UNIFORM CONVERGENCE OF FOURIER SERIES

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Abstract. This paper is an exposition on Fourier series that converge uniformly to functions. After some extensive preparation, it first shows how the nature of the function's derivative can give uniform convergence of its Fourier series. The paper then provides some insight into measuring how quickly the Fourier series of a function can converge uniformly.

Contents

1. Introduction

Fourier series represent a periodic function as an infinite trigonometric series such as one of the form

$$S(f)(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx \tag{1.1}$$

for a function f with period 2π (Though this paper considers periodic functions, it makes use of 2π-periodic functions in order to make equations less variable-heavy). However it is impossible to add up an infinite number terms in practice. Thus the question of how well the partial sums of the Fourier series

$$S_n(f)(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx + b_k \sin kx \tag{1.2}$$

approximate the function f for larger and larger n comes to the fore. In answering this question we could either consider the convergence of the partial sums to the $f(x)$ at a given point x, that is the point-wise convergence of the partial sums, or we could consider the much stronger case of uniform convergence, in which the partial sums approach f simultaneously for every point in an interval. According to the so-called Dirichlet conditions if f is absolutely integrable, then at a continuous point x where $f(x)$ has both a left-hand and right-hand derivative $S(f)(x)$ would

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convergence point-wise to $f(x)$ (If x is a discontinuous point then $S(f)(x)$ will converge halfway between the left-hand and right-hand limits of x, if they both exist). With regard to uniform convergence, our most obvious condition can be observed by noticing that the function f would need to be continuous in order to be the uniform limit of the partial sums $S_n(f)(x)$. However, it was shown by Du Bois-Reymond that continuity alone does not ensure point-wise convergence much less uniform convergence [1, p.38].This paper thus aims to throw more light on the uniform convergence of the Fourier series of a function f and the nature of this uniform convergence.

Section 2 focuses exclusively on formulating and proving the Riemann-Lebesgue lemma in order not to interrupt the flow towards the proofs of our main theorems which would come in later sections and to highlight the lemma's foundational role in the study of Fourier series. In Section 3, we will then state and prove two lemmas which will be directly applied to proving our main theorems of uniform convergence. In Section 4, we first consider uniform convergence to a periodic continuous function with an absolutely integrable derivative before looking at the more general case of an absolutely integrable periodic function continuous on some closed interval $[a, b]$ and possessing an absolutely integrable derivative. Finally in section 5, we consider how quickly the Fourier series of a function with $m-1$ continuous derivatives and an m^{th} absolutely integrable derivative converges uniformly to that function, in the process illustrating the process of improving the convergence of a Fourier series.

2. The Riemann-Lebesgue Lemma

The coefficients of the Fourier series, a_k and b_k as shown in Equation (1.1) are defined as

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\, dx \tag{2.1}$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\, dx \tag{2.2}$$

A simple application of the Weierstrass M-test will show that if

$$\sum_{k=1}^{\infty} |a_k| + |b_k| \tag{2.3}$$

converges, then Equation (1.1) will converge uniformly and absolutely. As it is a necessary condition for a convergent series that the terms tend to zero as k tends to infinity, we can reasonably conclude that Equations (2.1) and (2.2) tend to zero as k tends to infinity in any situation where (2.3) holds. This observation turns out to be a lot more general than just the case where (2.3) holds. In fact if $f(x)$ is absolutely integrable then (2.1) and (2.2) will tend to zero as k tends to infinity regardless of the limits of integration chosen. This fact is referred to as Riemann-Lebesgue lemma.

