DOMINO TILING

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Abstract. In this paper we explore the problem of domino tiling: tessellating a region with 1x2 rectangular dominoes. First we address the question of existence for domino tilings of rectangular grids. Then we count the number of possible domino tilings when one exists.

Contents

1. Introduction

Definition 1.1. (Domino) A domino is a rectangle formed by connecting two unit squares along an edge.

Definition 1.2. (Domino Tiling) A domino tiling is a covering of a grid using dominoes such that all dominoes are disjoint and contained inside the boundary of the grid.

Tilings were originally studied in statistical mechanics as a model for molecules on a lattice. Our dominoes become equivalent to dimers, which are two molecules connected by a bond, and our grid becomes equivalent to a lattice. Arrangements of dominoes on a lattice are useful because thermodynamical properties can be calculated from the number of arrangements when there is a zero energy of mixing. Domino tiling is also useful as a model for finding the free energy of a liquid, because this calculation requires the number of ways that a volume of liquid can be divided into certain sized 'cells,' that are equivalent to our grid's tiles.

Additionally, our grid can also be seen as equivalent to a particular bipartite graph, as illustrated in the figure below. On the left we see a possible domino tiling of a 2×3 grid, and on the right we see the equivalent graph, with vertices representing tiles and edges representing dominoes.

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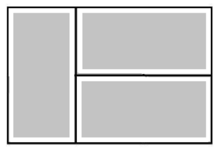
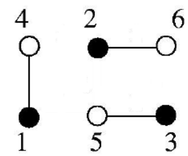




Thus our primary goal of counting the number of domino tilings of a grid is equivalent to finding the number of perfect matchings for the corresponding graph, which we will actually use to find our expression for counting domino tilings.

Definition 1.3. (Matching) A matching of a graph $G = (V, E)$ is a set of edges such that no two edges share a common vertex. A perfect matching is a matching where every vertex of the graph is incident to exactly one edge of the matching.

Remark 1.4. For the grids we examine in this paper, we will only orient dominoes horizontally or vertically.

2. Rectangular Grids

Before we can attempt to count domino tilings of an $m \times n$ rectangle, we must first see whether a tiling even exists.

Theorem 2.1. *Let Q be an $m \times n$ grid. Then Q has a domino tiling if and only if mn is even.*

Proof. Suppose Q has a domino tiling. Let k be the number of dominoes this tiling. Each domino occupies 2 unit squares. Thus the area of Q must be equal to $2k$ square units. Thus $2k = mn$, so m is even or n is even, and thus mn is even.

Now suppose mn is even. Thus m is even or n is even. Without loss of generality, suppose m is even. This means that Q has an even number of rows, so we can tile each column with $\frac{m}{2}$ vertically aligned dominoes to create a domino tiling. □

Corollary 2.2. *Let Q be an $m \times n$ grid. If Q has a domino tiling, then Q is tiled by $\frac{1}{2}mn$ dominoes.*

Proof. Suppose Q has a domino tiling. The number of dominoes must be the total area, mn, divided by the area of each domino, i.e., 2 unit squares. Thus the number of dominoes is $\frac{1}{2}mn$, which we know is an integer because mn is even. □

Thus we are only interested in grids where m or n is even. However, while existence of domino tilings is easy to verify, the number of tilings is much more challenging to compute. Before we examine general grids, we address a specific case: tilings of $2 \times n$ grids.

Theorem 2.3. *Let Q_n be a $2 \times n$ grid and let a_n be the number of domino tilings for Q_n. Then a_n is the $(n + 1)$th Fibonacci number, where $F_1 = 1$ and $F_2 = 1$.*

Proof. It is obvious that $a_1 = 1$ as the only tiling for a 2×1 grid is one vertical domino. a_2 is also easy to compute. The only ways to tile a 2×2 grid are to place two dominoes both horizontally or both vertically. Thus $a_2 = 2$.

We could compute a_3 with relative ease as well, but instead we make an observation about the ways dominoes can be arranged in Q_n. Because Q_n has only two rows, if a domino is placed horizontally in one row, another must be aligned with it in the other row. Staggering the dominoes would eventually lead to a lone square, which cannot be tiled by a domino. As a result, we can observe the following recurrence relation between tilings of Q_n.

