

THE EULER CHARACTERISTIC OF FINITE TOPOLOGICAL SPACES

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ABSTRACT. The purpose of this paper is to illustrate the relationship between the topological property of the Euler characteristic and a combinatorial object, the Möbius function, in the context of finite T_0 -spaces. To do this I first explain the fundamental connection between such spaces and finite partially ordered sets by proving some facts fundamental to the study of finite spaces. Then I define the Euler characteristic and provide some elementary facts pertaining to the Euler characteristic of finite T_0 -spaces. Finally, I introduce the Möbius function and prove its relationship to the Euler characteristic.

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1. PRELIMINARY RESULTS

1.1. The Relationship Between T_0 Finite Topological Spaces and Finite Posets.

To begin we will establish some basic results concerning the relationship between finite topological spaces and finite preordered sets. Let X be some finite topological space and for some $x \in X$ let U_x be the intersection of all of the open sets containing x , which may be referred to as the *minimal open set* of that point. This allows us to define a preorder on X where $x \leq y$ if $x \in U_y$. A space is said to satisfy the T_0 separation axiom if the topology distinguishes points, that is for any two distinct points in X there exists an open set containing one of the points but not the other. Note then that for $x, y \in X$ for some T_0 space X , if $x \leq y$ and $y \leq x$, x and y share all of their neighborhoods, and thus must be the same point. This

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means that $x \leq y$ and $y \leq x$ implies that $x = y$ so if a space is T_0 , the ordered set determined by its topology is a partially ordered set, henceforth referred to as a poset. Thus, *every T_0 finite topological space determines a poset.*

Similarly, beginning with a finite preordered set X we may define a topology on X via the basis of sets of the form $\{y \in X \mid x \leq y\}$. If $y \leq x$, then $y \in U_x$ and conversely if $y \in U_x$ then $y \in \{z \in X \mid z \leq x\}$ and thus $y \leq x$, implying that the constructions relating finite preorders and finite topological spaces are mutually inverse. As before, if X is a poset then the topology generated from that poset structure is T_0 because if $x \leq y$ and $y \leq x$ then x and y must share all of their neighborhoods so if $x \leq y$ and $y \leq x$ implies that $x = y$, then x and y are equal when they share all of the same neighborhoods.

In summary, we record the following result:

Proposition 1.1. *There exists a bijection between finite topological spaces and preorders. Moreover, there exists a bijection between T_0 finite topological spaces and posets.*

Note that this claim could be extended to infinite sets by way of Alexandroff spaces, spaces in which arbitrary intersections of open sets are open, but this paper will be confined to the finite case.

An important aspect of this relation is the following:

Lemma 1.2. *A function $f : X \rightarrow Y$ between finite spaces is continuous if and only if it is order-preserving.*

Proof. Let f be continuous and let $w \leq x$. By the definition of continuity, $f^{-1}(U_{f(x)})$ must be open in X , and because U_x is the smallest open set containing x , $U_x \subset f^{-1}(U_{f(x)})$ where $w \in U_x$. This implies that $f(w) \in U_{f(x)}$ so $f(w) \leq f(x)$. Let f be order preserving and let V be open in Y . For some $f(x) \in V$, $U_{f(x)} \subset V$ so if $w \in U_x$ then $w \leq x$ implying that $f(x) \leq f(w)$ so $f(w) \in U_{f(x)}$, meaning $w \in f^{-1}(V)$. We may then write $f^{-1}(V)$ as the union of all U_x for $f(x) \in V$ so it is open. \square

1.2. Finite T_0 Spaces and Simplicial Complexes.

Within a poset, a *chain* is a subset of elements where any two elements are comparable. Using this notion of chains, we may construct a simplicial complex in the following way:

Definition 1.3. Let X be a finite T_0 -space. The *simplicial complex associated* with X is the simplicial complex whose simplices are the non-empty chains of X , where these chains are formed by considering X as a poset. This shall be denoted $\mathcal{K}(X)$.