Theorem 2.4. *(The Riemann-Lebesgue lemma) For any function $f(x)$ absolutely integrable on an interval $[a, b]$,*

$$\lim_{k\to\infty}\int_a^b f(x)\cos kx\, dx = \lim_{k\to\infty}\int_a^b f(x)\sin kx\, dx = 0$$

Proof. We can safely prove the theorem for just the $\sin kx$ case as the proof for $\cos kx$ is essentially the same. We begin by first proving the theorem for proper integrals. Notice

$$\lim_{k\to\infty}\int_{t_{i-1}}^{t_i}\sin kx\,dx = \lim_{k\to\infty}\frac{\cos kt_{i-1}-\cos kt_i}{k} = 0$$

We create a partition of the interval $[a,b]$
$a < t_1 < ... < t_{n-1} < b$ and define a step function $s(x) = \inf_{y\in[t_{i-1},t_i]} f(y)$ for all $x \in (t_{i-1}, t_i)$ leaving $s(x)$ undefined for all t_i. Thus for any partition of $[a,b]$

$$\lim_{k\to\infty}\int_a^b s(x)\sin kx\,dx = 0$$

and for all $\epsilon > 0$, there exists some partition of $[a,b]$ that gives

$$\int_a^b |f(x)-s(x)|\,dx = \int_a^b f(x)-s(x)\,dx < \frac{\epsilon}{2}$$

Hence through the triangle inequality, we find

$$\left|\int_a^b f(x)\sin kx\,dx\right| \le \int_a^b f(x)-s(x)\,dx + \left|\int_a^b s(x)\sin kx\,dx\right| < \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon$$

for large enough k.

For proper integrals, it can be seen that we need only require that f be integrable (If a proper integral of f exists, then the proper integral of $|f|$ will also exist. A proof of this is found in Stein and Sharkarchi's book [1, pp.283-284]). In the case of an improper integral on $[a,b]$, we would now explicitly require that f be absolutely integrable. Without loss of generality assume the improper integral in question is

$$\lim_{p\to b}\int_a^p f(x)\sin kx\,dx \tag{2.5}$$

As f is absolutely integrable, for any $\epsilon > 0$ we can choose some $t \in [a,b]$ which gives

$$\lim_{p\to b}\int_t^p |f(x)|\,dx < \frac{\epsilon}{2} \tag{2.6}$$

The triangle inequality and a little work will then show that the absolute value of (2.5) is less than or equal to the absolute value of a proper integral of $f(x)\sin kx$ on $[a,t]$ and the improper integral shown in the inequality (2.6). Thus through our proof of the Riemann-Lebesgue lemma for proper integrals and (2.6), we have that for any $\epsilon > 0$

$$\left|\lim_{p\to b}\int_a^p f(x)\sin kx\,dx\right| < \epsilon$$

for large enough k, completing the proof ☐

With regard to Fourier Series, the Riemann-Lebesgue lemma tells us that the terms of the series approach zero as k gets larger. Thus the Fourier series of any absolutely integrable function already satisfies a necessary (but not sufficient) condition for point-wise and uniform convergence.

3. Auxiliary Lemmas

The lemmas in this section are directly applicable to the proofs of our main theorems in Section 4.

Lemma 3.1. *Let f(x) be an absolutely integrable, periodic function and let $\omega(u)$ be a function with a continuous derivative on $[a, b]$. Then for any $\epsilon > 0$, the inequality*

$$\left| \int_a^b f(x+u)\omega(u) \sin mu \, du \right| < \epsilon$$

holds for all x provided m is large enough.

Proof. The proof given here uses ideas similar to those in the proof of the Riemann-Lebesgue lemma in Section 2. First notice that for $t_i, t_{i-1} \in [a, b]$

$$\int_{t_{i-1}}^{t_i} \omega(u) \sin mu \, du = \frac{1}{m} \left(\int_{t_{i-1}}^{t_i} \omega'(u) \cos mu \, du - \omega(t_i) \cos mt_i + \omega(t_{i-1}) \cos mt_{i-1} \right)$$

The terms in parenthesis are bounded since there exists M such that $M \geq |\omega(u)|$ and $M \geq |\omega'(u)|$ on $[a, b]$. Thus as m tends to infinity our integral would tend to zero. This implies that for any step function defined with respect to our periodic function f,

$$\left| \int_a^b s(x+u)\omega(u) \sin mu \, du \right| < \epsilon$$

for large enough m, regardless of x. It thus follows for a proper integral that,

$$\left| \int_a^b f(x+u)\omega(u) \sin mu \, du \right| < \epsilon$$

for large enough m, regardless of x.