If we take a $2 \times n$ grid, we can place either a single vertical domino or two horizontal dominoes in the leftmost columns. When we place a vertical domino, we are left with a Q_{n-1} grid, which we know has a_{n-1} tilings. Similarly, when we place horizontal dominoes, we are left with a Q_{n-2} grid, which has a_{n-2} tilings. Thus we get

$$a_n = a_{n-1} + a_{n-2}$$

which is the same recurrence relation as for the Fibonacci sequence. Because a_1 and a_2 are the second and third Fibonacci numbers, respectively, they will generate the rest of the Fibonacci sequence. □

We examine the $2 \times n$ case not merely because it is a neat pattern to find, but because it is easy to compute the nth Fibonacci number. Now we aim to find the number of domino tilings for a general rectangular grid. This result was computed independently by Kasteleyn and by Temperley and Fisher. Both proofs rely on computations of Pfaffians of certain matrices.

Definition 2.4. (Pfaffian) Let A be a triangular array of numbers $a_{i,j}$ such that $1 \leq i < j \leq N$ and N is even. Let P be the set of permutations $\sigma \in S_N$ satisfying

$$\sigma_1 < \sigma_3 < \ldots < \sigma_{N-1} \text{ and } \sigma_1 < \sigma_2; \sigma_3 < \sigma_4; \ldots \sigma_{N-1} < \sigma_N.$$

The Pfaffian $\mathrm{Pf}(A) = \sum_{\sigma \in P} \mathrm{sgn}(\sigma) a_{\sigma_1,\sigma_2} a_{\sigma_3,\sigma_4} \ldots a_{\sigma_{N-1},\sigma_N}$. $(*)$

Remark 2.5. Sometimes $\mathrm{Pf}(A)$ is defined the same way for skew-symmetric $N \times N$ matrices. Additionally, we will use the important property [3] of Pfaffians that if A is a skew-symmetric matrix, then $\mathrm{Pf}(A) = \sqrt{\det A}$.

Theorem 2.6. *(Kasteleyn, 1961). Let $Q_{m,n}$ be an $m \times n$ grid with m even. Then the number of domino tilings for $Q_{m,n}$ is* $\prod_{k=1}^{\frac{1}{2}m} \prod_{l=1}^{n} 2\sqrt{\cos^2 \frac{k\pi}{m+1} + \cos^2 \frac{l\pi}{n+1}}$.

Proof. As mentioned in the introduction, we can think of our grid as a graph, with vertices representing the squares. If an edge belongs to a bipartite matching, then it represents a domino covering two tiles. If we assign an ordered pair (i, j) with $1 \leq i \leq m$ and $1 \leq j \leq n$ to each vertex, then it becomes obvious that (i, j) is connected to (i', j') if and only if $|i - i'| = 1$ and $j = j'$ or else $i = i'$ and $|j - j'| = 1$.

It will be useful for us to find a generating function for the number of domino tilings of a grid. To do this, we will define a few variables: let $h \geq 0$ be the number of horizontal dominoes of a tiling and let $v \geq 0$ be the number of vertical dominoes of that tiling. Define the function $g(h, v)$ to be the number of tilings with h vertical dominoes and v vertical dominoes. We get our generating function

$$Z_{m,n}(z, z') = \sum_{h,v} g(h, v) z^h z'^v$$

which is a polynomial in z, z' where the sum ranges over $h, v \geq 0$ satisfying $2(h + v) = mn$ (which is necessary by Corollary 2.2 and ensures we have exactly enough dominoes for a tiling). Our generating function counts the number of domino tilings for every valid combination of horizontal and vertical dominoes. Thus $Z_{m,n}(1,1)$ returns the total number of domino tilings for an $m \times n$ grid.

