matrix $\varphi(r)$ by the diagonal matrix $e^{-r\lambda}Id$, the claim follows.

It is straightforward to check that the function $A \mapsto e^A$ is a continuous bijection from lower triangular matrices with 0's in the diagonal to lower triangular matrices with 1s there. Hence, taking logarithms is well defined, and doing so in the initial equality, we can conclude the stated result for rational numbers and by continuity, also for real numbers.

3 Finite groups

In this section, G will always denote a finite group. Finite groups are the most well behaved groups in representation theory, in part due to the existence of an average operation.

For instance every representation of a finite group is "unitarizable", that is, the space can be equipped with an inner product such that the representation is unitary. Indeed, suppose that the group G is finite and (V, π) is a representation, with an arbitrary inner product $\langle\cdot,\cdot\rangle$. We can construct an invariant one by the formula $\langle u, v\rangle_{\text{new}} = \sum\langle g\cdot u, g\cdot v\rangle$.

The following statement asserts that representations of finite groups are made up of irreducible representations. Observe that in particular, it proves that irreducible representations for finite groups are finite-dimensional.

Theorem 3.1. *Any representation (V, π) of a finite group G is decomposable as the orthogonal direct sum of irreducible finite-dimensional representations.*

Proof. Recall that we can assume the representation to be unitary. By Zorn's lemma, consider a subspace W which is maximal with respect to the property of being direct sum of irreducible finite-dimensional subspaces. Since the representation is unitary, it follows that $W^\perp$ is also an invariant subspace. If $W^\perp$ is not trivial, then consider the orbit of a vector $w^\perp$ in $W^\perp$. Since G is finite, the span of this orbit is a finite-dimensional invariant subspace and this contradicts the maximality assumption. □

Definition 3.2. For a finite group G, we define the *regular representation* as the vector space of functions $\mathbb{C}[G] := \{f : G \to \mathbb{C}\}$ with action $(g\cdot f)(x) = f(g^{-1}x)$.

The regular representation has the natural inner product defined by $f_1 \cdot f_2 = \sum f_1(g)\overline{f_2(g)}$. Moreover, under this inner product $\mathbb{C}[G]$ is an unitary representation.

Definition 3.3. Let (V, π) be a representation of a finite group G. For any $\varphi \in V^*$, $v \in V$ we call the function $f(g) = \varphi(g\cdot v)$ a *matrix coefficient* of the representation. Further, we define the *character* of the representation by the function $\chi(g) = \mathrm{tr}(\pi(g))$.

When a representation has finite dimension, a matrix coefficient can be thought of as a fixed entry of the matrices which represent the group's elements. Indeed, because of the morphism, the group action can be realized as a subgroup of matrices. Then, if φ is the projection on the i-th entry, and v is the j-th vector of the space basis, the respective matrix coefficient gives us the ij-th entry of the matrix for each element of the group.

Given a finite-dimensional representation (V, π), with orthonormal basis $\{v_1, v_2, ..., v_n\}$ and dual basis $\{\varphi_1, \varphi_2, ..., \varphi_n\}$, define $f^\pi_{ij}(g)$ to be the matrix coefficient of v_i and φ_j. From the definition, the character of a representation is given by the formula $\chi^\pi = \sum f^\pi_{ii}$.

Theorem 3.4. *Every irreducible representation occurs as many times as its dimension in the decomposition of the regular representation as direct sum of irreducible representations. Moreover the matrix coefficients f^π_{ij}, where π runs over all non-isomorphic irreducible representations (V, π) of G, form an orthogonal basis of $\mathbb{C}[G]$.*

Proof. Let V be an irreducible representation and let $\bigoplus V_i$ be the decomposition of $\mathbb{C}[G]$ as direct sum of irreducible representations. If G is finite, $L^2(G) = \mathbb{C}[G]$, and in section two we said that $L^2(G)$ is the induction of the trivial representation for the trivial subgroup. Therefore, by Frobenius, we have:

$$\bigoplus \mathrm{Hom}_G(V, V_i) = \mathrm{Hom}_G(V, \mathbb{C}[G]) = \mathrm{Hom}(V, \mathbb{C}) = V^*.$$

On the other hand, by Schur's lemma $\mathrm{Hom}_G(V_i, V)$ is $\mathbb{C}$ or null depending on whether or not each V_i is isomorphic to V. Taking dimensions to both sides, we obtain that $\dim V$ equals the number of V_i's isomorphic to V, as desired.