Now, without loss of generality, assume we have an improper integral

$$\lim_{p \to b} \int_a^p f(x+u)\omega(u) \sin mu \, du$$

The absolute value of the above improper integral will be less than sum of the absolute value of a proper integral of $f(x+u)\omega(u) \sin mu$ on $[a, t]$, where $t < b$, and the improper integral

$$M \lim_{p \to b} \int_t^p |f(x+u)| \, du$$

which we can make as small as we please by choosing t close enough to b. We thus have

$$\left| \lim_{p \to b} \int_a^p f(x+u)\omega(u) \sin mu \, du \right| < \epsilon$$

for all x provided m is large enough, completing the proof. □

Lemma 3.2. *The integral*

$$I(u) = \int_0^u \frac{\sin mt}{2 \sin \frac{t}{2}} \, dt$$

is bounded on $[-\pi, \pi]$

Proof. We should first note that

$$\frac{\sin mt}{2\sin\frac{t}{2}} \tag{3.3}$$

is continuous on $[-\pi, 0)$ and $(0, \pi]$, therefore $I(u)$ will be bounded on any closed sub-interval of these intervals. Thus in order to show that $I(u)$ is bounded on $[-\pi, \pi]$, it suffices to show that (3.3) tends to a definite value as t tends to 0. An application of L'Hospital's rule reveals that (3.3) approaches m as t tends to zero. $I(u)$ is therefore bounded on $[-\pi, 0)$ and $(0, \pi]$ and hence $[-\pi, \pi]$ (regardless of how $I(0)$ is defined). □

4. Continuous Functions with an Absolutely Integrable Derivative

As was mentioned in the introduction, our first condition for the uniform convergence of the Fourier series $S(f)(x)$ to the function $f(x)$ would necessarily be that f be continuous on the interval in question. What is more, the Dirichlet conditions for a continuous point x hint at the role that the derivative of the function may have to play in the convergence of $S(f)(x)$ to $f(x)$. The nature of the derivative of a function does indeed play an important part in ensuring uniform convergence of the function's Fourier series. In this section, we will show that if the derivative of absolutely integrable periodic f is absolutely integrable over the interval, we obtain uniform convergence. A few preparations need be made first. Substituting Equations (2.1) and (2.2) into Equation (1.2) and using

$$\frac{1}{2} + \cos x + \cos 2x + ... + \cos nx = \frac{\sin{(n+\frac{1}{2})x}}{2\sin\frac{x}{2}} \tag{4.1}$$

which can be derived from applying the trigonometric factor formulae to the product of the denominator on the right-hand side and the sums on the left-hand side, the partial sums of the Fourier series represented by Equation (1.2) can be reformulated as

$$S_n(f)(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+u)\frac{\sin{(n+\frac{1}{2})u}}{2\sin\frac{u}{2}}\, du \tag{4.2}$$

Also notice from integrating the left-side of Equation (4.1) that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\sin{(n+\frac{1}{2})u}}{2\sin\frac{u}{2}}\, du = 1 \tag{4.3}$$

Theorem 4.4. *If $f(x)$ is continuous and periodic and $f'(x)$ is absolutely integrable, then $S_n(f)(x)$ converges uniformly to $f(x)$ for all x.*

Proof. Using Equations (4.2) and (4.3),

$$|S_n(f)(x) - f(x)| = \frac{1}{\pi}\left|\int_{-\pi}^{\pi} [f(x+u) - f(x)]\frac{\sin{(n+\frac{1}{2})u}}{2\sin\frac{u}{2}}\, du\right|$$

We now split the integral on the right-hand side into three integrals along sub intervals $[-\pi, -\delta]$, $[-\delta, \delta]$ and $[\delta, \pi]$ where $0 < \delta < \pi$. We will thus have that $|S_n(f)(x) - f(x)|$ is less than the sum of the absolute values of these three integral by the triangle inequality.

Now notice that $\frac{1}{2\sin\frac{u}{2}}$ has a continuous derivative on $[-\pi, -\delta]$ and $[\delta, \pi]$ thus by Lemma 3.1, we can make the absolute value of two of our three integrals as small as we please by choosing n large enough, regardless of x.

