TOWARDS THE PRIME NUMBER THEOREM

ERIC ANTLEY

ABSTRACT. Originally proven by Jacques Salomon Hadamard and Charles Jean de la Valle-Poussin independently in 1896, the prime number theorem shows the number of primes less than or equal to a given number x approaches $\frac{x}{\log x}$ as x tends towards infinity. In this paper, a proof using complex analysis will be given to prove this statement.

CONTENTS

Let $\mathcal{P}$ be the set of primes. We define the function $\pi(x)$ to be the prime counting function.

Definition 0.1. Given a real number x, $\pi(x)$ is the the cardinality of the set $\{p \in \mathcal{P} | p \leq x\}$.

Theorem 0.2 (The Prime Number Theorem)**.** *We have the following limit behavior:*

$$\pi(x) \sim \frac{x}{\log x}. \tag{0.3}$$

1. THE CHEBYSHEV FUNCTIONS AND THEIR PROPERTIES

We define the function $\nu(x)$ to be the first Chebyshev function.

Definition 1.1. Given a real number x, we define $\nu(x)$ to be the sum $\sum_{\{p \in \mathcal{P} | p \leq x\}} \log p$.

We define the function $\psi(x)$ to be the second Chebyshev function.

Definition 1.2. Given a real number x, we define $\psi(x)$ to be the sum over primes p and natural numbers k of $\sum_{\{p \in \mathcal{P} | p^k \leq x\}} \log p$.

We define the function $\Lambda(x)$ to be the von Mangoldt function.

Definition 1.3. Given a real number x, we define $\Lambda(x)$ to be $\log p$ if $p^k = x$ where p is some prime and 0 else, where $k \in \mathbb{Z}$.

Clearly, $\psi(x) = \sum_{n \leq x} \Lambda(n)$.

The prime counting function $\pi(x)$ is by definition what we are investigating, yet this function is very "unnatural." For now, the reader can understand this as simply meaning that it is a difficult function to work with, although easy to comprehend. On the other hand, the function $\psi(x)$ is a very "natural" function. We will see later that $\psi(x)$ is strongly related to a deceptively simple looking infinite sum known as the Riemann zeta function which is deeply rooted in number theory. Our first goal will be to examine $\psi(x)$ and related functions in order to show that properties of $\psi(x)$ imply the Prime Number Theorem, but first we will show that $\psi(x)$ is $\Theta(x)$. This means that for constants A and A' with sufficiently large x, we have $Ax \leq \psi(x) \leq A'x$. We will use this in section 3. From now on, we assume the symbol p denotes a prime number.

Lemma 1.4. $\psi(x) = \sum_{p \leq x} \lfloor \frac{\log x}{\log p} \rfloor \log p$.

Proof. We now attempt to find an expression that assigns the correct "weight" to each $\log p$. By definition, $\psi(x) = \sum_{p \in \mathcal{P}, p^k \leq x} \log p$. Let $p \in \mathcal{P}$ and $k \geq 0$ be such that $p^k \leq x$ and $p^{k+1} > x$. Note p contributes $k \log p$ to $\psi(x)$. Let $x = p^k \alpha$. Then we see that $\log_p x = \log_p (p^k \alpha) = k \log_p p + \log_p \alpha = k + \log_p \alpha$. Now, α is necessarily less than k, so $\lfloor k + \log_y \alpha \rfloor = k$. Thus, we can express $\psi(x)$ as the sum: $\sum_{p \leq x} \lfloor \log_p x \rfloor \log p$. Finally, we apply the change of log base formula to get $\sum_{p \leq x} \lfloor \log_p x \rfloor \log p = \psi(x) = \sum_{p \leq x} \lfloor \frac{\log x}{\log p} \rfloor \log p$. □

We now turn our attention to estimating the size of $\psi(x)$, but first we will require a few more definitions that will be used to describe the limiting behavior of functions.

Definition 1.5. $o(f(x))$

Let $f(x)$ and $g(x)$ be functions. We say that $g(x) = o(f(x))$ if for all $\epsilon > 0$ there exists an x_0 such that for all $x > x_0$ we have $|g(x)| \leq \epsilon |f(x)|$.

Definition 1.6. $O(f(x))$

Let $f(x)$ and $g(x)$ be functions. We say that $g(x) = O(f(x))$ if there exists an $M > 0$ such that for all $x > x_0$ we have $|g(x)| \leq M|f(x)|$.

Definition 1.7. $\Theta(f(x))$

Let $f(x)$ and $g(x)$ be functions. We say that $g(x) = \Theta(f(x))$ if there exist A and A' such that for all $x > x_0$ we have $Ag(x) \leq f(x) \leq A'g(x)$.

Lemma 1.8. $\psi(x) = \nu(x) + O(x^{\frac{1}{2}} (\log x)^2)$.

Proof. First, we note that $\psi(x) = \sum_{m=1}^{\infty} \nu(x^{\frac{1}{m}})$. This is a quick check which follows from the fact that $p^k \leq x$ is equivalent to $p \leq x^{\frac{1}{k}}$. Now, it is trivial to see that for $x > 1$ we have $\nu(x) < x \log x$. Thus for $m > 2$, $\nu(x^{\frac{1}{m}}) < x^{\frac{1}{m}} \log x \leq x^{\frac{1}{2}} \log x$. Now consider the sum $\sum_{2 \leq m} \nu(x^{\frac{1}{m}})$. There are $O(\log x)$ terms of this sum because $\nu(x^{\frac{1}{m}}) = 0$ if $m > \log_2 x$, and each term is $O(x^{\frac{1}{2}} \log x)$. Therefore:

$$\psi(x) = \sum_{m=1}^{\infty} \nu(x^{\frac{1}{m}}) = \nu(x) + \sum_{2 \leq m} \nu(x^{\frac{1}{m}}) = \nu(x) + O(x^{\frac{1}{2}} \log x) O(\log x) \tag{1.9}$$

$$= \nu(x) + O(x^{\frac{1}{2}} (\log x)^2).$$

□

Theorem 1.10. *There exist constants A and A' such that for large $x \in \mathbb{N}$ we have $Ax < \psi(x) < A'x$. Therefore $\psi(x) = \Theta(x)$.*

Proof. By the binomial theorem, $\binom{2m+1}{m} = \frac{(2m+1)!}{m!(m+1)!} = \frac{(2m+1)(2m)\ldots(m+2)}{m!}$. Set M equal to this expression. Then since M occurs twice as a term of the expansion of $(1+1)^{2m+1}$, $2M < 2^{2m+1}$. Thus, $M < 2^{2m}$, and $\log M < 2m \log 2$. Now, it's obvious that if $m+1 < p \leq 2m+1$, then p must be equal to only one of the terms in the numerator of M. Thus the product of all primes satisfying the previous inequality must divide the numerator of M, so $\prod_{m+1<p\leq 2m+1} p \leq M$. We have,

$$\nu(2m+1) - \nu(m+1) = \sum_{m+1<p\leq 2m+1} \log p \leq \log M < 2m \log 2. \tag{1.11}$$