Before we proceed, we want a way to clearly represent configurations of dominoes. We can use the ordered pairs assigned to each tile, but we need a way to establish uniqueness. Currently we can say that (i,j) is connected to (i',j') or alternatively that (i',j') is connected to (i,j), though both of these statements describe the same edge. Thus we will create a numbering convention for our tiles, and then we will create conditions based on this numbering so that we can describe each edge in just one way. For each tile (i,j), we define a number $p = (i-1)n + j$. This number is in fact unique to the tile, so we could even represent each tile only by its number p. The figure below illustrates an example of this numbering convention for a 4×4 grid. Notice that the numbers run uniquely from 1 to $mn = 16$.

p=1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

Now we create conditions for ordering tiles based on their p numbers:

$$p_1 < p_3 < \ldots < p_{mn-1} \tag{2.7}$$

$$p_1 < p_2 \ ; \ p_3 < p_4 \ ; \ \ldots \ ; \ p_{mn-1} < p_{mn}. \tag{2.8}$$

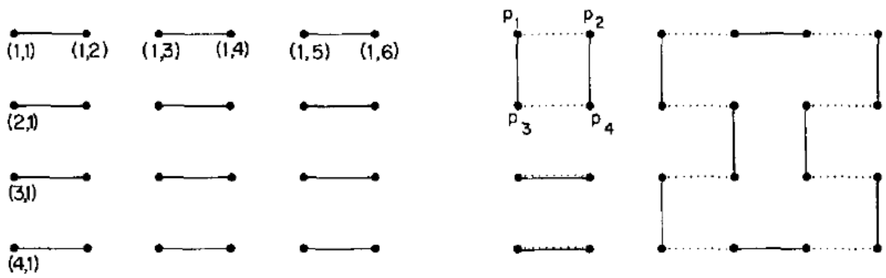
This allows us to *uniquely* write a configuration $C = |p_1, p_2|p_3, p_4| \ldots |p_{mn-1}, p_{mn}|$ as our conditions forbid a configuration such as $C = |p_3, p_4|p_2, p_1| \ldots$. Additionally, we define a "standard configuration":

$$C_0 = |(1,1),(1,2)|(1,3),(1,4)| \ldots |(m-1,n),(m,n)|.$$

Also notice that our conditions for configurations resemble the conditions for permutations used in computing a Pfaffian. These configurations will become important in our next step, where we attempt to design a triangular array such that every non-zero term in the Pfaffian corresponds to a configuration, and every configuration corresponds to a non-zero term of the Pfaffian.

Our goal now is to create a triangular array (or skew-symmetric matrix), D, such that the Pfaffian of this array is equal to our generating function $Z_{m,n}(z,z')$. We want to design D such that each non-zero term in the Pfaffian of D corresponds to a configuration C, and every configuration C corresponds to a non-zero term of the Pfaffian of D. This way, the number of terms in the Pfaffian of D is the number domino tiling of $Q_{m,n}$. This suggests making D a weighted $mn \times mn$ adjacency matrix. First we can define $D_{p,p'} = 0$ if p and p' are not connected. As a result, each term of the Pfaffian with a pair of unconnected vertices becomes 0, as it cannot correspond to a domino tiling. By the definition of a Pfaffian, each term is multiplied by the sign of its respective permutation in the sum $(*)$. Because we wish to find the number of non-zero terms, we want to make all the coefficients in $(*)$ positive so that their summation returns the desired result. Thus we want non-zero terms whose permutation has an even sign to be 1 and non-zero terms whose permutation has an odd sign to be -1. For the purpose of our generating functions, we also wish to weight horizontal and vertical connections with z and z', respectively, though this is not particularly important for this paper as our ultimate goal is to set $z = z' = 1$.

In order to examine the parity of a configuration, we use the following technique. We draw our vertices and then use dotted lines as the edges of our standard configuration, C_0. Then we can draw solid lines in the diagram to represent some configuration, C. An example of this process below is taken from Kasteleyn.



One vertex of our graph cannot have multiple solid edges attached to it because that would represent overlapping dominoes. Thus we can observe that the solid and dotted lines of our diagram either directly overlap or else they form closed polygons. Furthermore, if we were to cyclically shift all the solid edges in each polygon, we would notice that our configuration actually returns to the standard configuration. Now we wish to prove a lemma about these polygons.