For the remaining integral on $[-\delta, \delta]$, integrating by parts gives

$$\left| \left([f(x+u) - f(x)] \int_0^u \frac{\sin mt}{2\sin\frac{t}{2}}\, dt \right) \Big|_{-\delta}^{\delta} - \int_{-\delta}^{\delta} f'(x+u) \int_0^u \frac{\sin mt}{2\sin\frac{t}{2}} dt\, du \right|$$

By Lemma 3.2 there exists $K \geq |I(u)|$ on $[-\pi, \pi]$, thus if we also note that $I(u)$ is an even function, then the above will be less than

$$K\left(|(f(x+\delta) + f(x-\delta) - 2f(x)| + \int_{-\delta}^{\delta} |f'(x+u)|\, du \right)$$

Since $f(x)$ is continuous and $f'(x)$ is absolutely integrable, the above expression can be made as small as we please by choosing δ small enough. Put together this means that we can make $|S_n(f)(x) - f(x)|$ as small as we please by choosing n (which has no dependence on x) large enough, as claimed. □

The function f in Theorem 4.4 is "too nice" in that it is continuous on the entire real line. Our aim will now be to prove uniform convergence to a function continuous on some arbitrary interval $[a, b]$. To do this we first make an observation that is important to the study of Fourier series.

Lemma 4.5. *(The Riemann Localization Principle) If $f(x)$ and $g(x)$ are absolutely integrable functions with the same period that are equal on some interval $[a, b]$ then*

$$S_n(f)(x) - S_n(g)(x)$$

converges uniformly to zero for all $x \in (a, b)$.

Proof. Taking $x \in (a, b)$

$$S_n(f)(x) - S_n(g)(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} [f(x+u) - g(x+u)] \frac{\sin{(n+\frac{1}{2})u}}{2\sin\frac{u}{2}} du$$

As with the proof of Theorem 4.4 we split the right-hand integral into three integrals on the intervals $[-\pi, -\delta]$, $[-\delta,\ \delta]$ and $[\delta,\ \pi]$ where δ is such that $0 < \delta < \pi$ and $[x-\delta, x+\delta] \subset (a, b)$. As with the proof of Theorem 4.4, the absolute values of the integrals on $[-\pi, -\delta]$ and $[\delta,\ \pi]$ can be made as small as we please with large enough n, regardless of x, by Lemma 3.1.

The integral on $[-\delta,\ \delta]$ is obviously zero since $f(x+u) = g(x+u)$ whilst

$$\frac{\sin{(n+\frac{1}{2})u}}{2\sin\frac{u}{2}}$$

is bounded on $[-\delta, \delta]$ (see proof of Lemma 3.2). Thus using the triangle inequality we find that for any $\epsilon > 0$, large enough n gives

$$|S_n(f)(x) - S_n(g)(x)| < \epsilon$$

for every $x \in (a, b)$, thus proving the statement. □

Corollary 4.6. *If*

$$\lim_{n\to\infty} S_n(f)(x)$$

exists for $x \in (a, b)$ then

$$\lim_{n\to\infty} S_n(g)(x) = \lim_{n\to\infty} S_n(f)(x)$$

Theorem 4.7. *If a periodic, absolutely integrable function $f(x)$ is continuous on $[a, b]$ with an absolutely integrable derivative, then the Fourier series $S(f)(x)$ converges uniformly to $f(x)$ on (a, b).*