Now, for $n = 1$ or $n = 2$, it is clear that $\nu(n) < 2n \log 2$. To show this inequality holds for all $n \in \mathbb{N}$, we proceed by induction. Assume it holds for all $n \leq x-1$ for some natural number $x > 2$. If x is even, then x is not prime. Thus $\nu(x) = \nu(x-1)$ and $\nu(x) < 2(x-1)\log 2 < 2x \log 2$. On the other hand, if x is odd, then for some n we have $x = 2n+1$. By our previous work, we see that:

$$\nu(x) = \nu(2n+1) - \nu(n+1) + \nu(n+1) < 2n \log 2 + 2(n+1)\log 2 = 2x \log 2. \tag{1.12}$$

Note that we were able to bound $\nu(n+1)$ because $n+1 < x$, so our induction held up to $n+1$ by assumption. Since we have shown the inductive step holds for x odd or even, $\nu(n) < 2n \log 2 = O(n)$, which in turn implies $\psi(n) = O(n)$. We will now show that the bound is tight. Recall that $x! = \prod_p p^{\lfloor x/p \rfloor + \lfloor x/p^2 \rfloor + \lfloor x/p^3 \rfloor + \cdots}$. Let $N = \binom{2n}{n} = \frac{2n!}{n!^2}$. We can write this as the product: $\prod_p p^{\sum_{i=1}^{\infty} \lfloor \frac{2n}{p^i} \rfloor - 2\lfloor \frac{n}{p^i} \rfloor}$. Notice that each term of the sum is either one or zero; this follows from simple properties of the floor function. Furthermore, every term of the sum is zero after p^i exceeds $2n$. Therefore:

$$\sum_{i=1}^{\infty} \lfloor \frac{2n}{p^i} \rfloor - 2\lfloor \frac{n}{p^i} \rfloor \leq \sum_{i=1}^{\lfloor \log_p 2n \rfloor} 1 = \lfloor \log_p 2n \rfloor = \lfloor \frac{\log 2n}{\log p} \rfloor. \tag{1.13}$$

We see that

$$\log N = \sum_{p \leq 2n} \left(\sum_{i=1}^{2n} (\lfloor \frac{2n}{p^i} \rfloor - 2\lfloor \frac{n}{p^i} \rfloor) \right) \log p \leq \sum_{p\leq 2n} \lfloor \frac{\log 2n}{\log p} \rfloor \log p = \psi(2n). \tag{1.14}$$

Finally, we see that $N = \frac{n+1}{1}\frac{n+2}{2}\ldots\frac{2n}{n}$. Observe each term in this product is greater than or equal to 2 and there are n terms. Thus we have $2^n \leq N$ and $n \log 2 \leq \log N \leq \psi(2n)$. If $1 \leq n = \lfloor \frac{1}{2} x \rfloor$ we see that:

$$\frac{1}{4} x \log 2 \leq n \log 2 \leq \psi(2n) \leq \psi(x). \tag{1.15}$$

Therefore, $Ax < \psi(x) < A'x$. ∎

Corollary 1.16. *$\frac{\psi(x)}{x}$ is bounded.*

Proof. By Theorem 1.10, for natural numbers x we have $0 \leq \psi(x) = O(x)$. Therefore there exists an M such that for all $x > x_0$ we have $\psi(x) \leq Mx$. Division by nonzero x gives $\frac{\psi(x)}{x} \leq M$ for $x > x_0$. For the remaining number of finite natural numbers n less than or equal to x_0, $\psi(n) < \infty$. Therefore $\frac{\psi(x)}{x}$ is bounded. □

We now wish to restate the Prime Number Theorem in the language of a function whose properties are now familiar to us, $\psi(x)$.

Theorem 1.17. *The Prime Number Theorem is equivalent to* $\lim_{x\to\infty} \frac{\psi(x)}{x} = 1$.

Proof. We must show that $\lim_{x\to\infty} \frac{\psi(x)}{x} = \lim_{x\to\infty} \frac{\pi(x)}{x/(\log(x))}$. For that, we take the difference:

(1.18)
$$\left|\frac{\pi(x)}{x/(\log(x))} - \frac{\psi(x)}{x}\right| = \left|\frac{\sum_{p\leq x}\log(x)}{x} - \frac{\sum_{p\leq x}\lfloor\frac{\log x}{\log p}\rfloor \log p}{x}\right| \leq \frac{1}{x}\sum_{p\leq x}\log(x) - \log(p).$$

Now, fix a $\epsilon > 0$. We split the sum as follows:

(1.19)
$$\frac{1}{x}\sum_{p\leq x}\log(x) - \log(p) = \frac{1}{x}\sum_{p\leq x^{1-\epsilon}}\log(x) - \log(p) + \frac{1}{x}\sum_{x^{1-\epsilon}<p\leq x}\log(x) - \log(p).$$

Now for the first term, we note that $-\log p < 0$ and there are at most $x^{1-\epsilon}$ terms in the sum. Thus, $\frac{1}{x}\sum_{p\leq x^{1-\epsilon}}\log(x) - \log(p) \leq \frac{1}{x}x^{1-\epsilon}\log x = \frac{\log x}{x^\epsilon} = o(1)$. For the latter term, we note that for some constant c, $\log x - \log p < \log x - \log x^{1-\epsilon} = \epsilon\log x = c\epsilon\log p$. Namely we can take $c = c(\epsilon) = \frac{1}{(1-\epsilon)}$. Hence:

(1.20)
$$\frac{1}{x}\sum_{x^{1-\epsilon}<p\leq x}\log(x) - \log(p) \leq \frac{1}{x}\sum_{x^{1-\epsilon}<p\leq x}\epsilon c(\epsilon)\log p = \epsilon c(\epsilon)\sum_{x^{1-\epsilon}<p\leq x}\frac{\log p}{x} \leq \epsilon c(\epsilon)\frac{\psi(x)}{x}.$$

Thus $|\frac{\pi(x)}{x/(\log(x))} - \frac{\psi(x)}{x}| \leq o(1) + \epsilon c(\epsilon)\frac{\psi(x)}{x}$, and by applying Corollary 1.16, $|\frac{\pi(x)}{x/(\log(x)} - \frac{\psi(x)}{x}| \leq o(1) + \epsilon c(\epsilon)M$. Note that as ϵ goes to 0, $c(\epsilon)$ goes to 1. Since ϵ was arbitrary, for any positive quantity δ, we can pick ϵ close enough to 0 such that $|\lim_{x\to\infty}\frac{\pi(x)}{x/(\log(x))} - \frac{\psi(x)}{x}| < \delta$ which implies $\lim_{x\to\infty}\frac{\psi(x)}{x} = 1$. Conversely, assuming $\lim_{x\to\infty}\frac{\psi(x)}{x} = 1$ implies $\lim_{x\to\infty}\frac{\pi(x)}{x/(\log(x))} = 1$. Thus PNT is equivalent to $\lim_{x\to\infty}\frac{\psi(x)}{x} = 1$. □

This momentarily concludes our study of $\psi(x)$, and we now turn our attention to the Riemann Zeta Function.