It will often be useful to think of simplicial complexes as geometric objects so we give the following definition:

Definition 1.4. The *geometric realization* of a simplicial complex is the set of convex combinations of the form $t_1x_1 + \dots + t_r x_r$ where $x_1 < \dots < x_r$ is a chain

in X , $\sum_{i=1}^r t_i = 1$, and $t_i > 0$ for all i . In this way, we may realize the simplices of a simplicial complex as subsets of \mathbb{R}^N , each chain giving a simplex. We give this the metric topology with metric:

$$d\left(\sum_{i=0}^n t_i x_i, \sum_{i=0}^n s_i y_i\right) = \sqrt{\sum_{i=0}^n (t_i - s_i)^2}$$

This construction shall be denoted $|\mathcal{K}(X)|$.

We will need the following result due to McCord:

Theorem 1.5. *A finite T_0 -space X is of the same weak homotopy type as the geometric realization of its associated simplicial complex.*

The proof is fairly involved but can be found in [1] starting on page 12.

1.3. Cores of Finite Spaces.

Following May's terminology,

Definition 1.6. Let X be a finite space. An *upbeat* point is a point x for which there exists some $y > x$ such that if $z > x$ then $z \geq y$. A *downbeat* point is a point x for which there exists some $y < x$ such that if $z < x$ then $z \leq y$. A point that is either upbeat or downbeat is referred to as a *beat point*.

Definition 1.7. A finite T_0 -space is *minimal* if it has no beat points. A *core* of a finite space X is a subspace Y that is minimal and is a deformation retract of X .

We need the following fact to prove an important result about cores:

Lemma 1.8. *If f and g are functions between two finite spaces and $f \leq g$ then $f \simeq g$. Furthermore, if $f \simeq g$ then there exists a sequence of functions $\{f = f_1, f_2, \dots, f_q = g\}$ such that either $f_i \leq f_{i+1}$ or $f_{i+1} \leq f_i$ for $i < q$.*

The above result requires quite a few lemmas that are not directly related to the topic at hand, but a good exposition can be found in [3] on page 18.

Theorem 1.9. *Every finite T_0 -space has a core.*

Proof. Suppose that a finite T_0 -space X has an upbeat point and consider $X - \{x\}$ which we shall show is a deformation retract of X . Define $f : X \rightarrow X - \{x\}$ such that $f(z) = z$ if $z \neq x$ and $f(x) = y$ where y is the point that makes x upbeat. Suppose that for $u, v \in X$, $u \leq v$. Clearly if $u = v = x$ or if $u \neq x, v \neq x$ then $f(u) \leq f(v)$. Alternatively if $u = x$ and $x < v$ then because x is an upbeat point $f(u) = y \leq f(v) = v$ or if $v = x$ and $u < x$ then $f(u) = u < x < y = f(v)$ again because x is an upbeat point. Because clearly $f \geq id$, by the lemma above, $f \simeq id$ so it is a deformation retract. An analogous argument applies to downbeat points. Thus, by successively removing the beat points of X , we obtain in finitely many steps a deformation retract of X with no beat points. \square

We shall also need the following result recorded in both [3] (page 22) and [1] (page 8):

Theorem 1.10. *Let X and Y be finite spaces. X and Y are homotopy equivalent if and only if their cores are homeomorphic.*

2. THE EULER CHARACTERISTIC

2.1. Defining the Euler Characteristic.

We now have the machinery we need to define the Euler characteristic for finite spaces. For a space X with homology groups that are finitely-generated graded abelian groups, the *Euler characteristic* is defined as $\chi(X) = \sum_{n \geq 0} (-1)^n \cdot b_n(X)$

where b_i is the i -th Betti number of X , that is $b_i(X) = \text{rank}(H_i(X))$. For our purposes it will be useful to adopt a different definition of the Euler characteristic. To do this we require the purely algebraic fact that for a short exact sequence of finitely generated abelian groups $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$, $\text{rank}(B) = \text{rank}(A) + \text{rank}(C)$.