Next, we prove that matrix coefficients of non-isomorphic representations are orthogonal. Let $\varphi_1(g \cdot v_1)$ and $\varphi_2(g \cdot v_2)$ be matrix coefficients of two non-isomorphic irreducible representations V_1 and V_2. Define $L : V_2^* \to V_1$ by $L(\varphi) = \sum \overline{\varphi(g \cdot v_2)} g \cdot v_1$. Observe that

$$h \cdot L(\varphi) = \sum_{g \in G} \overline{\varphi_2(g \cdot v_2)}(hg) \cdot v_1 = \sum_{g \in G} \overline{\varphi_2(g \cdot v_2)}(hg) \cdot v_1 = L(h \cdot \varphi).$$

Hence, L commutes with the group action and since V_1 and V_2^* are not isomorphic, Schur's lemma implies that L is the null map. In particular, we have the following equality

$$\sum_{g \in G} \varphi_1(g \cdot v_1)\overline{\varphi_2(g \cdot v_2)} = \varphi_1(\sum_{g \in G} g \cdot \overline{(\varphi_2(g \cdot v_2))} v_1) = \varphi_1(L(\varphi_2)) = 0.$$

An analogous computation shows that functions $f_{ij}^{\pi_l}$ are also orthogonal as we run over different values of the parameters i and j, for a constant value of l.

Suppose that $V_1, \ldots, V_k$ have dimensions $n_1, \ldots, n_k$ respectively. By considering the functions $f_{ij}^{\pi_l}$ we obtain $n_1^2 + ... + n_k^2$ linearly independent matrix coefficients. Furthermore, the first part implies that the dimension of $\mathbb{C}[G]$ is $n_1^2 + ... + n_k^2$ and we conclude the desired. □

Observe that matrix coefficients give us a map from $V^* \otimes V$ to $\mathbb{C}[G]$, which sends an element $\varphi \otimes v$ to the function $\varphi(g \cdot v)$. The above implies that this map is injective if V is irreducible and moreover, that the regular representation decomposes as $\bigoplus V_i^* \otimes V_i$, where V_i runs over all non-isomorphic irreducible representations of our finite group G.

If we endow $\mathbb{C}[G]$ with a product given by the *convolution* of functions, we obtain an algebra. Explicitly, if f_1, f_2 belong to $\mathbb{C}[G]$, then

$$(f_1 * f_2)(x) := \sum_{g \in G} f_1(g) f_2(g^{-1}x).$$

Observe that the action of G on V induces an action of $\mathbb{C}[G]$ on V as algebra, that is, it induces a homomorphism from $\mathbb{C}[G]$ to $\mathrm{End}(V)$. The corresponding action is

$$f \cdot v = \sum_{g \in G} f(g) g \cdot v.$$

It is easy to check that $(f_1 + f_2) \cdot v = f_1 \cdot v + f_2 \cdot v$ and also $(f_1 * f_2) \cdot v = f_1 \cdot (f_2 \cdot v)$. It follows that V is a $\mathbb{C}[G]$-module. On the other hand, given a $\mathbb{C}[G]$-module, we can restrict the morphism from $\mathbb{C}[G]$ into $\mathrm{End}(V)$ to G. Indeed, $\mathbb{C}[G]$ has a copy of G inside given by the functions which map one element to 1 and any other to 0. In particular, from this we observe that giving a representation of G is the same as giving a $\mathbb{C}[G]$-module.

Then, inducing a representation, is the same as extending scalars from $\mathbb{C}[H]$ to a $\mathbb{C}[G]$. It is also clear that Frobenius translates into the usual adjunction of the tensor product.