2. The Riemann Zeta Function

Definition 2.1. Dirichlet Series and $\zeta(z)$

A Dirichlet series is a series of the form $F(z) = \sum_{n=1}^{\infty}\frac{\alpha_n}{n^z}$. In particular, we will be interested in the simplest to define infinite Dirichlet series, the Riemann zeta function: $\zeta(z) = \sum_{n=1}^{\infty}\frac{1}{n^z}$.

Definition 2.2. Re(z)

Let $z \in \mathbb{C}$. We may write z as $a + bi$. Then we define $\text{Re}(z)$ to be the fuction such that $\text{Re}(z) = a$.

Definition 2.3. Complex Logarithm

For $z \in \mathbb{C}$, $\log z = w$ iff $z = e^w$.

Though not apparent at first glance, the Prime Number Theorem is deeply connected to the series $\sum_{n=1}^{\infty} \frac{1}{n^z}$. We will see that PNT is strongly related to the fact that the zeta function has no zeroes for values of z with real part 1, and thus showing this will be crucial to our proof. We begin with a short investigation of some properties of complex functions and relate the zeta function to a product of primes.

The first thing we need in order to prove the function $\zeta(z)$ has no zeroes for z with real part 1 is to examine some properties of the complex logarithm.

Lemma 2.4. $\log|z| = \text{Re}(\log z)$ *and* $|e^z| = \text{Re}(e^z)$ *for* $z \in \mathbb{C}$.

Proof. Let z be a complex number such that $z = a + bi$. For some r and angle θ, we can write z in polar coordinates as $re^{i\theta}$. Thus $\log z = \log re^{i\theta} = i\theta + \log r$. Recall that r is simply a distance defined by the hypotenuse of a right triangle with a base a units in the real direction and a side of b units in the imaginary direction. Therefore, $r = \sqrt{a^2 + b^2}$. Now, consider $\log|z|$. By the definition of absolute value, we have:

$$\log|z| = \log\sqrt{a^2 + b^2} = \log r = \text{Re}(\log w). \tag{2.5}$$

Thus, the logarithm of an absolute value of a complex number is just the real part of the log. Similarly, consider $|e^z| = |e^{a+bi}|$. By Euler's formula, we have $e^{bi} = \cos b + i \sin b$. Thus we have:

$$|e^{a+bi}| = |e^a e^{bi}| = |e^a|\sqrt{\cos b^2 + \sin b^2} = |e^a|. \tag{2.6}$$

But for real numbers $e^a > 0$ so $|e^z| = \text{Re}(e^z)$. □

Lemma 2.7. *The Riemann Zeta function converges for* $z \in \mathbb{C}$ *satisfying* $\text{Re}(z) > 1$.

Proof. Let $z = a + bi$ be as in the statement of the lemma. By definition $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$. Let $\epsilon > 0$ be given. We need to show that there exists $m > 0$ such that for all $n \geq m$ we have $|\sum_{n=m}^{\infty} \frac{1}{n^z}| < \epsilon$. By the triangle inequality, we have:

$$|\sum_{n=m}^{\infty} \frac{1}{n^z}| \leq \sum_{n=m}^{\infty} |\frac{1}{n^z}| = \sum_{n=m}^{\infty} \frac{1}{n^a} < \int_m^{\infty} \frac{1}{x^a} dx. \tag{2.8}$$

But for $m > 0$, the integral in the above inequality is convergent. Therefore there must exist an m satisfying $\int_m^{\infty} \frac{1}{x^a} dx < \epsilon$. □

Theorem 2.9. *For* $\text{Re}(z) > 1$, $\prod_p \frac{1}{1-p^{-z}}$ *converges and is equal to* $\sum_{n=1}^{\infty} \frac{1}{n^z}$.

Proof. Let $z = a + bi$ be such that $\text{Re}(z) > 1$. Because every prime is greater than 1, we have the following:

$$\frac{1}{1 - p^{-z}} = 1 + p^{-z} + p^{-2z} + p^{-3z} + \cdots \tag{2.10}$$

Let $\mathbb{P}_i$ be the series generated from the ith prime. Now consider the product of the first j such series. We may write this as $\prod_{i=2}^{j} \mathbb{P}_i$. Before simplifying, this product is equal to the sum of complex numbers in the form of $n^{-z} = 2^{-\alpha_2 z} 3^{-\alpha_3 z} 5^{-\alpha_2 z} \cdots p_j^{-\alpha_j z}$, where each α is some whole number greater than or equal to 0 and n is some natural number. Due to the fundamental theorem of arithmetic each term of the series is distinct. Furthermore, we note that a number in the base of $n^{-\alpha_i z}$ will be a term of this series iff n has no prime factors greater than p_j. We now define the following index set:

(2.11)
$$(\mathbb{W}_j) = \{n \in \mathbb{N} |\ \text{n has a prime factorization with no prime greater than } p_j\}.$$

Therefore the product of the series generated by the first j primes is as such:

$$\prod_{i=1}^{j} \frac{1}{1 - p^{-z}} = \prod_{i=1}^{j} \mathbb{W}_i = \sum_{(\mathbb{W}_j)} n^{-z}. \tag{2.12}$$

Finally, note that the index set $(\mathbb{W}_j)$ contains every natural number up to p_j in addition to infinitely more. Thus we have the following inequality:

(2.13)
$$\left| \sum_{i=1}^{\infty} n^{-z} - \sum_{(\mathbb{W}_j)} n^{-z} \right| \leq \left| \sum_{n=p_j+1}^{\infty} n^{-z} \right| \leq \sum_{n=p_j+1}^{\infty} \left| n^{-z} \right| = \sum_{n=p_j+1}^{\infty} \left| \frac{1}{e^{z \log n}} \right| = \sum_{n=p_j+1}^{\infty} \left| \frac{1}{n^a} \right|.$$

By applying the comparison test to $\int_1^\infty \frac{1}{x^a} dx$ with $a > 1$, the rightmost sum converges to 0 as p_j tends towards ∞. We conclude $\sum_{n=1}^{\infty} \frac{1}{n^z} = \prod_p \frac{1}{1-p^{-z}}$. □

We may now proceed with showing that $\zeta(z) \neq 0$ for any z with real part greater than 1.