Theorem 2.1. *Let X be a compact CW-complex. Then $\chi(X) = \sum_{n \geq 0} (-1)^n c_n$ where c_n is the number of n -cells contained in the complex.*

Proof. Let

$$0 \rightarrow C_k \xrightarrow{d_k} C_{k-1} \rightarrow \dots \rightarrow C_1 \xrightarrow{d_1} 0$$

be the chain complex of chain groups of the CW-complex and the d_i are the boundary maps. Letting $B_i = \text{im}(d_{i+1})$ and $Z_i = \text{ker}(d_i)$ we have the following short exact sequences:

$$\begin{aligned} 0 \rightarrow Z_i \xrightarrow{i} C_i \xrightarrow{d_i} B_{i-1} \rightarrow 0 \\ 0 \rightarrow B_i \xrightarrow{d_{i+1}} Z_i \xrightarrow{q} H_i \rightarrow 0 \end{aligned}$$

By the lemma mentioned above, we have that

$$\begin{aligned} \text{rank}(C_i) &= \text{rank}(Z_i) + \text{rank}(B_{i-1}) \\ \text{rank}(Z_i) &= \text{rank}(B_i) + \text{rank}(H_i) \end{aligned}$$

By substituting the second equation into the first, multiplying the resulting equality by $(-1)^i$ and then summing over i , the B_i terms will cancel, giving $\sum_{n \geq 0} (-1)^n c_n = \sum_{n \geq 0} (-1)^n \cdot b_n(X)$ as desired. \square

By regarding simplicial complexes as special cases of CW-complexes, we may use this result to deduce that for a finite T_0 -space, $\chi(X) = \sum_{C \in \mathcal{C}(X)} (-1)^{\sharp C}$ where $\mathcal{C}(X)$ is the set of non-empty chains of X and $\sharp C$ is the cardinality of some element of that set.

2.2. Homotopy Invariance.

Using this last definition we can prove for finite spaces that the Euler characteristic is homotopy invariant.

Theorem 2.2. *Let X and Y be finite T_0 -spaces that are homotopy equivalent. Then $\chi(X) = \chi(Y)$.*

Proof. Let X_c and Y_c be the cores of X and Y respectively, which must exist by 1.9. 1.10 implies that X_c and Y_c are homeomorphic and thus $\chi(X_c) = \chi(Y_c)$. As per 1.9, we may think of X_c as part of a sequence of subspaces of X , where each successive element in the sequence is generated by removing a beat point. Thus, it

suffices to show that removing a beat point does not affect the Euler characteristic. Let P be a finite poset with beat point p , where there must exist some q such that if r is comparable with p then r is also comparable with q . We can then construct a bijection

$$\begin{aligned} \varphi : \{C \in \mathcal{C}(P) \mid p \in C, q \notin C\} &\rightarrow \{C \in \mathcal{C}(P) \mid p \in C, q \notin C\} \\ C &\mapsto C \cup \{q\} \end{aligned}$$

We may thus write:

$$\begin{aligned} \chi(P) - \chi(P - \{p\}) &= \sum_{p \in C \in \mathcal{C}(P)} (-1)^{\sharp C+1} = \sum_{q \notin C \ni p} (-1)^{\sharp C+1} + \sum_{q \in C \ni p} (-1)^{\sharp C+1} = \\ &= \sum_{q \notin C \ni p} (-1)^{\sharp C+1} + \sum_{q \notin C \ni p} (-1)^{\sharp \varphi(C)+1} = \sum_{q \notin C \ni p} (-1)^{\sharp C+1} + \sum_{q \notin C \ni p} (-1)^{\sharp C} = 0 \end{aligned}$$

□

3. THE MÖBIUS FUNCTION

The Euler characteristic of finite T_0 -spaces is particularly interesting because of its relationship to the Möbius function of posets. To define the Möbius function we first define an *incidence algebra* \mathfrak{A} on P . $\mathfrak{A}(P)$ is the set of functions $P \times P \rightarrow \mathbb{R}$ such that for $f \in \mathfrak{A}(P)$, $f(x, y) = 0$ if $x \not\leq y$. This forms a vector space over \mathbb{R} where we have a product defined as

$$fg(x, y) = \sum_{z \in P} f(x, z)g(z, y)$$

We let $\zeta_p \in \mathfrak{A}$ be the function such that $\zeta_p(x, y) = 1$ whenever $x \leq y$. This function has an inverse in \mathfrak{A} which we call the *Möbius function* and denote μ_p . Note that $\zeta_p(x, y)$ is invertible according to [1] page 26. The identity of \mathfrak{A} is

$$\delta(s, t) = \begin{cases} 1 & : s = t \\ 0 & : s \neq t \end{cases}$$

Note that in equations where elements of \mathfrak{A} are added to some integer that integer simply denotes a multiple of δ .