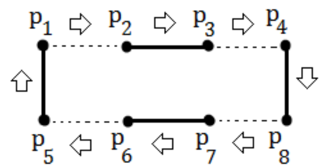
Lemma 2.9. *Let C be a configuration corresponding to some permutation σ that satisfies conditions (2.7) and (2.8). Each polygon in C contributes a factor of -1 to* $\operatorname{sgn}(\sigma)$.

Proof. Consider a configuration C that has at least one polygon. By definition, the polygon must be enclosed by a cycle of dotted and solid lines. If we take a starting point and travel along the cycle, then for any column in the polygon, we travel "forward" along dotted lines the same amount of times as we travel "backward" along them. In the same way, for any row, we travel "forward" along dotted lines the same amount of times as "backward".

Now we aim to express the permutation of C as the product of several rearrangements and reversals. To illustrate this more clearly, we consider the square $p_1p_3p_2p_4$ in Kasteleyn's diagram.

Firstly, the order of our dotted lines in C_0 will be increasing based on the ordering we established. To make it cyclic, we will need to reverse some number of pairs of vertices. We call this number r. For our square, the dotted lines are represented by $p_1p_2p_3p_4$, and we must reverse 1 pair of vertices to make it cyclic and get $p_1p_2p_4p_3$. Now we can travel "forward" from p_1 to p_2, and "backward" from p_4 to p_3.

Secondly, we need to rearrange $2r$ horizontal lines so that our travel along them also follows a cyclic order. A square is too small to demonstrate an example of this, but we can see it with a 1×3 rectangle. So far, we have reversed 2 pairs of vertices to get go from $p_1p_2p_3p_4p_5p_6p_7p_8$ to $p_1p_2p_3p_4p_6p_5p_8p_7$. However, this does not cyclically form a rectangle. We need to rearrange our 4 pairs of vertices to form rectangle $p_1p_2p_3p_4p_8p_7p_6p_5$, shown in the following figure.



Thirdly, we need to cyclically shift the $4r$ vertices in our permutation so it actually represents our polygon. With our square, we do not want to connect p_1 to p_2 and p_4 to p_3. We shift the 4 vertices and connect p_2 to p_4 and p_3 to p_1 with solid lines, so we have the square $p_2p_4p_3p_1$.

Next we want to satisfy conditions (2.7) and (2.8). For this step, we must look at two vertices at a time and compare their p numbering. We must reverse r pairs of vertices to satisfy condition (2.8). If we look at our square, we have p_2 connected to p_4 and p_3 connected to p_1. However, condition (2.8) requires $p_1 < p_3$, so we must reverse this 1 pair of vertices to get the square $p_2p_4p_1p_3$.

Finally, to satisfy condition (2.7), we must us rearrange $2r$ pairs of vertices. As we can see in our square, we have $p_2 < p_1$, so we must rearrange 2 pairs of vertices so that $p_1 < p_2$. Thus we go from square $p_2p_4p_1p_3$ to $p_1p_3p_2p_4$.

Now we compute how a polygon affects the parity of our permutation. Reversing the vertices within a pair contributes a factor of -1. We did this r times in the first step, and r times in the fourth step. Rearranging pairs, as in the second and fifth steps, contributes a factor of 1. Cyclically permuting, as in step three, contributes a factor of $(-1)^{4r-1}$. Thus the total contribution of a polygon to the parity of a permutation is $(-1)^r(-1)^r(1)(1)(-1)^{4r-1} = (-1)^{6r-1} = -1$ because $6r-1$ must be odd. Finally, if there are multiple polygons, then the permutations described for each polygon can be performed for one polygon at a time. Therefore each polygon will contribute a factor of -1 to $\mathrm{sgn}(\sigma)$. $\square$

Now we make an observation about the number of squares contained in a polygon. Let us first break it up into horizontal strips. If we move along a side of a strip, a horizontal solid line must be proceeded by a horizontal dotted line (there are no vertical dotted lines). For the same reason, it must be preceeded by one as well. Thus if there are k solid lines on one side, there must be $k+1$ dotted lines, and thus an odd number of unit squares contained in the row. Because the dotted lines are all aligned in the columns, there is no way to construct a polygon made of an even number of strips; it would either require a vertical dotted line or a dotted line in a column that has none. Because the number of squares in each strip and the number of strips are both odd, the number of squares contained in each polygon is odd.