Corollary 2.14. *For z with real part greater than 1 we have $\zeta(z) \neq 0$.*

Proof. Notice that $\zeta(z) \prod_p (1 - p^{-z}) = 1$ for all z in this region. We know $\zeta(z)$ is convergent for $\mathrm{Re}(z) > 1$, so the first term of this product is finite. Moreover, we have $\prod_p (1 - p^{-z})$ is finite by Theorem 2.9. Thus, if $\zeta(z)$ was 0, we'd have a contradiction since a finite value times 0 would be 0 and $0 \neq 1$. □

But what about $\zeta(z)$ for z with real part 1? Interestingly enough, $\zeta(z)$ has no zeroes for these values either, but to prove this we are going to need to do some analysis. In fact, this critical theorem is equivalent to the Prime Number Theorem! Hence showing $\zeta(z) \neq 0$ for z such that $\mathrm{Re}(z) = 1$ will be essential to our goal of proving PNT. Notice that our current definition of ζ fails to converge when we try to evaluate z with real part less than or equal to 1. Therefore, our first step must be finding an analytic continuation for ζ to investigate its behavior in this part of the complex plane. By showing that $\zeta(z) - \frac{1}{z-1}$ has an analytic continuation on $\mathbb{C}$, we will show $\zeta(z)$ has an analytic continuation with a singularity at $z = 1$. We will first require a couple of lemmata in order to find an analytic extension for $\zeta(z) - \frac{1}{z-1}$. But first we will state, though not prove, the following theorem about convergence from Vitali.

Theorem 2.15. *Let $\{f_n\}$ be a locally bounded sequence of analytic functions in a domain Ω such that $\lim_{n\to\infty} f_n(z)$ exists for each z belonging to a set $E \subset \Omega$ which has an accumulation point in Ω. Then $\{f_n\}$ converges uniformly on compact subsets of Ω to an analytic function.*

Using this we may prove the following theorem.

Theorem 2.16. *$\zeta(z) - \frac{1}{z-1}$ has an analytical continuation for $\mathrm{Re}(z) > 0$.*

Proof. Recall the partial summation formula:

$$\sum_{k=m}^{n} f_k(g_{k+1} - g_k) = (f_{n+1}g_{n+1} - f_m g_m) - \sum_{k=m}^{n} (f_{k+1} - f_k)g_{k+1} \tag{2.17}$$

where f and g are sequences. Now, let f_k be k and $g_k = \frac{1}{k^z}$. Let $\mathrm{Re}(z) > 1$. We see that:

$$\sum_{k=1}^{n-1} k((k+1)^{-z} - k^{-z}) = \frac{1}{n^{z-1}} - 1 - \sum_{k=1}^{n-1} \frac{1}{(k+1)^z}. \tag{2.18}$$

Notice that each term of the left hand side is:

$$k((k+1)^{-z} - k^{-z}) = -kz \int_k^{k+1} t^{-z-1}dt = -z \int_k^{k+1} \lfloor t \rfloor t^{-z-1}dt. \tag{2.19}$$

Thus we have:

$$\begin{aligned} \sum_{k=1}^{n} \frac{1}{k^z} = 1 + \sum_{k=1}^{n-1} \frac{1}{(k+1)^z} &= \frac{1}{n^{z-1}} - \sum_{k=1}^{n-1} -z \int_k^{k+1} \lfloor t \rfloor t^{-z-1}dt \\ &= \frac{1}{n^{z-1}} + z \int_1^{n} \lfloor t \rfloor t^{-z-1}dt. \end{aligned} \tag{2.20}$$

Let n tend towards ∞ pointwise to obtain:

$$\zeta(z) = z \int_1^{\infty} \lfloor t \rfloor t^{-z-1}dt. \tag{2.21}$$

Furthermore, consider the following:

$$\frac{z}{z-1} + 1 = z \int_1^{\infty} t^{-z}dt \tag{2.22}$$

and,

$$\zeta(z) - \frac{1}{z-1} = 1 + z \int_1^{\infty} t^{-z-1}(\lfloor t \rfloor - t)dt. \tag{2.23}$$

Recall from complex analysis that for $s > 0$:

$$|\int s^z ds| \leq \int s^{Re(w)}ds. \tag{2.24}$$

We may apply this to obtain the following inequality:

$$|\int_1^n (\lfloor t \rfloor - t)t^{-z-1}dt| \le \int_1^n t^{Re(-z-1)}dt \le \int_1^\infty t^{Re(-z-1)}dt = \frac{1}{Re(z)}. \tag{2.25}$$

Now consider the sequence of analytic and single valued functions $f_n(z) = \int_1^n (\lfloor t \rfloor - t)t^{-z-1}dt$ for $z > 0$. We have shown there exists an M such that $|f_n(z)| \le M$ for all n, z on compact subsets. Then by Vitali's Convergence Theorem, $\lim_{n\to\infty} f_n(z) = f(z) = \int_1^\infty (\lfloor t \rfloor - t)t^{-z-1}dt$ is an analytic function. Thus, $1 + z\int_1^\infty t^{-z-1}(\lfloor -t \rfloor - \lfloor t \rfloor)dt$ is an analytic function and moreover is the analytic extension of $\zeta(z) - \frac{1}{z-1}$. This provides us with a way to evaluate ζ on the line $Re(z) = 1$. □

The reader may be wondering what all of this has to do with $\psi(x)$, as the investigation as of yet does not seem to have any relationship with section 1. This could not be further from the truth. We will see in our next theorem there exists a stunning relationship between $\psi(x)$ and the logarithmic derivative of the Riemann zeta function.

Theorem 2.26. *For* $\mathrm{Re}(z) > 1$ *the logarithmic derivative of* $\zeta(z)$ *is* $\frac{\zeta'(z)}{\zeta(z)} = -z\int_1^\infty \psi(x)x^{-z-1}dx$*.*

To prove this, we will unsurprisingly need to use the formula for the logarithmic derivative of a product.

Lemma 2.27. *Let* f_n *be a sequence of functions such that each* f_n *belongs to* $A(\Omega)$ *and each* $\sum |f_n|$ *converges uniformly on compact subsets of* Ω*. Then for every point where* $g(z) = \prod_{n=1}^\infty (1+f_n(z))$ *is nonzero, we have* $\frac{g'(z)}{g(z)} = \sum_{n=1}^\infty f_i'(x)\frac{\prod_{i\ne k}^j (1+f_i(x))}{\prod_{i=1}^j (1+f_i(x))}$*.*

Proof. Let $g_j(z) = \prod_{i=1}^j (1 + f_i(z))$. Recall the product rule for a collection of functions, f_1 through f_j:

$$\frac{g_j'(z)}{g_j(z)} = \frac{\frac{d}{dx}\prod_{i=1}^j (1 + f_i(x))}{\prod_{i=1}^j (1 + f_i(x))} = \sum_{n=1}^j f_i'(x)\frac{\prod_{i\ne k}^j (1 + f_i(x))}{\prod_{i=1}^j (1 + f_i(x))}. \tag{2.28}$$

Because $g_j(z)$ is nonzero, we are allowed to divide with impunity. Since g_n converges uniformly to g because each g_n is analytic, g_n' converges uniformly to g' on each compact subset of Ω. Therefore, letting j tend towards infinity, we have $\frac{g_(' z)}{g_(z)} = \sum_{n=1}^\infty f_i'(x)\frac{\prod_{i\ne k}^j (1+f_i(x))}{\prod_{i=1}^j (1+f_i(x))}$. □