Proof. Notice that if $b - a \geq 2\pi$ then $f(x)$ is continuous on the entire real line and this will be Theorem 4.4. If $b - a < 2\pi$, it suffices to construct a 2π-periodic function $F(x)$ that is continuous on the entire real line and equals $f(x)$ on $[a, b]$ (One way to do this is by linking $f(b)$ and $f(a + 2\pi)$ with a continuous function and extending this function defined on $[a, a + 2\pi]$ periodically over the real line). By Theorem 4.4, the Fourier series of $F(x)$ will converge uniformly to $F(x)$. Thus the Riemann Localization Principle shows that for any $\epsilon > 0$

$$\epsilon > |S_n(F)(x) - F(x)| = |S_n(f)(x) - f(x)|$$

for large enough n, for all $x \in (a, b)$, as claimed. □

5. The Absolutely Integrable m^{th} Derivative

This section will seek to throw some light on how quickly the Fourier series of a function can converge uniformly. We begin by noting that if a continuous periodic function f has an absolutely integrable derivative f', then the Fourier series corresponding to f' is given by

$$S(f')(x) = \sum_{k=1}^{\infty} k[b_k \cos kx - a_k \sin kx]$$

(No claim is made as to whether this series converges). This is simply a result of integrating

$$a'_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f'(x) \cos kx \, dx$$
$$b'_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f'(x) \sin kx \, dx$$

by parts. Thus we find $a'_k = k\, b_k$ and $b'_k = -k\, a_k$. If the notation $a_k^{(m)}$ and $b_k^{(m)}$ to represent the Fourier coefficients of the absolutely integrable m^{th} derivative of a function is allowed, one can easily check that $|a_k^{(m)}| = |k^m\, a_k|$ for even m and $|k^m\, b_k|$ for odd m whilst $|b_k^{(m)}| = |k^m\, b_k|$ for even m and $|k^m\, a_k|$ for odd m, provided that the $m - 1$ preceding derivatives are also continuous. From these considerations, we immediately obtain the following result.

Theorem 5.1. *If a continuous function $f(x)$ has $m - 1$ continuous derivatives and an m^{th} absolutely integrable derivative, then the Fourier coefficients of $f(x)$ satisfy the relation*

$$\lim_{k\to\infty} k^m a_k = \lim_{k\to\infty} k^m b_k = 0$$

Proof. As a result of the Riemann-Lebesgue lemma,

$$\lim_{k\to\infty} a_k^{(m)} = \lim_{k\to\infty} b_k^{(m)} = 0$$

Since the $m - 1$ preceding derivatives are continuous we thus have

$$\lim_{k\to\infty} k^m a_k = \lim_{k\to\infty} k^m b_k = 0$$

. □

By Theorem 4.4, the Fourier series of the function f in the Theorem 5.1 will converge uniformly to f everywhere (In fact the Fourier series of f's first $m-1$ derivatives will also converge uniformly). Since Theorem 5.1 shows that the Fourier coefficients of f converge to 0 faster than k^{-m}, we now see a way to observe how quickly the partial sums of the Fourier series converge to a function. The larger m such that

$$\lim_{k\to\infty} k^m a_k = \lim_{k\to\infty} k^m b_k = 0 \tag{5.2}$$

is for a given Fourier series, the better the partial sums of the Fourier series $S_n(f)$ approximate the function for a given n since the subsequent terms in the series will decay to zero more quickly. Increasing m is thus the main consideration when improving the convergence of a given Fourier series, that is when improving the quality of each approximation $S_n(f)(x)$ of $f(x)$.

Example 5.3 (2, p.145)**.** For the following series, which is uniformly convergent on $x \in (-\pi, \pi)$ (a fact we will not prove here),

$$f(x) = \sum_{k=2}^{\infty} (-1)^k \frac{k^3}{k^4 - 1} \sin kx \tag{5.4}$$

the largest m satisfying the condition (5.2) is zero, but by using

$$\frac{k^3}{k^4-1} - \frac{1}{k} = \frac{1}{k^5 - k}$$

and the Fourier series

$$\frac{x}{2} = \sum_{k=1}^{\infty} (-1)^k \frac{\sin kx}{k}$$

for $x \in (-\pi, \pi)$, we can rewrite Equation (5.2) as

$$f(x) = -\frac{x}{2} + \sin x + \sum_{k=2}^{\infty} (-1)^k \frac{\sin kx}{k^5 - k}$$

for $x \in (-\pi, \pi)$, thus improving the convergence by making $m = 4$.

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References

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[2] Georgi P. Tolstov. Fourier Series. Translated from Russian by Richard A. Silverman. Dover Publications, Inc. 1976.