Now, we claim that matrix coefficients for an irreducible representation (V, π) act trivially on every non-isomorphic-to-V representation (W, ϕ). Indeed, given $\varphi \in W^*$ and $v \in V$, consider the map $w \mapsto \sum \varphi(g \cdot w) g \cdot v$ from W to V. It is clearly a morphism of representations and hence by Schur's lemma it is the null map. But this is precisely the action of a general matrix coefficient for W, which proves the claim.

We conclude that the morphism constructed above from $\mathbb{C}[G]$ to $\mathrm{End}(V)$ is not null only in the component $V^* \otimes V$ of $\mathbb{C}[G]$. It is also easy to check that the functions f_{ij}^{π} are mapped to elementary matrices in $\mathrm{End}(V)$. We finally obtain the following result

Corollary 3.5. *We have the following isomorphisms of algebras*

$$\mathbb{C}[G] \cong \bigoplus V_i^* \otimes V_i \cong \bigoplus End(V_i)$$

Where V_i runs over all the irreducible non-isomorphic representations of G.

From the study of these equivalences we conclude the following corollary, which is a handy device to compute the possible irreducible representations for a given group.

Corollary 3.6. *If G is a finite group and n_1, n_2,..., n_k are the dimensions of all its non-isomorphic irreducible representations, then*

$$|G| = n_1^2 + n_2^2 + ... + n_k^2$$

and k equals the number of conjugacy classes of the group G.

Proof. The first part is immediate by taking dimension to both sides in the previous corollary. If we instead take the center, we obtain $Z(\mathbb{C}[G]) = \bigoplus Z(\mathrm{End}(V_i))$. On the other hand, since in general $Z(\mathrm{End}(V)) = \mathbb{C}$, then the dimension of the center of $\mathbb{C}[G]$ is k.

The center of $\mathbb{C}[G]$ consists of all functions which do not depend on the conjugacy class. Then $\dim Z(\mathbb{C}[G])$ equals the number of different conjugacy classes of G, as desired. □

Example 3.7. Consider the problem of the introduction. Up to conjugation, determine how many pairs of matrices A, B belonging to $\mathrm{GL}_6(\mathbb{C})$ satisfy $A^2 = B^2 = (AB)^3 = Id$.

Observe that giving such pair of matrices is the same as giving a representation of $\mathbb{S}_3$ in $\mathbb{C}^6$. Indeed, a way to characterize $\mathbb{S}_3$ is by the property of being the group with two generators, namely a and b, such that $a^2 = b^2 = (ab)^3 = 1$. On the other hand, the condition "up to conjugacy" is equivalent to considering two representations equal if they are isomorphic.

Now since $\mathbb{S}_3$ has three different conjugacy classes, by corollary 3.6, there must exist three non-isomorphic irreducible representations of $\mathbb{S}_3$, called V_1, V_2 and V_3, and if they have dimensions n_1, n_2 and n_3, they must satisfy $n_1^2 + n_2^2 + n_3^2 = 6$. Observe that $\mathbb{S}_3$ admits the trivial one-dimensional representation, and also admits the one-dimensional representation given by $a \cdot x = b \cdot x = -1$. This implies $n_1 = 1$, $n_2 = 1$ and therefore $n_3 = 2$.

Then, theorem 3.1 applied to the considered representation gives:

$$\mathbb{C}^6 = V_1^{i_1} \oplus V_2^{i_2} \oplus V_3^{i_3}$$

where $i_1 + i_2 + 2i_3 = 6$. Now, if $i_3 = 3$, then $i_1 = i_2 = 0$. If $i_3 = 2$, then there are three possibilities. If instead we put $i_3 = 1$, there are five possibilities, and if $i_3 = 0$, there are seven. We conclude that there exist $1 + 3 + 5 + 7 = 16$ such pairs of matrices A and B.

4 Compact groups

In this section, G denotes a compact group with Haar measure μ, so that $\mu(G) = 1$. Compact groups resemble finite groups in several aspects, in particular many of the theorems which were true for finite group representations carry over to the compact case.

For example, replacing "sums" by "integrals" we obtain that, for compact groups, any representation is unitarizable. In particular, since we can restrict our study to unitary representations, irreducible and idecomposable representations coincide.

