AN OVERVIEW OF KNOT INVARIANTS

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ABSTRACT. The central question of knot theory is whether two knots are isotopic. This question has a simple answer in the Reidemeister moves, a set of three operations that preserve isotopy and can transform a knot into any isotopic knot. While this characterizes isotopy, it is useless for proving inequivalence. Instead, a number of quantities have been discovered that are isotopy invariant. While these invariants are not perfect, they are powerful tools for distinguishing knots. This paper will describe a number of such invariants, including the knot group, some elementary invariants, and the Jones polynomial.

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1. INTRODUCTION

A large portion of knot theory is devoted to verifying whether or knot two knots are isotopic. This paper provides an overview of several knot invariants used to distinguish between knots that are not isotopic. The Reidemeister moves, a set of diagrammatic operations that completely describe isotopy, are the first ingredients to any discussion of isotopy invariants; they provide a simple way to show that a quantity is invariant over isotopy. Next come two simpler invariants, crossing number and tricolorability. Algebraic topology provides the next invariant: the knot group, defined as the fundamental group of the knot complement. The knot group will be defined, along with an algorithmic method to calculate it. Finally comes the Jones polynomial; discovered by Vaughn Jones, it provides an invariant that is both powerful and easy to calculate. We finish with a brief description of the HOMFLY polynomial, a generalization of the Jones polynomial. This paper assumes basic knowledge of algebraic topology, in particular the concepts of homeomorphism, homotopy, and homotopy group. While by no means comprehensive, this paper describes a number of intuitive, powerful, and useful invariants.

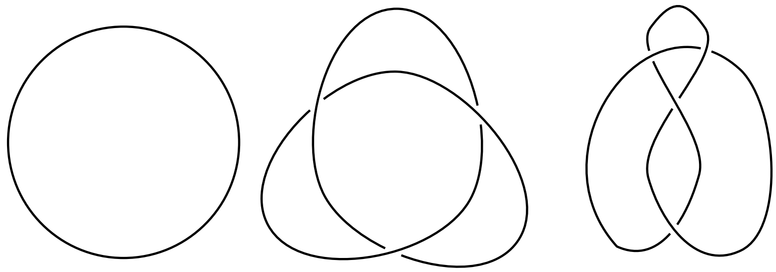


FIGURE 1. Examples of Knots: The Unknot, Trefoil Knot, and Figure-Eight Knot

2. MATHEMATICAL KNOTS AND ISOTOPY

The concept behind a mathematical knot is simple: imagine taking a piece of rope, tying a knot in it, and then sealing the ends together. Formally:

Definition 2.1. A *knot* is an embedding of the circle S^1 into $\mathbb{R}^3$.

Here, instead of a rope, we have the segment $[0,1]$. It is allowed to wrap around itself in R^3, and the points corresponding to 0 and 1 are then identified. A mathematical knot is sometimes represented as an embedding into the sphere S^3 instead; the notions are equivalent.

Intuitively, two knots are equivalent if they can be transformed into the other without untying the knot or self-intersection. We can make that notion mathematically rigorous with the following definition.

Definition 2.2. An *ambient isotopy* mapping a knot K to a knot K' is a continuous map $H : \mathbb{R}^3 \times I \to \mathbb{R}^3$ such that H_0 is the identity map, for each t, H_t is a homeomorphism from $\mathbb{R}^3$ to itself, and $H_1 \circ K = K'$.

Two knots K and K' are *isotopic* if there exists an ambient isotopy between them.

It is easy to verify that this is an equivalence relation; this paper will use the terms 'isotopic' and 'equivalent' interchangeably.

The trivial knot, the embedding $K : S^1 \to \mathbb{R}^3$ sending (x, y) to $(x, y, 0)$ is called the unknot.

Definition 2.3. A n-component link is an embedding of n disjoint circles into $\mathbb{R}^3$.

A knot is simply a 1-component link. The trivial example of a link, the embedding $K : S_n \to \mathbb{R}^3$ sending (x, y) to $(x, y, 0)$ where S_n is the union of n disjoint circles, is called the n-component unlink.

These definitions can lead to problematic results. Consider the knot in Figure 2, where each circle represents some nontrivial knotted area with a segment entering and leaving, decreasing in size exponentially. This is a perfectly valid knot, but it has an infinite number of crossings. Many invariants are impossible to compute on these kinds of knots. To avoid this pathology, we introduce the idea of polygonal knots.

Definition 2.4. A *polygonal knot* is a knot that consists of the union of finitely many line segments in R^3.

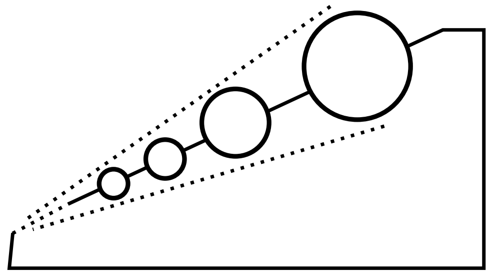


FIGURE 2. A wild knot

Definition 2.5. A knot is *tame* if it is isotopic to a polygonal knot, and a link is tame if all of its components are tame. A knot or link is *wild* if it is not tame.