Proof. With that in mind, it is evident that we must take the derivative of a single term of the product $\prod_p \frac{1}{1-p^{-z}}$ to calculate the derivative of the whole product. An easy calculation shows that $\frac{d}{dz}\frac{1}{1-p^{-z}} = -\frac{p^z \log p}{(1-p^z)^2}$. Now we just plug into our formula to get:

$$\begin{aligned}\frac{\zeta'(z)}{\zeta(z)} &= \sum_p -\frac{p^z \log p}{(1-p^z)^2}\frac{\prod_{q\in\mathcal{P}:q\neq p}\frac{1}{1-q^{-z}}}{\zeta(z)}\\ &= \sum_p -\frac{p^z \log p}{(1-p^{-z})^2}\zeta(z)(1-p^{-z})\frac{1}{\zeta(z)}\\ &= \sum_p -\frac{p^z \log p}{1-p^{-z}}.\end{aligned} \tag{2.29}$$

Recall the following fact about infinite series: If $|r| < 1$ then $\sum_{k=1}^{\infty}\frac{r}{1-r}$ converges. We take p^z to be r. Then we have $-\sum_p \frac{p^z \log p}{1-p^z} = -\sum_p(\sum_{k=1}^{\infty} p^{-zk})\log p$. We note that this is the sum over all ordered pairs of prime numbers p and natural numbers k. Thus we can change the index set as follows:

$$-\sum_p(\sum_{k=1}^{\infty} p^{-zk})\log p = -\sum_{(p,k):k\in\mathbb{N}} p^{-zk}\log p. \tag{2.30}$$

Recall the definition of $\Lambda(x)$ in section 1. Now let $t = p^k$ for some natural number k and prime number p. Then we have:

$$-\sum_{(p,k):k\in\mathbb{N}} p^{-zk}\log p = -\sum_{t=1}^{\infty} t^{-z}\Lambda(t). \tag{2.31}$$

We'd like to once again use equation (2.17). Sadly, there is no difference in this series yet. However, using the relationship between $\psi(x)$ and $\Lambda(x)$ we can express the series as such: $\psi(x) = \sum_{n\leq x}\Lambda(n)$, so $\psi(x)-\psi(x-1) = \sum_{x-1<n\leq x}\Lambda(n) = \Lambda(x)$. Therefore for some large natural number L:

(2.32)

$$\begin{aligned}-\sum_{t=1}^{L} t^k(\psi(t)-\psi(t-1)) &= -((\psi(L)(L+1)^{-z}-0)-(\sum_{t=1}^{L}((t+1)^{-z}-t^{-z})\psi(t)))\\ &= -\psi(L)(L+1)^{-z}-\sum_{t=1}^{L}(t^{-z}-(t+1)^{-z})\psi(t).\end{aligned}$$

Since we know $\psi(x)$ is of order x from before, and $Re(z) > 1$, the first term goes to zero as L tends towards infinity. We can write the term $(t^{-z}-(t+1)^{-z})$ inside the sum as the integral $z\int_t^{t+1} x^{-z-1}dx$. Moreover, because over an interval of 1 unit, $\psi(x)$ is constant, we may move it inside the integral. Hence:

$$\begin{aligned}-\sum_{t=1}^{\infty}(t^{-z}-(t+1)^{-z})\psi(t) &= -\sum_{t=1}^{\infty} z\int_t^{t+1}\psi(x)x^{-z-1}dx\\ &= -z\int_1^{\infty}\psi(x)x^{-z-1}dx.\end{aligned} \tag{2.33}$$

This is what was desired. □

Before showing that $\zeta(1+bi) \neq 0$, we examine the behavior of ζ as we approach 1 from above by real numbers.

Lemma 2.34. *For $a \in \mathbb{R}$ and $a > 1$, $\zeta(a) - \frac{1}{a-1} = O(1)$.*

Proof. It is helpful to consider $\zeta(u)$ in the rather silly integral form:

$$\zeta(a) = \sum_1^\infty \int_n^{n+1} (n^{-a}) dx. \tag{2.35}$$

Now, we add and subtract the same term to the right hand side to get:

$$\zeta(a) = \sum_1^\infty \int_n^{n+1} (n^{-a} - x^{-a}) dx + \int_1^\infty x^{-a} dx. \tag{2.36}$$

We'd like to integrate the final term. Notice: $\int_1^\infty x^{-a}dx = 0 - \frac{1}{-a+1} = \frac{1}{a-1}$ since $a > 1$. So to finish our lemma, we must simply show $\sum_1^\infty \int_n^{n+1}(n^{-a} - x^{-a})dx = O(1)$. First we bound the term, recalling first that we restricted x such that it satisfied $n < x < n+1$ and have $a > 1$. Thus we can use the following series of inequalities to help bound the term:

$$n^{-a} - x^{-a} < \int_n^x at^{-a-1} dt < \frac{a}{n^2}. \tag{2.37}$$

But then $\sum_1^\infty \int_n^{n+1}(n^{-a} - x^{-a})dx$ is less than $a\sum_1^\infty n^{-2}$, a convergent sum. Therefore we have $\zeta(a) - \frac{1}{a-1} = O(1)$. This implies that ζ has a simple pole of order 1 at $a = 1$, a fact that will be crucial in the next proof. □

Finally, we may show that ζ is zero free on the line $Re(z) = 1$.

Theorem 2.38. *$\zeta(z) \neq 0$ for $z = 1 + bi$ where $b \in \mathbb{R}$.*

Proof. Our proof will consists of three parts: first we will use our experience with the complex logarithm to come up with a new function $h(z)$ related to $\zeta(z)$, show $h(z)$ obeys certain inequalities, and then assume $\zeta(z)$ has a zero for some z with real part 1 and use limits to derive a contradiction. First, let $z = a + bi$, and use the Euler product to express $\log \zeta(z)$ as:

$$\log(\zeta(z)) = \log(\prod_p (1 - p^{-z})^{-1}) = -\sum_p \log(1 - p^{-z}) \tag{2.39}$$

where the sum exists iff $\prod_p (1 - p^{-z})^{-1}$ exists. Recall from analysis the Taylor series expansion for $\log(1 - x)$. This is $-\sum_{k=1}^\infty \frac{(-1)^k(-x)^k}{k}$. We substitute this in for $\log(1 - p^{-z})$ to get:

$$-\sum_p \log(1 - p^{-z}) = \sum_p \sum_{k=1}^\infty \frac{(p^{-z})^k}{k}. \tag{2.40}$$

Now we apply our previous lemma about complex logarithms composed with absolute values to see that $\log|\zeta(z)|$ is equivalent to:

$$\text{(2.41)} \qquad \mathrm{Re}(\sum_p \sum_{k=1}^{\infty} \frac{(p^{-z})^k}{k}) = \mathrm{Re}(\sum_p \sum_{k=1}^{\infty} \frac{(p^{-ak-bik})}{k}) = \sum_p \sum_{k=1}^{\infty} \frac{\cos(bk \log p)}{kp^{ak}}.$$