Now we can assign coefficients for our array D. Because each polygon contains an odd number of squares, we can choose signs for $D_{p,p'}$ in such a way that an odd number of lines around each square has a negative sign. Because the standard configuration C_0 has a positive sign and is composed entirely of horizontal connections, we must attribute a negative sign to vertical connections. To ensure that one side of each square has a negative sign, we will make every other vertical connection negative. Formally, this gives us the following set of conditions for our coefficients:

$$D_{(i,j),(i+1,j)} = z \qquad \text{for } 1 \le i \le m-1 \text{ and } 1 \le j \le n \tag{2.10}$$

$$D_{(i,j),(i,j+1)} = (-1)^i z' \qquad \text{for } 1 \le i \le m \text{ and } 1 \le j \le n-1 \tag{2.11}$$

As established earlier, $D_{p,p'} = 0$ in all other cases.

From here, we wish to extend our triangular array into a skew-symmetric matrix. We simply do this by applying the definition for skew-symmetry: $D_{(i,j),(i',j')} = -D_{(i',j'),(i,j)}$. Now that we have our skew-symmetric matrix D, we attempt to diagonalize it so that we can find its determinant and thus its Pfaffian.

Definition 2.12. (Kronecker product) Let A be an $n \times n$ matrix and let B be an $m \times m$ matrix. The Kronecker product $A \otimes B$ is an $mn \times mn$ matrix that can be written in block form as

$$\begin{bmatrix} a_{1,1}B & a_{1,2}B & \dots & a_{1,n}B \\ a_{2,1}B & a_{2,2}B & \dots & a_{2,n}B \\ \vdots & \vdots & \dots & \vdots \\ a_{n,1}B & a_{n,2}B & \dots & a_{n,n}B \end{bmatrix}$$

Remark 2.13. We will use several properties [4] of Kronecker products.

(1) $(A+B)\otimes C = (A\otimes C) + (B \otimes C)$ and $A \otimes (B+C) = (A\otimes B) + (A \otimes C)$.
(2) $(A\otimes B)(C \otimes D) = (AC)\otimes(BD)$.
(3) If λ is a scalar, then $(\lambda A)\otimes B = \lambda(A\otimes B) = A \otimes(\lambda B)$.
(4) If A and B are invertible, then $(A\otimes B)^{-1} = A^{-1}\otimes B^{-1}$.

We can actually take D and decompose it into the sum of two $mn\times mn$ matrices. Recall our conditions for coefficients of D. Condition (2.10) gives us $z(I_n \otimes Q_m)$ where I_n is the $n \times n$ identity matrix and

$$Q = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & -1 & 0 \end{bmatrix}.$$

Similarly, condition (2.11) gives us $z'(Q_n \otimes F_m)$ where

$$F = \begin{bmatrix} -1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}.$$

Thus $D = z(I_n \otimes Q_m) + z'(Q_n \otimes F_m)$. Now we wish to transform D in such a way that we can easily compute the determinant. We begin by finding the eigenvectors of Q, and use these to create a matrix U. Thus we have entries of U and U^{-1}:

$$U_{n(l,l')} = \Big(\frac{2}{n+1}\Big)^{\frac{1}{2}} i^l \sin\Big(\frac{ll'\pi}{n+1}\Big), \tag{2.14}$$

$$U^{-1}_{n(l,l')} = \Big(\frac{2}{n+1}\Big)^{\frac{1}{2}} (-i)^{l'} \sin\Big(\frac{ll'\pi}{n+1}\Big). \tag{2.15}$$

Note that here i is the square root of -1 rather than a counter variable. Because U_n is formed from eigenvectors of Q_n, the transformation $Q'_n = U_n^{-1} Q_n U_n$ will diagonalize Q_n, and furthermore the diagonal elements of Q'_n will be the eigenvalues of Q_n. In the same way, we can obtain $Q'_m = U_m^{-1} Q_m U_m$. The transformation of Q_n will not affect I_n because it is the identity matrix. To see how transformation of Q_m affects F_m, we make a few observations.