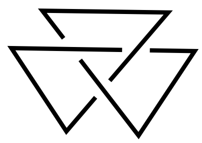


FIGURE 3. A polygonal trefoil knot

For the rest of the paper we will only refer to tame knots.

While knots exist in three-dimensional space, it is useful to consider drawings, as in Figure 1, of knots projected onto a plane. Note that it is impossible to do so without overlapping arcs, unless the knot in question is the unknot.

Definition 2.6. A projection of a knot or a link onto a plane is in *regular position* if no point in the projection is shared by more than two arcs, and no arcs in the image of the embedding lie tangent to each other.

The second condition is to ensure that every shared point is a genuine crossing. These shared points will have different z coordinates in the preimage; that is, one arc has a higher z coordinate and so passes 'over' the other in the preimage of the projection. That arc forms the *overcrossing*, while the arc with a lower z coordinate the *undercrossing*. The arc that passes under the crossing is drawn with a gap, as in Figure1.

Definition 2.7. A *knot diagram* is a projection of a knot into a plane with arcs at crossings differentiated into overcrossings and undercrossings. A *link diagram* is a similar projection of a link.

Proposition 2.8. *Every tame knot or link is isotopic to a knot or link that can be projected into a regular position on the plane.*

A full proof of this fact can be found in [3]; the general idea of the proof is to take an isotopic polygonal knot, and show that the projections that fail to place the knot in regular position are nowhere dense in the set of possible projections. Thus there must exist a valid projection of the knot.

3. Reidemeister Moves

The Reidemeister moves are a useful tool for showing that two knots are isotopic, and for proving that a quantity is isotopy-invariant. The Reidemeister moves reduce the knot isotopy from a complicated topological problem to a diagrammatic one, and enable easily verified proofs of knot isotopy. While they cannot easily prove that two knots are not isotopic, they enable simple proofs that quantities are isotopy invariant, and are thus critical to this paper.

Definition 3.1. The Reidemeister moves are a set of three types of local moves on a link diagram, characterized as follows. They are illustrated in Figure 4.
Type 1: Twisting an arc.
Type 2: Passing one arc over or under another arc.
Type 3: Passing an arc over or under a crossing.

These moves are denoted R_1, R_2, and R_3, respectively.

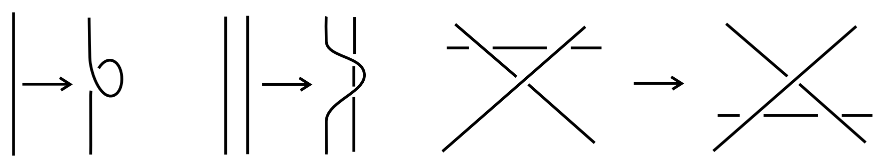


FIGURE 4. Type 1, Type 2, and Type 3 Reidemeister moves.

The importance of the Reidemeister moves lies in the following theorem:

Theorem 3.2. *Two diagrams of links are isotopic if and only if one can be transformed into the other by a finite sequence of Reidemeister moves.*

It is easy to see that all of the Reidemeister moves preserve isotopy. The converse is far less simple, and a proof can be found in Reidemeister's book[1]; a slightly easier proof for polygonal diagrams can be found in [4].

4. Elementary Invariants

Elementary invariants are the simplest knot invariants, assigning a numerical value to knots. This section will define two of them: the crossing number and tricolorability. These invariants are simple and not particularly powerful, but each has uses; the crossing number is used to list and categorize knots, and to speak of the relative complexity of a knot. Tricolorability is a fairly quick way to show that a knot is nontrivial, since we will show that the unknot is not tricolorable.

Definition 4.1. The crossing number of a knot is the minimum number of crossings in a diagram of any isotopic knot.

This is clearly an invariant, because for any knot diagram with the minimum number of crossings for that knot, the Reidemeister moves will either preserve the number of crossings or add more.

The crossing number is primarily useful for categorizing knots; tables that list known knots are often organized by crossing number.

Definition 4.2. A knot is *three-colorable,* or *tricolorable,* if every knot diagram of any isotopic knot admits a coloring of its arcs such that all of its arcs can be colored with three colors such that at every crossing, either all three colors meet or only one color is used. To prevent trivial colorings, all colors must be used at least once for a valid coloring.

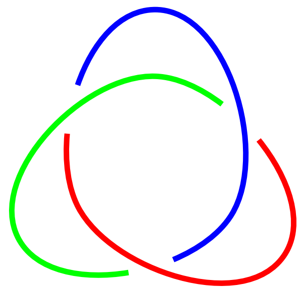


FIGURE 5. A valid coloration of the trefoil knot

Theorem 4.3. *Tricolorability is a knot invariant.*

Proof. An example of the preservation of tricolorability is shown in Figure 6. We see that R_1 preserves tricolorability, since the new arc can be colored with the same color as the arc it was formed out of. Likewise, R_2 preserves tricolorability as illustrated in Figure 6. There are several cases to show that R_3 preserves tricolorability, based on the initial coloring; one is shown below, and the rest resolve themselves similarly. □

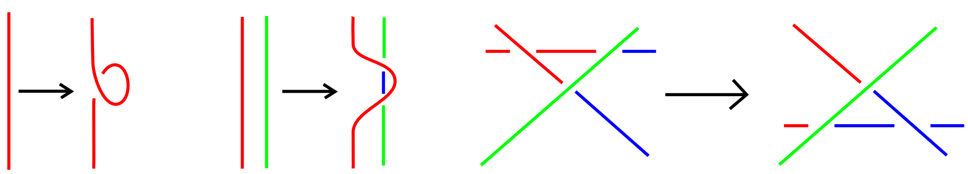


FIGURE 6. Reidemeister moves preserve tricolorability

The only possible coloring of the basic diagram of the unknot is a trivial coloring using only one color; since tricolorability is an isotopy invariant, the unknot is not tricolorable. This provides an easily verified way to distinguish a knot from the unknot; if a three-coloring can be found, the knot is nontrivial.