Now we define our auxiliary function $\alpha(a+bi) = \log|\zeta(a+2bi)| + 3\log|\zeta(a)| + 4\log|\zeta(a+bi)|$ where a, b are real numbers. Conveniently, we can use equation (2.41) to see the equality between:

$$\text{(2.42)} \qquad \alpha(a+bi) = \sum_p \sum_{k=1}^{\infty} \frac{\cos(2bk\log p) + 3 + 4\cos(bk \log p)}{kp^{ak}}.$$

This function may seem to be pulled out of thin air, but in fact we can see that it is the result of some elementary trigonometry. Notice for all θ, we have the following: $0 \leq 2(1+\cos(\theta))^2 = 2 + 4\cos\theta + 2(\cos\theta)^2 = 3 + 4\cos\theta + \cos 2\theta$, from the use of a double angle formula. Setting bk as our θ, we see that $\alpha(a+bi)$ has the very important property that $0 \leq \alpha(a+bi)$ for all a where $\zeta(a)$ is convergent. Since we are trying to show that $\zeta(z)$ is zero free for z with real part 1, it doesn't help us too much if $0 \leq \alpha(a+bi)$. An easy fix to this is to let both sides of this inequality be exponents of Euler's number. Thus we have:

(2.43)

$$e^0 = 1 \leq e^{\alpha(a+bi)} = e^{\log|\zeta(a+2bi)| + 3\log|\zeta(a)| + 4\log|\zeta(a+bi)|} = |\zeta(a+2ib)||\zeta(a)|^3|\zeta(a+bi)|^4.$$

Finally, we have everything in order to set up our contradiction. Here we will use our lemma that examines the behavior $\zeta(a)$ as a approaches 1 to show that $\zeta(1+bi)$ could not possibly be 0. Assume for the sake of contradiction there exists some bi such that $\zeta(1+bi) = 0$. We know from our previous study of $\zeta(a)$ that it behaves like $\frac{1}{a-1} + \gamma$ as a approaches 1. Hence, if we multiply it by $a-1$, it will behave like 1 as it approaches 1. Thus if $b \neq 0$:

(2.44)

$$\begin{aligned}
\lim_{a\to 1^+} \frac{1}{a-1}|\zeta(a+2bi)||\zeta(a)|^3|\zeta(a+bi)|^4 &= \lim_{a\to 1^+} |\zeta(a+2bi)||\zeta(a)(a-1)|^3 \left|\frac{\zeta(a+bi)}{a-1}\right|^4 \\
&= \lim_{a\to 1^+} |\zeta(a+2bi)| \left|\frac{\zeta((a-1)+1+bi)}{a-1} - \frac{\zeta(1+bi)}{-1}\right|^4 \\
&= \lim_{a\to 1^+} |\zeta(a+2bi)||\zeta'(1+bi)| \\
&= L < \infty.
\end{aligned}$$

The fact that the limit must be finite follows from our analytic continuation of $\frac{\zeta'(z)}{\zeta(z)} - \frac{1}{z-1}$. However, since we have $a > 1$, there is the following contradiction: $\lim_{a\to 1^+} \frac{1}{a-1} \leq \lim_{a\to 1^+} \frac{1}{a-1}|\zeta(a+2bi)||\zeta(a)|^3|\zeta(a+bi)|^4 = L$ which is absurd. We conclude that $\zeta(z) \neq 0$ for $z = 1+bi$ with $b \neq 0$. $\square$

3. Some Integral Transforms

We are nearing the end of our proof; however, we will require a bit more complex analysis to finish the proof of the prime number theorem. We will prove Cauchy's Integral Formula, and then apply it to two integral transformations to show that $\frac{\psi(x)}{x}$ goes to 1 as x tends towards infinity, thus proving the Prime Number Theorem. Now we turn our attention to stating Cauchy's Integral Theorem and proving Cauchy's Integral formula.

Theorem 3.1. *Let U be an open, simply connected subset of $\mathbb{C}$. Let $f : U \to \mathbb{C}$ be a holomorphic function, and let γ be a rectifiable path i.e. a curve with finite length in U whose initial point is equal to its end point. Then $\oint_\gamma f(z)dz = 0$.*

Theorem 3.2. *Let U be an open subset of the complex plane, let f be analytic such that $f : U \to \mathbb{C}$, and let C be a closed set contained in U. If we let γ be the boundary of C, then we have for all z in the interior of C:*

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(t)}{t-z} dt. \tag{3.3}$$

Proof. Let D_r be a disc with radius r about z such that r is small enough to be completely contained by the closed set C and be such that within the disk $|f(t) - f(z)| < \epsilon$ for all t in the interior of D. Eventually we will take the limit as r tends towards 0. We wish to evaluate $\oint_C \frac{f(t)}{t-z} dt$. Because f is analytic, the contour integral along D_r is the same as the contour integral along C by Cauchy's Integral Theorem. Thus we have:

$$\frac{1}{2\pi i} \oint_D \frac{f(t)}{t-z} dt = \frac{1}{2\pi i} \oint_{D_r} \frac{f(t)}{t-z} dt. \tag{3.4}$$

We split the above integral on the right hand side in two by adding and subtracting a term:

$$\frac{1}{2\pi i} \oint_{D_r} \frac{f(t)}{t-z} dt = \frac{1}{2\pi i} \oint_{D_r} \frac{f(t) - f(z)}{t-z} + f(z) \frac{1}{2\pi i} \oint_{D_r} \frac{1}{t-z} dt. \tag{3.5}$$

Our task is to show that the right integral is in fact $f(a)$. This can be seen by letting $t - z = re^{i\theta}$. Then by substitution the right integral is $\int_0^{2\pi} \frac{ire^{i\theta}}{re^{i\theta}} d\theta = 2\pi i$, and everything cancels nicely and does not depend on the radius r. This just leaves us with showing the first term tends towards 0 as r tends towards 0, but this holds because:

$$|\frac{1}{2\pi i} \oint_{D_r} \frac{f(t) - f(z)}{t-z}| \leq \frac{1}{2\pi} \oint_{D_r} \left| \frac{\epsilon dt}{r} \right| = \epsilon. \tag{3.6}$$

This completes the proof. □

Definition 3.7. Laplace Transform

Let $f(t)$ be defined on $t \in [0, \infty)$. The Laplace Transform of $f(t)$, denoted by $\mathcal{L}f(t)$, is the function $G(z) = \int_0^\infty e^{-zt} f(t)dt$, where $G(z)$ is a function of the complex numbers.

Our current goal to prove a theorem by Tauber. A word of caution: it is likely the most difficult part of our proof of the Prime Number Theorem.