It follows directly from the definition of the multiplication that:

$$\zeta^2(s, u) = \sum_{s \leq t \leq u} 1$$

so we may deduce that $\zeta^2(s, u)$ is the number of chains of length 2 between s and u (note that the length is given as the number of elements minus 1). Similarly

$$\zeta^k(s, u) = \sum_{s=s_0 \leq s_1 \leq \dots \leq s_k=u} 1$$

which is the number of chains of length k . Observing that

$$(\zeta - 1)(s, u) = \begin{cases} 1 & : s < u \\ 0 & : s = u \end{cases}$$

we can use $(\zeta - 1)^k$ to count the number of strictly-increasing chains. Note furthermore that

$$(2 - \zeta)(s, t) = \begin{cases} 1 & : s = t \\ -1 & : s < t \end{cases}$$

Proposition 3.1. $(2 - \zeta)^{-1}(s, t)$ gives the total number of strictly increasing chains from s to t .

Proof. Let ℓ be the length of the longest chain between s and t so that $(\zeta - 1)^{\ell+1}(u, v) = 0$ for $s \leq u \leq v \leq t$. For such u and v

$$\begin{aligned} (2 - \zeta)[1 + (\zeta - 1) + (\zeta - 1)^2 + \dots + (\zeta - 1)^\ell](u, v) &= \\ [1 - (\zeta - 1)][1 + (\zeta - 1) + (\zeta - 1)^2 + \dots + (\zeta - 1)^\ell](u, v) &= \\ [1 - (\zeta - 1)^{\ell+1}](u, v) &= \delta(u, v) \end{aligned}$$

The equality from the second line to the third comes from multiplying out so that all of the central terms cancel. Because δ is the identity, $(2 - \zeta)^{-1} = 1 + (\zeta - 1) + (\zeta - 1)^2 + \dots + (\zeta - 1)^\ell$ when restricted to the elements between some s and t . But as explained above, $(\zeta - 1)^k$ are just the chains of length k between s and t so it follows that $(2 - \zeta)^{-1}$ is the total number of chains from s to t . \square

The following theorem connects the combinatorial notion of the Möbius function to the topological notion of the Euler characteristic:

Theorem 3.2 (Hall's Theorem). *Let P be a finite poset and let \hat{P} be $P \cup \{\hat{0}, \hat{1}\}$ where $\hat{0}$ and $\hat{1}$ are minimum and maximum elements. Let c_i be the number of strictly increasing chains between $\hat{0}$ and $\hat{1}$ of length i . Then*

$$(3.3) \quad \mu_{\hat{P}}(\hat{0}, \hat{1}) = c_0 - c_1 + c_2 - c_3 + \dots$$

Proof.

$$\begin{aligned} \mu_{\hat{P}}(\hat{0}, \hat{1}) &= (1 + (\zeta - 1))^{-1}(\hat{0}, \hat{1}) \\ &= (1 - (\zeta - 1) + (\zeta - 1)^2 - \dots)(\hat{0}, \hat{1}) \\ &= 1(\hat{0}, \hat{1}) - (\zeta - 1)(\hat{0}, \hat{1}) + (\zeta - 1)^2(\hat{0}, \hat{1}) - \dots \\ &= c_0 - c_1 + c_2 - \dots \end{aligned}$$

\square

This expression is very close to the expression developed for the Euler characteristic. Indeed the only difference is that when computing the Euler characteristic, the empty-set is not regarded as a face of the simplicial complex whereas in this expression it is, entering the sum as -1 . Thus by defining the reduced Euler characteristic, $\tilde{\chi}(X) = \chi(X) - 1$ we have the following remarkable fact:

Proposition 3.4. *Let P be a finite poset.*

$$(3.5) \quad \mu_{\hat{P}}(\hat{0}, \hat{1}) = \tilde{\chi}(\mathcal{K}(P)).$$

For more information on Hall's theorem or Proposition 3.4 see [4] pages 307-8.

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