Theorem 3.1 also carry over to the compact case. Notice that its proof was simply pointing out that the orbit of any vector under the action of the group in finite, here those orbits are possibly non-countable and for that reason we require a more elaborated approach.

The statement would be immediate if we knew that any irreducible representation is finite-dimensional. However, we first need to prove this result for the regular representation in order to prove it in general, as we shall see in theorem 4.4 below.

Theorem 4.1. *(Peter-Weyl) Any representation (V, π) of a compact group G is decomposable as the orthogonal direct sum of irreducible finite-dimensional representations.*

Next we present the regular representation for compact groups. Similar to that defined in the previous section, it has an important role for studying irreducible representations.

Definition 4.2. For a compact group G, we define the *regular representation* as the vector space of functions $L^2(G) := \{f : G \to \mathbb{C} \text{ s.t } \|f\|_2 < \infty\}$ with the action $(g \cdot f)(x) = f(g^{-1}x)$.

We extend the inner product defined for finite representations by $\int f_1(x)\overline{f_2(x)}\,dx$. The integrability assumption inside the previous definition guarantees the convergence of the integral. Moreover, the regular representation is unitary under this inner product.

Definition 4.3. Let (V, π) be a representation of a compact group G. For any $\varphi \in V^*$, $v \in V$ we call the function $f(g) = \varphi(g \cdot v)$ a *matrix coefficient* of the representation.

As before, given a representation (V, π), with orthonormal basis $\{v_1, v_2, \dots\}$ and dual basis $\{\varphi_1, \varphi_2, \dots\}$, we also define $f^{\pi}_{ij}(g)$ to be the matrix coefficient associated to v_i and φ_j.

Theorem 4.4. *(Peter-Weyl) Every irreducible representation occurs as many times as its dimension in the decomposition of $L^2(G)$ as direct sum of irreducible representations. Moreover the matrix coefficients f^{π}_{ij}, where π runs over the set of all non-isomorphic irreducible representations (V, π) of G, form an orthogonal basis of $L^2(G)$.*

Proof. The heart of the proof relies on proving that the span of the matrix coefficients for finite-dimensional representations is a dense subspace. Let us just pointing out that if we were working with compact groups of matrices, these result would follow from the Stone-Weiertstrass theorem, by considering the algebra structure of the matrix coefficients.

Let us proceed by contradiction. We need to find a continuous non-zero function F in $U^{\perp}$ such that $F(yxy^{-1}) = F(x)$ and $F(x) = \overline{F(x^{-1})}$. Indeed, assume we can find such function, then we can consider the operator

$$Tf(x) = \int_G F(x^{-1}y)f(y)\,dy.$$

It turns out from the continuity of F that this is a non-zero Hilbert-Schmidt operator, and then it has an eigenvalue λ and a finite-dimensional eigenspace V associated to λ. This space is invariant under the action of the regular representation, and hence, it has a finite-dimensional irreducible subspace W. From the definition, and the properties of F, it is not hard to check that the product of F with a non-zero matrix coefficient of W is non-zero. This would lead to a contradiction since F belongs to $U^{\perp}$.

Then, let us construct the function F. Let I_N be the characteristic function of a neighborhood N of 1 in G, and H a function in $U^{\perp}$. Observe that the functions

$$\frac{1}{|N|}\int_G I_N(y)H(y^{-1}x)\,dy$$

are continuous, tend to H as N collapses to 1 and belong to $U^{\perp}$. In particular, some of them must be non-zero functions, taking one, we obtain a non-zero continuous function F_1 in $U^{\perp}$. Without loss of generality, we can suppose that $F_1(1)$ is a non-zero real number. Now, set

$$F_2(x) = \int_G F_1(yxy^{-1})\,dy \text{ and } F(x) = F_2(x) + \overline{F_2(x^{-1})}$$

It is straightforward that F satisfies the required conditions.