5. THE KNOT GROUP

Algebraic topology provides a useful tool for analyzing knots. The knot group, the fundamental group of the knot's complement, is a powerful invariant. The Wirtinger Presentation of the knot group provides an algorithmic way to calculate the knot group. Unfortunately, it can be very difficult to verify whether or not the resulting groups of two different knots are isomorphic, so the knot group is limited in its practical use. Nonetheless, it is a powerful invariant.

Definition 5.1. The knot group of a knot K is the fundamental group of the complement of K; that is, the knot group is $\pi_1(\mathbb{R}^3 \setminus K)$.

Theorem 5.2. *The knot group of the unknot is $\mathbb{Z}$; similarly, the knot group of the n-component unlink is the free group on n generators.*

An intuitive sketch of the proof is as follows. Let U be the unknot. Loops in the complement of U are trivial if they do not loop around U. Loops are homotopic if they loop around U the same number of times in the same direction. Thus the fundamental group of the complement of U is $\mathbb{Z}$.

Similarly, for the n-component unlink, loops are homotopic if they loop around each component the same number of times in the same direction. Thus each component of the unlink serves as a generator for the knot group; since the links are completely disjoint, there are no relations between these generators. Thus the knot group of the n-component unlink is the free group on n generators.

Theorem 5.3. *If K and K' are isotopic knots, they have isomorphic knot groups.*

Proof. The isotopy taking K to K' provides a homeomorphism $\mathbb{R}^3 \setminus K$ to $\mathbb{R}^3 \setminus K'$. Since the knot complements are homeomorphic, they have isomorphic knot groups. □

The *Wirtinger presentation* provides an algorithmic way of computing knot groups. Choose an isotopic knot that can be projected into regular position, and choose a base point for the fundamental group of the knot complement that lies above it with respect to this projection. The generators for the Wirtinger presentation are homotopy classes of loops from this point that pass around each arc, while the relations are given by the crossings of the knot.

We give an orientation to the knot; we then orient the loops around it following the right-hand rule, as shown below.

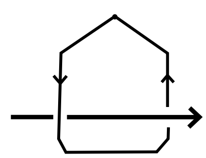


FIGURE 7. Orientation of loops around an arc

Loops around arcs with the same orientation are homotopic. Note the loop in Figure 8. The loop shown, which passes under c and a, is equivalent to a loop around b.

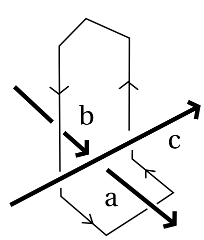


FIGURE 8. Relation given by a crossing

The corners of the loop in this diagram can be pulled up to the base point, forming three loops: a loop backwards around c, a loop forwards around a, and a another loop around c with the forwards orientation. When these loops are composed, we obtain a loop homotopic to the original loop in Figure 8, and thus homotopic to a loop around b. This is expressed by the relation $b = cac^{-1}$. Note that the convention used is for ab to signify first performing b and then a. The direction of the arcs a and b are independent of this relation. If we reverse the direction of these arcs as in Figure 9 the similar picture would give $b^{-1} = ca^{-1}c^{-1}$, and thus the same relation $b = cac^{-1}$.

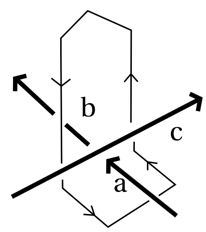


FIGURE 9. Relation given by a crossing with reversed arc direction

Thus the relation given by a crossing is $b = cac^{-1}$, where c separates a and b, and b is to the left of c relative to the orientation of c.

Example 5.4 (Wirtinger Presentation of a Trefoil Knot Group)**.** Giving the trefoil knot an orientation, we obtain the following diagram:

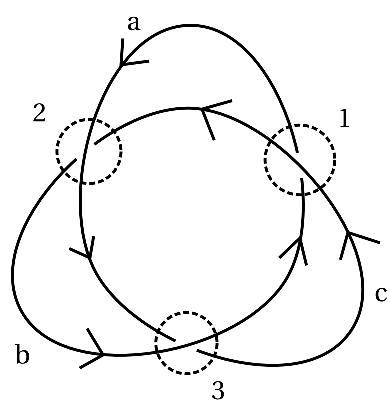


FIGURE 10. Calculating the knot group of a trefoil

We have three arcs, a, b, and c, which provide the generators for the group; likewise there are three crossings, marked with circles and numbered, that provide the relations. From crossing 1, we get the following relation:

$$b = cac^{-1}. \tag{5.5}$$

Likewise, from crossing 2, we get

$$c = aba^{-1}. \tag{5.6}$$

And from crossing 3, we obtain

$$a = bcb^{-1}. \tag{5.7}$$

Thus the knot group of the trefoil is isomorphic to the group $< a,b,c \,|\, a = cbc^{-1}, b = aca^{-1}, c = bab^{-1} >$.