Let $\mathbf{e}_i$ be the ith standard basis vector (all 0 entries with a 1 in the ith position) and let $\mathbf{u}_i$ be the ith column vector of U_m. We start with the fact that $U_m \mathbf{e}_i = \mathbf{u}_i$. Because the $\mathbf{u}$ are the eigenvectors of Q_m, we get

$$Q_m U_m \mathbf{e}_i = Q_m \mathbf{u}_i = \lambda_i \mathbf{u}_i,$$

where λ_i is the eigenvalue corresponding to $\mathbf{u}_i$. Multiplying by U_m^{-1}, we get

$$U_m^{-1} Q_m U_m \mathbf{e}_i = U_m^{-1}(\lambda_i \mathbf{u}_i) = \lambda_i (U_m^{-1} \mathbf{u}_i) = \lambda_i \mathbf{e}_i.$$

We know from (2.14) that the entries of $\mathbf{u}_{k'}$ have the form $i^k \sin(\frac{kk'\pi}{m+1})$, where k is the row number (we briefly ignore the $(\frac{2}{m+1})^{\frac{1}{2}}$ coefficient). Multiplying by F_m makes these entries take the form $(-i)^k \sin(\frac{kk'\pi}{m+1})$. Let $j = m+1-k'$. We see that $\mathbf{u}_j$ takes the form

$$i^k \sin(\frac{k(m+1-k')\pi}{m+1}),$$

which can be simplified to

$$i^k \sin(k\pi - \frac{kk'\pi}{m+1}) = -(-i)^k \sin(\frac{kk'\pi}{m+1}).$$

Thus we see that $F_m \mathbf{u}_{k'} = -\mathbf{u}_{m+1-k'}$. From our earlier result, we have $-\mathbf{u}_{m+1-k'} = -U_m \mathbf{e}_{m+1-k'}$, which means that $(U_m^{-1} F_m U_m)\mathbf{e}_{k'} = -\mathbf{e}_{m+1-k'}$. Thus $U_m^{-1} F_m U_m$, which we shall call F'_m, takes an anti-diagonal form where all the entries of the anti-diagonal are -1.

Now we can let $V = U_n \otimes U_m$ and take the transformation $D' = V^{-1} D V$ to attempt to diagonalize D. Applying the property that $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$, we get

$$D' = V^{-1} D V = (U_n^{-1} \otimes U_m^{-1})(z(I_n \otimes Q_m) + z'(Q_n \otimes F_m))(U_n \otimes U_m).$$

Distributing matrix multiplication and moving the scalars expands to

$$D' = z(U_n^{-1} \otimes U_m^{-1})(I_n \otimes Q_m)(U_n \otimes U_m) + z'(U_n^{-1} \otimes U_m^{-1})(Q_n \otimes F_m))(U_n \otimes U_m).$$

Applying $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$ twice, we get

$$D' = z(U_n^{-1} I_n \otimes U_m^{-1} Q_m)(U_n \otimes U_m) + z'(U_n^{-1} Q_n \otimes U_m^{-1} F_m)(U_n \otimes U_m),$$
$$D' = z(U_n^{-1} I_n U_n \otimes U_m^{-1} Q_m U_m) + z'(U_n^{-1} Q_n U_n \otimes U_m^{-1} F_m U_m).$$

By substitution, we finally have

$$D' = z(I_n \otimes Q'_m) + z'(Q'_n \otimes F'_m).$$

The entries of D' take the form

$$D'_{(k,l),(k',l')} = 2iz\delta_{k,k'}\delta_{l,l'} \cos\frac{k\pi}{m+1} - 2iz'\delta_{k+k',m+1}\delta_{l,l'} \cos\frac{l\pi}{n+1}$$

where δ_{ij} is the Kronecker delta, which is 0 when $i \neq j$ and 1 when $i = j$. Currently, D' has $m \times m$ blocks along the diagonal where the nonzero elements form a cross shape, with 0 entries everywhere else. We can bring this into block diagonal form without disturbing the determinant.