On the other hand, as we saw for finite groups, matrix coefficients for an irreducible representation V span a representation $V^* \otimes V$, that is also the direct sum of V as many times as its dimension. Moreover, the same way as for finite groups, the matrix coefficients f_{ij}^{π} form an orthogonal set, and this concludes the proof. □

The previous proof implies that the regular representation splits as the orthogonal direct sum of irreducible finite-dimensional representations. Then, if V is an irreducible infinite-dimensional representation, then $\mathrm{Hom}_G(V, L^2(G)) = 0$; but, by Frobenius and example 2.4

$$\mathrm{Hom}_G(V, L^2(G)) = \mathrm{Hom}_G(V, \mathrm{Ind}_e^G \mathbb{C}) = \mathrm{Hom}(V, \mathbb{C}) = V.$$

Then any irreducible representation is finite-dimensional, which proves theorem 4.1.

Now, recall that for every group G, the space $L^1(G)$ with the convolution of functions is naturally an algebra. Explicitly, if f_1, f_2 belong to $L^1(G)$, we define their convolution by

$$(f_1 * f_2)(x) := \int_G f_1(g) f_2(g^{-1}x)\,dg.$$

If the group G is compact, we know that $L^2(G) \subseteq L^1(G)$ and therefore, by Young inequality, the convolution of two functions in $L^2(G)$ is again in $L^2(G)$. Therefore, the space $L^2(G)$ is also an algebra with the convolution.

In addition, the action of G on the space V can be extended to an action of $L^1(G)$ by

$$f \cdot v = \int_G f(g) g \cdot v\,dg.$$

Finally, using exactly the same argument of the previous section, we conclude the following

Corollary 4.5. *(Peter-Weyl) We have the following isomorphisms of algebras*

$$L^2(G) \cong \bigoplus V_i \otimes V_i^* \cong \bigoplus End(V_i)$$

where V_i runs over all irreducible non isomorphic representations of G.

Example 4.6. Consider $G = \mathrm{SO}(2) = \mathbb{S}^1$ the group of rotations of the one-dimensional sphere. Since the group SO(2) is abelian, any of its irreducible representations is one-dimensional. Then, irreducible representations are defined by continuous homomorphisms $\rho : \mathbb{S}^1 \to \mathbb{C}$.

It is easy to check that all irreducible representations of SO(2) are in bijection with the integer numbers, and they are given by the morphisms $e^{i\theta} \mapsto e^{in\theta}$. Denote by $L(n)$ the irreducible representation of SO(2) associated to the map $e^{i\theta} \mapsto e^{in\theta}$, this map is also the only matrix coefficient of $L(n)$. In particular, observe that what Peter-Weyl theorem states in this case is what we already know from the theory of Fourier series: the space $L^2(\mathbb{S}^1)$ can be written as the orthogonal direct sum of exponential functions.

Example 4.7. Set $G = \mathrm{SO}(3)$, the compact group of all rotations of the two-dimensional sphere. As before, it turns out that the irreducible representations of SO(3) are classified.

Set an embedding from SO(2) to SO(3). Then, any representation V of SO(3) is a representation of SO(2). By the above example and Peter-Weyl theorem V splits as the orthogonal direct sum of some $L(k)$. Let us define the maximal weight of the representation as the maximal k such that $L(k)$ appears in the decomposition of V.

It can be shown that, for every natural number n, there exists a unique irreducible representation H_n of maximal weight n such that

$$H_n = L(-n) \oplus \cdots \oplus L(0) \oplus \cdots \oplus L(n).$$

In particular, $\dim H_n = 2n + 1$, and the subspace of H_n fixed by the action of SO(2) is $L(0)$, that has dimension 1. Moreover, these are all the irreducible representations of SO(3).

Observe that we could embed SO(2) in SO(3) as the stabilizer of the vector $(0, 0, 1)$. Since the action of SO(3) is transitive, it follows that $\mathbb{S}^2 \cong \mathrm{SO}(3)/\mathrm{SO}(2)$. On the other hand, as we said in the introduction, the group SO(3) acts linearly on $L^2(\mathbb{S}^2)$, and therefore, $L^2(\mathbb{S}^2)$ is a representation of SO(3). Moreover, by the example 2.4, we obtain

$$L^2(\mathbb{S}^2) = L^2(\mathrm{SO}(3)/\mathrm{SO}(2)) = \mathrm{Ind}_{\mathrm{SO}(2)}^{\mathrm{SO}(3)}\mathbb{C}.$$

Example 4.8. Finally, we return to the problem about the Laplacian on the sphere that we stated in the introduction. Explicitly, the Laplacian is defined by the formula $\Delta_{\mathbb{S}^2}(f) = \Delta(\tilde{f})$, where $\tilde{f}(x) = f(x/\|x\|)$ for every non-zero vector on the space.