However, we can simplify this further. From (5.6) and (5.7) we see that $ca = ab$ and $ca = bc$. From this we see $a = bcb^{-1}$, which means that (5.5) follows from (5.6) and (5.7) and thus can be omitted as a relation of the group. Returning to the equation $ca = ab$, we apply (5.6) to obtain $ca = ab = acac^{-1}$, and thus $cac = aca$.

The relations on the group are completely encoded in (5.6) and (5.7), which themselves are consequences of the relation $cac = aca$. Thus the knot group of the trefoil is isomorphic to the group $< a,b \,|\, bab = aba >$.

Unfortunately, a different diagram of the trefoil knot could have many more crossings and appear much more complicated. The Wirtinger presentations of the two knot groups will be isomorphic, but that need not be obvious from the presentations themselves. So while the knot group is easy to calculate, the difficulty in verifying that the presentations are isomorphic limits its usefulness. As an alternative, we turn to knot polynomials.

6. Knot Polynomials

An important subset of knot invariants assign a polynomial to knots. Knot polynomials have the advantage of being relatively easy to compute and, unlike with the knot group, it is easy to verify whether the the results of two such computations are the same. This section will define the Jones polynomial and describe several of its properties, as well as touching upon the more recent HOMFLY knot polynomial.

The Kauffman bracket polynomial is based around the smoothing of crossings in knot diagrams. For the rest of this section, we will consider knot diagrams that are identical outside of a small region, usually only encompassing a single crossing. To represent this we will draw the differing areas inside a dotted circle.

A crossing L of a link diagram can be smoothed in two ways, resulting in the diagrams L_A and L_B in Figure 11. The smoothing taking L to L_A will be referred to as *method A* for smoothing a crossing; likewise, *method B* takes L to L_B. The crossing can also be inverted, as shown by L'.

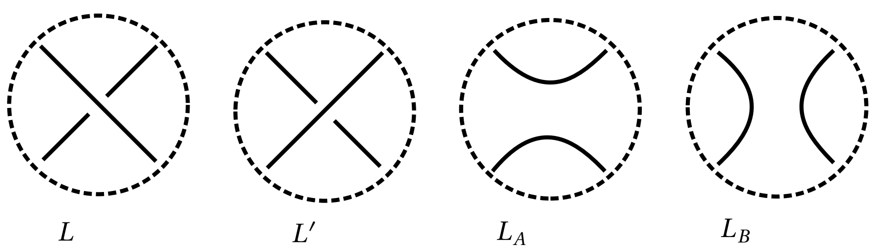


FIGURE 11. Smoothing of a crossing

Definition 6.1. The *Kauffman bracket polynomial* of a link diagram L, denoted $\langle L \rangle$, is the unique[1] polynomial that satisfies the following axioms:

[1]We will justify this momentarily.

(1) $\langle L\rangle = a\langle L_A\rangle + a^{-1}\langle L_B\rangle$
(2) $\langle L\cup O\rangle = (-a^2-a^{-2})\langle L\rangle$, where O is the basic diagram of the unknot.
(3) $\langle O\rangle = 1$.

Note that O refers to an unknot that has no crossings with the rest of the diagram.

Theorem 6.2. *There exists a unique polynomial that satisfies the axioms of the Kauffman bracket.*

Proof. To show this, we will define the polynomial explicitly.

Let L be a link diagram with n crossings.

A *state* of L is a diagram obtained from L by smoothing each of its crossings by method A or B. A diagram L thus has 2^n states. Let $\alpha(s)$ be the number of crossings in a state s smoothed by method A; likewise, let $\beta(s)$ be the number of crossings of s smoothed by method B. Let $\gamma(s)$ be the number of disjoint unknots in s. The bracket polynomial looks at all possible smoothings of every crossing, so our explicit definition will take the sum over all states of L.

Each state is reached by n smoothings, either by method A or B. Each time a crossing is smoothed by method A, by Axiom 1 we see that the term representing that state gains a coefficient of a in the sum; likewise, a smoothing by method B gives the term of coefficient of a^{-1}. Thus a state s contributes $a^{\alpha(s)-\beta(s)}\langle s\rangle$.

Each state will have no crossings, and will consist entirely of $\gamma(s)$ disjoint unknots. Thus for a state s, $\langle s\rangle = (-a^2-a^{-2})^{\gamma(s)-1}$ by Axioms 2 and 3. From this, we see that each state s provides the term $a^{\alpha(s)-\beta(s)}(-a^2-a^{-2})^{\gamma(s)-1}$ to the sum.

Taking the sum over all such states, we obtain the following equation:

$$\langle L\rangle = \sum_s a^{\alpha(s)-\beta(s)}(-a^2-a^{-2})^{\gamma(s)-1} \tag{6.3}$$

We must now check that this polynomial satisfies all of the axioms; that is, that the axioms are consistent.