Theorem 3.8. *Let $F(t)$ be a piecewise continuous function bounded by 1 defined on $t \in [0,\infty)$. Let $G(z) = \mathcal{L}F(t) = \int_0^\infty F(t)e^{-zt}dt$ for $z \in \mathbb{C}$. Note this function is holomorphic on $Re(t) > 0$. Assume G has an analytic extension to a neighborhood of the imaginary axis, $Re(t) = 0$. Then $\int_0^\infty F(t)dt$ exists and is equal to $G(0)$.*

Proof. Let $G_\alpha(z) = \int_0^\alpha F(t)e^{-zt}dt$. We need to show that $\lim_{\alpha\to\infty} G_\alpha(0) = G(0)$. Let R be large; eventually we will let it tend towards infinity. Let C be the boundary of the region $\{z \in \mathbb{C} : |z| < R, -\theta \le Re(z)\}$, where θ is a function of R such that $G(z)$ is analytic on C. We employ Cauchy's integral formula to estimate the size of $G(0) - G_\alpha(0)$. We have:

$$G(0) - G_\alpha(0) = \frac{1}{2\pi i}\oint_C (G(z) - G_\alpha(z))\frac{1}{z}dz. \tag{3.9}$$

If we define C_+ to be the portion of C with $\mathrm{Re}(t) > 0$ and let $v = \mathrm{Re}(z)$ we have:

$$\left|\frac{G(z) - G_\alpha(z)}{z}\right| = \frac{1}{R}\left|\int_\alpha^\infty F(t)e^{-zt}dt\right| \le \frac{1}{R}\frac{e^{-\alpha v}}{v}. \tag{3.10}$$

Observe that we may modify this equation by replacing the G, G_α with their products with $e^{\alpha z}$ without changing the value of the integral because $G(0)e^{\alpha 0} = G(0)$, and $G_\alpha(0)e^{\alpha 0} = G_\alpha(0)$. Furthermore, we can add $\frac{1}{R^2}\frac{1}{2\pi i}\oint_C (G(z) - G_\alpha(z))zdz$ to the right hand side because $(G(z) - G_\alpha(z))\frac{z}{R^2}$ is analytic and therefore doesn't contribute anything by Cauchy's Theorem as we integrate over C. Thus we have:

$$G(0) - G_\alpha(0) = \frac{1}{2\pi i}\oint_C (G(z) - G_\alpha(z))e^{\alpha z}(1 + \frac{z^2}{R^2})\frac{dz}{z}. \tag{3.11}$$

Suppose $|z| = R$ as it does on C_+. Then $\frac{1}{z} + \frac{z}{R^2} = \frac{2\,\mathrm{Re}(z)}{R^2}$. We see that:

$$|(G(z) - G_\alpha(z))e^{\alpha z}(\frac{1}{z} + \frac{z^2}{R^2})| \le \frac{1}{Re(z)}\frac{2Re(z)}{R^2}e^{-\alpha z}e^{\alpha z} = \frac{2}{R^2}. \tag{3.12}$$

Next we see that the contour integral over C_+ must be bounded due to the ML theorem which states that a contour integral is bounded by a function's maximum over a contour M times the arc length of the contour L. Applying this yields:

$$\frac{1}{2\pi i}\oint_{C_+} |(G(z) - G_\alpha(z))e^{\alpha z}(1 + \frac{z^2}{R^2})\frac{dz}{z}| \le \frac{1}{2\pi i}\frac{2}{R^2}\pi i R = \frac{1}{R}. \tag{3.13}$$

where $\pi i R$ is the arc length of C_+. Clearly as R tends towards infinity, this part of the difference between $G(0)$ and $G_\alpha(0)$ tends towards 0. We must now deal with $C \setminus C_+$ which we will call C_-. First we apply the triangle inequality to this part of the integral:

(3.14)
$$|\frac{1}{2\pi i}\oint_{C_-} (G(z) - G_\alpha(z))e^{\alpha z}(1 + \frac{z^2}{R^2})\frac{dz}{z}| \le |\frac{1}{2\pi i}\oint_{C_-} (G(z)e^{\alpha z}(1 + \frac{z^2}{R^2})\frac{dz}{z}|$$
$$+ |\frac{1}{2\pi i}\oint_{C_-} G_\alpha(z)e^{\alpha z}(1 + \frac{z^2}{R^2})\frac{dz}{z}|.$$

By definition, $|G_\alpha(z)e^{\alpha z}(1+\frac{z^2}{R^2})\frac{1}{z}| = |(\int_0^\alpha F(t)e^{-zt}dt)e^{\alpha z}(1+\frac{z^2}{R^2})\frac{1}{z}|$. Since $G_\alpha(z)$ is an entire function, when we take its contour integral the value is independent of the path and dependent on the initial and end points. Therefore we take the path to again be a semicircle of radius R from iR to $-iR$ in the second and third quadrants of the complex plane. Thus we once more have $\frac{1}{z}+\frac{z}{R^2}=\frac{2Re(z)}{R^2}$. Observe:

$$|(\int_0^\alpha F(t)e^{-zt}dt)e^{\alpha t}\frac{2Re(t)}{R^2}| \leq \frac{1}{|Re(z)|}\frac{2|Re(z)|}{R^2} = \frac{2}{R^2}. \tag{3.15}$$

Applying the ML theorem gives $|\frac{1}{2\pi i}\oint_{C_-} G_\alpha(z)e^{\alpha z}(1+\frac{z^2}{R^2})\frac{dz}{z}| \leq \frac{1}{R}$ which must tend towards 0 as R tends towards infinity. Finally we must find a way to estimate $|\frac{1}{2\pi i}\oint_{C_-}(G(z)e^{\alpha z}(1+\frac{z^2}{R^2})\frac{dz}{z}|$. To do this, we first find a constant B such that $|G(z)| \leq B$ for $z \in C_-$, and we pick a θ' such $0 < \theta' < \theta(R)$. We split the integral into pieces such that the first integral evaluates over $Re(z) \leq -\theta'$ and the other evaluates over $-\theta' < Re(z)$. We can bound the integral corresponding to $Re(z) \leq -\theta'$ by the ML theorem as follows:

(3.16)

$$\begin{aligned}|\frac{1}{2\pi i}\oint_{C_-\cap Re(z)\leq-\theta'}(G(z)e^{\alpha z}(1+\frac{z^2}{R^2})\frac{dz}{z}| &\leq |\frac{1}{2\pi}|B(\frac{1}{\theta(R)}+\frac{1}{R})e^{-\theta'\alpha}\oint_{C_-\cap Re(z)\leq-\theta'}dz \\ &\leq \frac{1}{2\pi}B(\frac{1}{\theta(R)}+\frac{1}{R})\pi Re^{-\theta'\alpha} \\ &= \frac{R}{2}B(\frac{1}{\theta(R)}+\frac{1}{R})e^{-\theta'\alpha}.\end{aligned}$$