By Peter-Weyl, $L^2(\mathbb{S}^2) = \bigoplus V_i$, where the V_i's are irreducible representations of SO(3). We want to determine explicitly which these V_i's are. This is equivalent to compute how many times each representation H_n occurs in the decomposition of $L^2(\mathbb{S}^2)$. By Schur's lemma

$$\mathrm{Hom}_{\mathrm{SO}(3)}(H_n, L^2(\mathbb{S}^2)) = \mathrm{Hom}_{\mathrm{SO}(3)}(H_n, \oplus V_i) = \mathbb{C}^m$$

where m is the number of V_i's that are isomorphic to H_n. Now, by Frobenius

$$\mathrm{Hom}_{\mathrm{SO}(3)}(H_n, L^2(\mathbb{S}^2)) = \mathrm{Hom}_{\mathrm{SO}(3)}(H_n, \mathrm{Ind}_{\mathrm{SO}(2)}^{\mathrm{SO}(3)}\mathbb{C}) = \mathrm{Hom}_{\mathrm{SO}(2)}(\mathrm{Res}H_n, \mathbb{C}) = H_n^{\mathrm{SO}(2)}.$$

Finally, since $\dim H_n^{\mathrm{SO}(2)} = 1$, we conclude that $m = 1$, which implies that every irreducible representation of SO(3) occurs exactly once on the decomposition of $L^2(\mathbb{S}^2)$. In other words

$$L^2(\mathbb{S}^2) = \bigoplus_{n \in \mathbb{N}} H_n.$$

Now, recall that $\Delta_{\mathbb{S}^2}$ commutes with the action of SO(3), and therefore, it is a morphism of representations. By Schur's lemma $\Delta_{\mathbb{S}^2}$ must send irreducible subspaces to isomorphic ones, but there is only one copy of H_n inside $L^2(\mathbb{S}^2)$! Therefore the Laplacian must keep invariant the spaces H_n, and moreover, again by Schur's lemma, it acts as a scalar map. In particular choosing an orthonormal basis for each H_n gives us the desired eigenfunctions.

On the other hand, although this proof implies the statement, it does not allow us to compute either the eigenvalues or the eigenvectors for the Laplacian.

In order to compute that, define by P_n the space of homogeneous polynomials of degree n in three variables. it is easy to check that, $\dim P_n = (n+2)(n+1)/2$.

Now, observe that the usual Laplacian satisfies that $\Delta : P_n \to P_{n-2}$ is surjective. Let $\tilde{H}_n$ be *the harmonic polynomials of degree* n, which are defined as the kernel of this map. By the first isomorphism theorem, $P_{n-2} \cong P_n/\tilde{H}_n$, then $\dim \tilde{H}_n = \dim P_n - \dim P_{n-2} = 2n+1$.

It turns out that $\tilde{H}_n$ is an invariant subspace under the action of SO(3). Moreover, it can be shown that the only fixed vectors of $\tilde{H}_n$ under the action of SO(2) are scalar multiples of the Legendre polynomial $d^n/dz^n(1-z^2)^n$. Since any irreducible representation of SO(3) has exactly one subspace fixed by the action of SO(2), we conclude that $\tilde{H}_n$ is irreducible, and by the previous example, it must be the space H_n.

In particular $\Delta_{\mathbb{S}^2}$ acts diagonally in $\tilde{H}_n$, so, computing the eigenvector of the Laplacian on this space is equivalent to compute the factor of scaling for the Laplacian on any vector there. In particular, if we do so with the Legendre polynomial the required eigenvalue is $-n(n+1)$ and because of the above, it has multiplicity $2n+1$. This solves the stated problem.

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