To check Axiom 1, consider a diagram L, along with L_A and L_B denoting smoothings of L at a specific crossing. Let s_A denote a state of L smoothed by method A at that crossing, and s_B a state smoothed by method B. Since those are the only available methods of smoothing, we have $s_A+s_B=s$. From (6.3) we obtain

$$\langle L_A\rangle = \sum_{s_A} a^{\alpha(s_A)-\beta(s_A)-1}(-a^2-a^{-2})^{\gamma(s_A)-1}$$

and

$$\langle L_B\rangle = \sum_{s_A} a^{\alpha(s_A)-\beta(s_B)+1}(-a^2-a^{-2})^{\gamma(s_B)-1}.$$

Thus we have

$$a\langle L_A\rangle + a^{-1}\langle L_B\rangle = \sum_{s_A} a^{\alpha(s_A)-\beta(s_A)}(-a^2-a^{-2})^{\gamma(s_A)-1} + \sum_{s_B} a^{\alpha(s_B)-\beta(s_B)}(-a^2-a^{-2})^{\gamma(s_B)-1}$$

Since $s_A+s_B=s$, we have

$$a\langle L_A\rangle + a^{-1}\langle L_B\rangle = \sum_s a^{\alpha(s)-\beta(s)}(-a^2-a^{-2})^{\gamma(s)-1}.$$

By Equation 6.3, we then have $a\langle L_A\rangle + a^{-1}\langle L_B$, and thus Axiom 1 is satisfied.

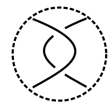
Adding a disjoint unknot to a diagram L will result in each state having an additional disjoint unknot, so we obtain

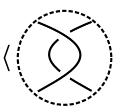
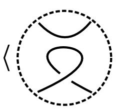
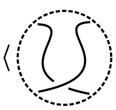
$$\langle L \cup O \rangle = \sum_{s} a^{\alpha(s)-\beta(s)}(-a^2 - a^{-2})^{\gamma(s)}.$$

Distributing, we can see that $\langle L \cup O \rangle = (-a^2 - a^{-2})\langle L \rangle$, so Axiom 2 is satisfied. Finally, $\langle O \rangle = (-a^2 - a^{-2})^0 = 1$, so Axiom 3 is satisfied.

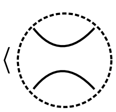
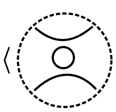
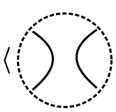
This polynomial was created directly from the axioms with no element of choice, and satisfies all of them, so it is unique. □

Theorem 6.4. *The Kauffman bracket polynomial is invariant over the second and third Reidemeister moves, but not the first Reidemeister move.*

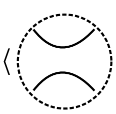
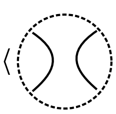
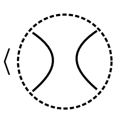
Proof. Consider a diagram containing the following crossings: . When we smooth the top crossing, by Axiom 1 we get:

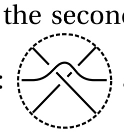
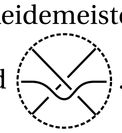
$\langle$  $\rangle = a \langle$  $\rangle + a^{-1} \langle$  $\rangle$

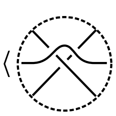
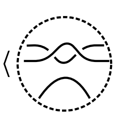
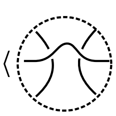
Smoothing the bottom crossing,

$= (a^2 + a^{-2})\langle$  $\rangle + \langle$  $\rangle + \langle$  $\rangle$

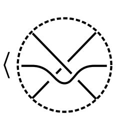
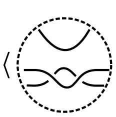
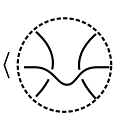
Applying Axiom 3,

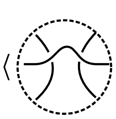
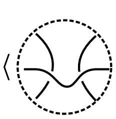
$= ((a^2 + a^{-2}) + (-a^2 - a^{-2}))\langle$  $\rangle + \langle$  $\rangle = \langle$  $\rangle$

Thus the bracket polynomial is invariant over the second Reidemeister move. Now consider the two diagrams connected by R_3:  and . Smoothing the central crossing point of each, we find

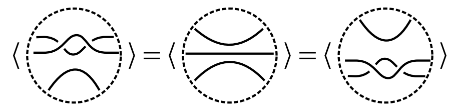
$\langle$  $\rangle = a \langle$  $\rangle + a^{-1} \langle$  $\rangle$

and

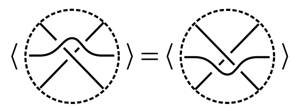
$\langle$  $\rangle = a \langle$  $\rangle + a^{-1} \langle$  $\rangle$

Consider the second terms of each equation. We can see $\langle$  $\rangle = \langle$  $\rangle$, since the two diagrams have identical crossings.

Since the bracket polynomial is invariant over R_2, we can connect the first terms:



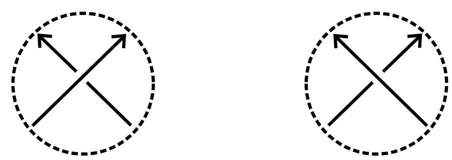
Thus,



and so the bracket polynomial is invariant over the third Reidemeister move.