Fixing R and θ', this again tends towards 0 as α goes to infinity. This leaves the final interval and the final step of our theorem. Recall we are integrating over the half circle from iR to $-iR$. There are two sections of this arc which satisfy $C_-\cap Re(z) \geq -\theta'$, thus we can bound the arc length of the integral over this set by $2R\arcsin\frac{\theta'}{R}$. Thus we estimate the final integral as follows:

$$|\frac{1}{2\pi i}\oint_{C_-\cap Re(z)\geq-\theta'}(G(z)e^{\alpha z}(1+\frac{z^2}{R^2})\frac{dz}{z}| \leq \frac{1}{2\pi}B(\frac{1}{\theta(R)}+\frac{1}{R})2R\arcsin\frac{\theta'}{R}. \tag{3.17}$$

Thus, by taking θ' arbitrarily close to 0 we can make this final term as small as we please. Let $\epsilon > 0$ be given. Let $R = \frac{4}{\epsilon}$, fix $\theta(R)$ as discussed such that G is analytic inside and on C. Then for sufficiently large α, we have $|G_\alpha(0) - G(0)| < \epsilon$. ☐

Definition 3.18. Mellin transform

Let $f(x)$ be a function defined on $x \in [0,\infty)$. We define the Mellin transform of the function f to be $\eta(s) = s\int_1^\infty f(x)x^{-s-1}dx$.

Theorem 3.19. *Let f be a nonnegative, piecewise continuous and nondecreasing function on $[1,\infty)$ such that $f(x) = O(x)$. Then $\eta(s)$, the Mellin transform of f, exists for $\mathrm{Re}(s) > 1$ and can be written as $\eta(s) = s\int_1^\infty f(x)x^{-s-1}dx$. Moreover η is an analytic function. Assume for some constant c, the function $\eta(s) - \frac{c}{s-1}$ has an analytic extension to a neighborhood of the line $Re(s) = 1$. Then as $x \to \infty$, $\frac{f(x)}{x} \to c$.*

Recall that $\frac{\zeta'(z)}{\zeta(z)}$ is the Mellin transform of $\psi(x)$. Hence, once we prove this penultimate theorem, PNT will be reduced to a series of applications of theorems we've already shown.

Proof. Let η, f be as in the statement of the theorem. Let $H(t) = e^{-t}f(e^t) - c$. H satisfies the first part of the hypothesis of the previous theorem. Therefore, the following Laplace transformation $G(s) = \int_1^\infty (e^{-t}f(e^t) - c)e^{-st}dt$ exists. Using the change of variable formula for $x = e^t$ we have $G(s) = \int_1^\infty (\frac{1}{x}f(x) - c)x^{-s}\frac{dx}{x} = \int_1^\infty f(x)x^{-s-2}dx - \frac{c}{s} = \frac{\eta(s+1)}{(s+1)} - \frac{c}{s}$. Since $\eta(s+1) - \frac{c}{s}$ has an analytic extension to a neighborhood of the line $Re(z) = 0$ so does G. Therefore by our previous theorem $\int_0^\infty H(t)dt$ converges to $G(0)$ because it satisfies the Tauberian theorem. Thus $\int_1^\infty (\frac{f(x)}{x} - c)\frac{dx}{x}$ exists. We stated that f is a nondecreasing function. Suppose for the sake of contradiction there exists x' such that $x' > 0$ and $\lfloor \frac{f(x')}{x'} \rfloor - c \geq 2\epsilon$. Then we have $x(c+\epsilon) \leq x'(c+2\epsilon) \leq f(x') \leq f(x)$ for $x' < x < \frac{c+2\epsilon}{c+\epsilon}x'$. We now take the integral from x' to $\frac{c+2\epsilon}{c+\epsilon}x'$:

$$(3.20) \qquad \int_{x'}^{\frac{c+2\epsilon}{c+\epsilon}x'} (f(x)/x - c)\frac{dx}{x} \geq \int_{x'}^{\frac{c+2\epsilon}{c+\epsilon}x'} (\epsilon + c - c)\frac{dx}{x} = \epsilon \log \frac{c+2\epsilon}{c+\epsilon}.$$

But because the integral $\int_1^\infty (\frac{f(x)}{x} - c)\frac{dx}{x}$ exists, $\int_a^b (\frac{f(x)}{x} - c)\frac{dx}{x}$ must go to 0 as a and b get arbitrarily large. Thus for fixed ϵ letting $x' \to \infty$, we have $\int_{x'}^{\frac{c+2\epsilon}{c+\epsilon}x'} (f(x)/x - c)\frac{dx}{x} < \epsilon \log \frac{c+2\epsilon}{c+\epsilon}$, a contradiction! Therefore, for all sufficiently large x', we have $\lfloor \frac{f(x')}{x'} \rfloor - c \leq 2\epsilon$. Again, assume for the sake of contradiction there exists x^* such that $x^* > 0$ and $\lfloor \frac{f(x^*)}{x^*} \rfloor - c \leq -2\epsilon$. Then for $\frac{c-2\epsilon}{c-\epsilon}x^* \leq x \leq x^*$ we have $f(x) \leq f(x^*) \leq x^*(c-2\epsilon) \leq x(c-\epsilon)$. Now we integrate over x between $\frac{c-2\epsilon}{c-\epsilon}x^*$ and x^*:

$$(3.21) \qquad \int_{\frac{c-2\epsilon}{c-\epsilon}x^*}^{x^*} (f(x)/x - c)\frac{dx}{x} \leq \int_{\frac{c-2\epsilon}{c-\epsilon}x^*}^{x^*} (-\epsilon + c - c)\frac{dx}{x} = \epsilon \log \frac{c-2\epsilon}{c-\epsilon}.$$

Note that the $\log \frac{c-2\epsilon}{c-\epsilon} < 0$ because $c - 2\epsilon < c - \epsilon$. But again, for fixed c, ϵ we have $\int_{\frac{c-2\epsilon}{c-\epsilon}x^*}^{x^*} (f(x)/x - c)\frac{dx}{x} > \epsilon \log \frac{c-2\epsilon}{c-\epsilon}$ for all x^* sufficiently large due to the convergence of the integral. Again we see for large enough x^*, $\frac{\lfloor f(x^*) \rfloor}{x^*} - c \geq -2\epsilon$ for fixed c, ϵ. Therefore $\frac{f(x)}{x} \to c$. □

Now we have the Prime Number Theorem.

Theorem 3.22. $\lim_{x\to\infty} \frac{\pi(x)}{x \log x} = 1$.

Proof. We recall that $-\frac{\zeta'(z)}{\zeta(z)} + \frac{1}{z-1} = z\int_1^\infty \psi(t)x^{-z-1}dx + \frac{1}{z-1}$ has an analytic extension to a region about the line $Re(z) = 1$. Moreover, $\psi(x)$ is $O(x)$, piecewise continuous, and nonnegative. By our previous theorem we have $\frac{\psi(x)}{x} \to 1$ as $x \to \infty$. Finally, we have shown the logical equivalence between the statement $\frac{\psi(x)}{x} \to 1$ as $x \to \infty$ and the Prime Number Theorem. □

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