First, we take any of our $m \times m$ "crosses". We can index our entries based on the corner they start from, as seen below.

$$\begin{bmatrix} a_1 & 0 & 0 & \dots & 0 & 0 & b_1 \\ 0 & a_2 & 0 & \dots & 0 & b_2 & 0 \\ 0 & 0 & a_3 & \dots & b_3 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & c_3 & \dots & d_3 & 0 & 0 \\ 0 & c_2 & 0 & \dots & 0 & d_2 & 0 \\ c_1 & 0 & 0 & \dots & 0 & 0 & d_1 \end{bmatrix}.$$

Each swap of two adjacent rows or columns changes the sign of the determinant by a factor of -1. We can use $m-2$ column swaps to move the mth column where the second was, and to shift the second through $(m-1)$th columns right. Since m is even, this will not affect the determinant.

$$\begin{bmatrix} a_1 & b_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & a_2 & 0 & \dots & 0 & b_2 \\ 0 & 0 & 0 & a_3 & \dots & b_3 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & c_3 & \dots & d_3 & 0 \\ 0 & 0 & c_2 & 0 & \dots & 0 & d_2 \\ c_1 & d_1 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

In the same way, we can make $m-2$ row swaps to move the bottom row where the second was, and shift the second through $(m-1)$th rows downwards.

$$\begin{bmatrix} a_1 & b_1 & 0 & 0 & \dots & 0 & 0 \\ c_1 & d_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & a_2 & 0 & \dots & 0 & b_2 \\ 0 & 0 & 0 & a_3 & \dots & b_3 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & c_3 & \dots & d_3 & 0 \\ 0 & 0 & c_2 & 0 & \dots & 0 & d_2 \end{bmatrix}.$$

As we can see, we have created a 2×2 block in the top-left corner, with a $(m-2) \times (m-2)$ "cross" of the same form as the original in the opposite corner. Inductively, we can continue producing 2×2 blocks along the diagonal until we run out of entries in the $m \times m$ block. Furthermore, we can repeat this process for every other $m \times m$ block until D' is a block diagonal matrix.

As D' is a block diagonal matrix composed of 2×2 blocks, our determinant becomes easy to compute because it is equal to the product of the determinants of each 2×2 block. This gives us:

$$\det D = \det D' = \prod_{k=1}^{\frac{1}{2}m} \prod_{l=1}^{n} \det \begin{bmatrix} 2iz \cos \dfrac{k\pi}{m+1} & -2iz' \cos \dfrac{l\pi}{n+1} \\ -2iz' \cos \dfrac{l\pi}{n+1} & -2iz \cos \dfrac{k\pi}{m+1} \end{bmatrix}.$$

Which simplifies to:

$$\det D = \prod_{k=1}^{\frac{1}{2}m} \prod_{l=1}^{n} 4\Big(z^2 \cos^2 \frac{k\pi}{m+1} + z'^2 \cos^2 \frac{l\pi}{n+1}\Big).$$

We established earlier that $Z_{m,n}(z,z') = \sqrt{\det D}$. Thus we get:

$$Z_{m,n}(z,z') = \prod_{k=1}^{\frac{1}{2}m} \prod_{l=1}^{n} 2\sqrt{z^2 \cos^2 \frac{k\pi}{m+1} + z'^2 \cos^2 \frac{l\pi}{n+1}}.$$

Evaluating $Z_{m,n}(1,1)$ gives us the total number of domino tilings. Therefore, the total number of domino tilings for an $m \times n$ grid is:

$$Z_{m,n}(1,1) = \prod_{k=1}^{\frac{1}{2}m} \prod_{l=1}^{n} 2\sqrt{\cos^2 \frac{k\pi}{m+1} + \cos^2 \frac{l\pi}{n+1}}.$$

$\square$

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