However, the same is not true for the first Reidemeister move, because the first Reidemeister move will create a new curl on a diagram L. By Axiom 1, this will change the polynomial from $\langle L\rangle$ to $p(-a^2-a^{-2})\langle L\rangle + p^{-1}\langle L\rangle$, where we define $p=\pm 1$, depending on the orientation of the curl. Thus we have $(-a^{\pm 3})\langle L\rangle$. □

In order to render this polynomial invariant over R_1, we need some way of accounting for these curls. We do this by giving the diagrams orientation.



(A) Positive crossing (B) Negative crossing

FIGURE 12. Signs given to crossings

Definition 6.5. To each oriented crossing we assign a positive or negative sign, as shown in Figure 12. Let w_+ be the number of positive crossings and w_- be the number of negative crossings. The *writhe number* of a link diagram D, denoted $w(D)$, is defined by the equation $w(D) = w_+ - w_-$.

Definition 6.6. We define the normalized bracket polynomial of an oriented link diagram D as

$$X(D) = (-a^3)^{-w(D)}\langle |D| \rangle$$

where $|D|$ is the unoriented diagram corresponding to D.

Theorem 6.7. *The normalized bracket is an invariant of oriented link diagrams.*

Proof. Since the bracket is invariant under R_2 and R_3, and neither move changes the writhe number, the normalized bracket is invariant under R_2 and R_3. Applying R_1 will multiply the bracket polynomial by $-a^{\pm 3}$; however, it will also change the writhe number by ± 1, which will exactly cancel this new multiplier. Since the normalized bracket is invariant under the Reidemeister moves, it is invariant under isotopy. □

Definition 6.8. A *Conway triple* is a set of three knots, differing as shown in Figure 13. A *skein relation* is an expression relating the different terms of a Conway triple.

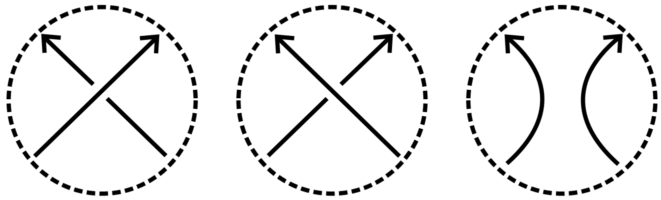
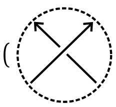
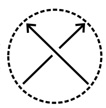


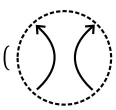
FIGURE 13. A Conway Triple

The normalized bracket polynomial satisfies a skein relation that aids in computation. Consider a Conway triple. Without loss of generality, we can assume that the writhe of the diagram outside of the crossing is 0, allowing us to obtain the following writhe numbers:

$w($  $) = 1$

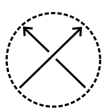
$w($  $) = -1$

and

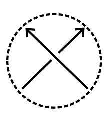
$w($  $) = 0$

.

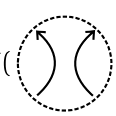
Let K_A and K_B be these crossings smoothed by methods A and B, respectively. By Axiom 1, we have that

$X($  $) = (-a)^{-3}(a\langle K_A\rangle + a^{-1}\langle K_B\rangle) = -a^{-2}\langle K_A\rangle - a^{-4}\langle K_B\rangle.$

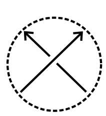
Likewise,

$X($  $) = -a^{2}\langle K_A\rangle - a^{4}\langle K_B\rangle$

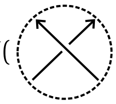
and

$X($  $) = \langle K_A\rangle.$

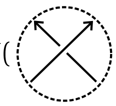
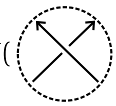
Thus we get

$a^4 X($  $) = -a^{2}\langle K_A\rangle - \langle K_B\rangle$

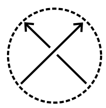
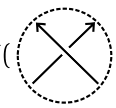
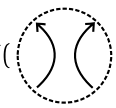
and

$a^{-4}X($  $) = -a^{-2}\langle K_A \rangle - \langle K_B \rangle.$

Taking their difference,

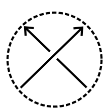
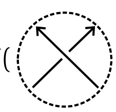
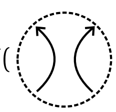
$a^{4}X($  $) - a^{-4}X($  $) = (a^{-2} - a^{2})X(K_A)$

and thus

$a^{4}X($  $) - a^{-4}X($  $) = (a^{-2} - a^{2})X($  $).$

Definition 6.9. A change of variable $q = a^{-4}$ in the normalized bracket polynomial gives the *Jones polynomial*, $V(K)$.

Since the normalized bracket polynomial was an invariant of oriented link diagrams, the Jones polynomial is as well. This change makes calculations with the skein relation slightly nicer, since skein relations are the primary method of computing the Jones polynomial. With this variable change we have the following skein relation for the Jones polynomial:

$q^{-1}V($  $) - qV($  $) = (q^{1/2} - q^{-1/2})V($  $).$

Note that the Jones polynomial is an invariant of oriented link diagrams, and is in general not independent of orientation. The exception to this is in the case of knots, where we have the following statement:

Theorem 6.10. *The Jones polynomial of a knot diagram is invariant over orientation changes.*

Proof. The Kauffman bracket polynomial is independent of orientation; in fact, the only part of the polynomial that the orientation affects is the writhe number. A knot has only one component, so reverse the orientation of the knot will reverse the orientation of both arcs at every crossing. Thus a crossing with writhe 1 will have the orientation of both the overcrossing and undercrossing switched; rotating the crossing reveals that the writhe hasn't actually changed. Likewise for crossings of writhe -1. □

Note that this does not hold for arbitrary link diagrams, because link diagrams can undergo orientation changes that don't invert the orientation of every crossing.

Thus it is meaningful to speak of the Jones polynomial of a knot without specifying orientation, while the same is not true of links.

Remark 6.11. The unknot is trivial; that is, $V(O) = 1$.

A useful application of the Jones polynomial is detecting knots that are not isotopic to their mirror images.

Definition 6.12. The *mirror image* of a knot is the composition $r \circ K$, where r is a reflection in $\mathbb{R}^3$.

A knot is called *amphicheiral* if it is isotopic to its mirror image.

Theorem 6.13. *The figure-eight knot is amphicheiral.*

FIGURE 14. Amphicheirality of the Figure-Eight knot

Proof. We begin with a simple figure-8 knot, as shown in Figure 14. We slide the top arc sideways by a planar isotopy. Next we rotate the leftmost arc around behind the rest of the diagram, onto the arc represented by a dotted line. This operation is a repeated application of R_1. A simple rotation results in a mirror image of the original knot. Since this mirror image was created solely through planar isotopy, R_1, and R_3, it is isotopic to the original knot. □

Not all knots are amphicheiral, however, and the Jones polynomial is a useful tool for detecting chirality.

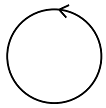
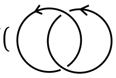
Theorem 6.14. *The Jones polynomial of the mirror image of a knot K is the Jones polynomial of K, with q replaced by q^{-1}.*

Proof. Let K' be the mirror image of K. In K', every crossing L is replaced with L', as in Figure 11. To smooth the crossing, we need to find an angle at which the crossing L' locally appears like the original crossing L. To do this, we rotate the crossing (and the whole knot diagram) right by 90 degrees. Smoothing the crossing by each method can occur as normal; once the crossing is smoothed, we rotate back 90 degrees. Thus a crossing smoothed by method A will, from the original angle, appear as L_B; likewise, a crossing smoothed by method B will appear as L_A. The states of the diagram are the same, but since the results of each of the smoothing methods are switched, the numbers of each type of crossings will be switched as well. Thus the Kauffman bracket polynomial for K' is that of K, with a replaced with a^{-1}. Likewise, the mirror image will switch the sign of the writhe number, so the normalization term becomes $(-a^3)^{-w(K')} = (-a^3)^{w(K)} = (-a^{-3})^{-w(K)}$. Substituting q for a^4 gives us that the Jones polynomial of K' is the Jones polynomial of K with q replaced by q^{-1}. □

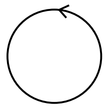
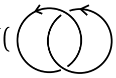
Example 6.15. As an example, we will compute the Jones polynomial of a trefoil knot.

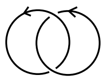
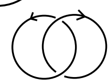
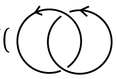
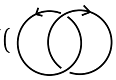
Using the skein relation shown above, we have

$V($  $) = q^2 V($  $) + q(q^{1/2} - q^{-1/2}) V($  $)$

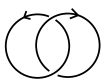
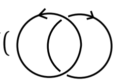
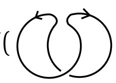
$= q^2 V($  $) + q(q^{1/2} - q^{-1/2})V($  $)$

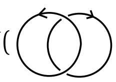
by isotopy.

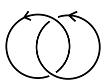
As previously stated, $V($  $) = 1$. Now it remains to compute $V($  $)$.

The link  is isotopic to , as can be seen by vertically rotating the right component 180 degrees. Thus we have $V($  $) = V($  $)$.

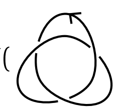
Thus,

$V($  $) = q^2 V($  $) + q(q^{1/2} - q^{-1/2})V($  $)$.

The Jones polynomial of the n-component unlink is $(-q^{1/2} - q^{-1/2})^{n-1}$, so $V($  $) = (-q^{1/2} - q^{-1/2})$. Therefore,

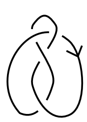
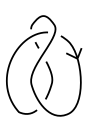
$V($  $) = q^2(-q^{1/2} - q^{-1/2}) + q(q^{1/2} - q^{-1/2}) = -q^{5/2} - q^{1/2}$.

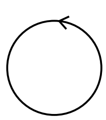
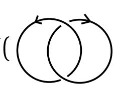
Thus we have

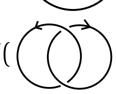
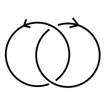
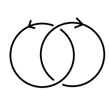
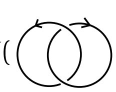
$V($  $) = q^2 + q(q^{1/2} - q^{-1/2})(-q^5/2 - q^{1/2}) = -q^4 + q^3 + q$.

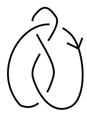
The Jones polynomial of the mirror image of this knot is thus $V(T') = -q^{-4} + q^{-3} + q^{-1}$. Clearly, the two are not equal. Thus the trefoil knot is not amphicheiral. With this in mind, we will show that this method is consistent with the earlier claim that the figure-eight knot is amphicheiral by calculating its Jones polynomial and that of its mirror image.

Example 6.16. As before, we use the skein relation, obtaining

$V($  $) = q^2 V($  $) + q(q^{1/2} - q^{-1/2})V($  $)$

$= q^2 V($  $) + q(q^{1/2} - q^{-1/2})V($  $)$

The calculation of $V($  $)$ need not be done explicitly; it suffices to notice that  is the mirror image of , and so $V($  $) = -q^{-5/2} - q^{-1/2}$. Thus,

$V($  $) = q^2 + q(q^{1/2} - q^{-1/2})(-q^{-5/2} - q^{-1/2}) = q^2 + q^{-2} - q - q^{-1} + 1$

Notice that substituting q^{-1} for q returns exactly the same expression, so the Jones polynomial does not detect chirality. This does not constitute a proof of amphicheirality, however; Figure 15 is a chiral knot with Jones polynomial $V(K) = t^3 + t^{-3} - t^2 - t^{-2} + t + t^{-1} - 1$.

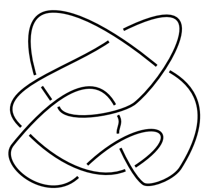


FIGURE 15. A chiral knot with symmetric Jones polynomial

Looking at the computation, one sees that the Jones polynomial of a knot is built off of the Jones polynomials of other, smaller knots. Computing the Jones polynomial of a knot is vastly simplified if the polynomials of the knots created through the skein relations are already known. This motivates a formal way of constructing large knots out of other knots.

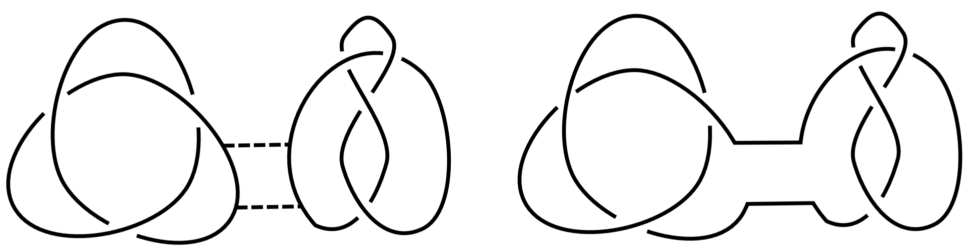


FIGURE 16. Adding a trefoil and a figure-eight knot

Definition 6.17. The connected sum of two knots K and K', denoted $K\#K'$, is formed by attaching the knots with respect to the orientation of each knot. See Figure 16 for an example. This is done by removing a small arc on each knot, then gluing the knots together by their boundary, respecting orientation.

As an example of knot addition, consider the square knot and the granny knot, as seen in Figure 17. The square knot is the sum of a trefoil knot and its reflection; the granny knot is the sum of a trefoil knot to itself.

This idea of a connected sum can be used to aid computation as follows:

Theorem 6.18. *Let K_1 and K_2 be knots. Then $V(K_1\#K_2) = V(K_1)V(K_2)$*

Proof. Computing the Jones polynomial of a knot sum follows the same procedure as computing the polynomial of K_1, except that when K_1 would be reduced to the unknot in the computation of $V(K_1)$, here it is instead reduced to K_2. Thus

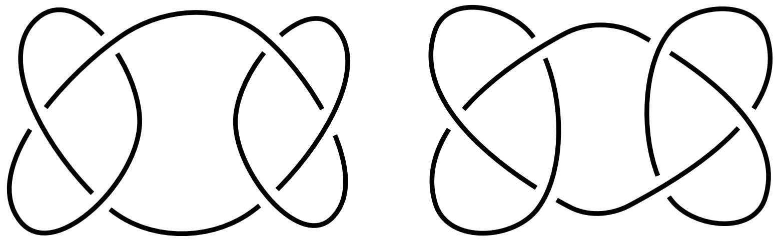
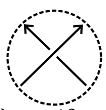
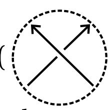
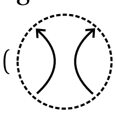


FIGURE 17. The Square Knot and the Granny Knot

the Jones polynomial of $K_1 \# K_2$ is the Jones polynomial of K_1 with an added coefficient of $V(K_2)$ on each term. A factoring gives the desired result. $\square$

Several more knot polynomials have been discovered, including a variant of the Jones polynomial in two variables. One of the most powerful is the HOMFLY polynomial, named after the initials of a number of mathematicians who discovered the polynomial more or less simultaneously[2].

Definition 6.19. The HOMFLY polynomial P of a link diagram L is defined as the polynomial in variables x, y, and z satisfying the following axioms:

a) $xP($  $) + yP($  $) + zP($  $) = 0$

b) $P(L) = 1$ if L is the unknot.

A proof of the existence and uniqueness of this polynomial is beyond the scope of this paper, but can be found in the original paper detailing the polynomial [2].

The HOMFLY polynomial is a generalization of the Jones polynomial and several other polynomials. In particular, the Jones polynomial of a link diagram is the HOMFLY polynomial evaluated at $(t, -t^{-1}, t^{1/2} - t^{-1/2})$.

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