CHARACTERIZING THE ORBITS OF THE ROTATION MAP

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Abstract. We begin by developing the necessary tools to analyze subsets of topological spaces. We then define a topological space and compare continuity in the topological sense to the traditional $\epsilon - \delta$ method. Lastly, we use the tools we have developed to characterize the orbits of the rotation map on the circle.

Contents

1. Introduction

In this paper, we study the rotation map on the circle, an important example in the theory of dynamical systems. In order to study the rotation map we must first develop basic topological concepts, including bounds, limit points, openness, and continuity. We then apply these concepts to study the orbits of the rotation map, which, for a given angle θ, rotates all the points in the circle by θ. We show that when the angle is an irrational multiple of 2π, the orbits are dense, and when the angle is a rational multiple of 2π, they are finite.

The organization of the paper is as follows: in Section 2, we discuss metric spaces, upper and lower bounds, limit points and some applications, and open sets and their properties. In Section 3, we define a topological space and give examples of different topological spaces. In Section 4, we examine the $\epsilon - \delta$ definition of continuity of a real-valued function on $\mathbb{R}$ and explore how it relates to the topological definition of continuity. We conclude in Section 5 by studying the orbits of the rotation map.

2. Bounds, Limit Points, and Openness

We begin by examining bounds on subsets of ordered sets. Next, we define a metric space and examine limit points, which we then use to examine openness in sets.

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Definition 1. An *ordered set* S is a set which has an order relation, $<$, satisfying the following two properties:

1. If $x \in S$ and $y \in S$, then exactly one of the following is true:

$$x < y, y < x, \text{or } y = x.$$

2. (Transitivity) If $x, y, z \in S$, then $x < y$ and $y < z$ implies $x < z$.

Definition 2. Let M be an ordered set, and $A \subset M$. An element $\beta \in M$ is an *upper bound* for A if for all $x \in A, x \leq \beta$. A is called *bounded above* if it has an upper bound. An *upper bound* β is called the *least upper bound* or *supremum* if for any α which is an upper bound, $\beta \leq \alpha$.

Definition 3. Let N be an ordered set, and $B \subset N$. An element $\alpha \in N$ is a *lower bound* for B if for all $x \in B, x \geq \alpha$. B is called *bounded below* if it has a lower bound. A *lower bound* α is called the *greatest lower bound* or *infimum* if for any γ which is a lower bound, $\alpha \geq \gamma$.

Claim 1. If E is a nonempty subset of an ordered set, X, $\alpha \in X$ is a lower bound of E and $\beta \in X$ is an upper bound of E, then $\alpha \leq \beta$.

Proof. Let $\alpha \in X$ be a lower bound of E. E is nonempty, so let $x \in E$. By definition of a lower bound, $\alpha \leq x$. Let β be an upper bound of E. By definition of an upper bound, $x \leq \beta$. By transitivity, $\alpha \leq \beta$. □

Definition 4. Take a set S with its elements called points. S is a *metric space* if any two points x and y in S have a real number distance associated with them. The real number distance between x and y is written as $d(x, y)$ and satisfies the following three properties:
1. $d(x, y) > 0$ if $x \neq y$, and $d(x, x) = 0$.
2. $d(x, y) = d(y, x)$.
3. $d(x, y) \leq d(x, z) + d(y, z)$ for any $z \in S$.

Example 1. An example of a metric space is the real line with standard distance. Take $x, y \in \mathbb{R}^1$. The distance is defined by $d(x, y) = |x-y|$ where $|\cdot|$ is the absolute value on $\mathbb{R}$.

Example 2. Another example of a metric space is arclength on S^1. Take $x, y \in S^1$, with $x = e^{i\theta}$ and $y = e^{i\phi}$. The arclength is defined by $d(x, y) = |\theta - \phi|$, where θ and ϕ are in radians.

Definition 5. Let $B_\epsilon(p)$ denote the points x around a point p in a metric space S such that $d(x, p) < \epsilon$. A point $p \in S$ is a *limit point* of a subset A of S if for any $\epsilon > 0$, there exists some $q \in A$ such that $q \in B_\epsilon(p)$ and $q \neq p$.

Definition 6. A subset X of a metric space Y is *dense* in Y if for all $x \in X$, $x \in Y$ or x is a limit point of Y.

Example 3. We can construct a bounded set of real numbers with one limit point.

Proof. Let

$$S = \{\frac{1}{n} | n \in \mathbb{N} - \{0\}\}.$$

Let $\epsilon > 0, \epsilon \in \mathbb{R}$. By the Archimedean Property, we know that given $\epsilon > 0$, there exists an n such that $\frac{1}{n} < \epsilon$. Because $\frac{1}{n} \in S$ and $\frac{1}{n} \neq 0$, 0 is a limit point of S.

We can show that 0 is also the only limit point of S. Let $x \in \mathbb{R}, x \neq 0$.

Case 1. $x < 0$. Set $\epsilon = \frac{-x}{2}$. Then $B_\epsilon(x) \cap S = \emptyset$, so x is not a limit point.

Case 2. $x > 1$. Set $\epsilon = \frac{x}{2}$. Then $B_\epsilon(x) \cap S = \emptyset$, so x is not a limit point.

Case 3. $x = 1$. Set $\epsilon = \frac{1}{3}$. Then $B_\epsilon(x) \cap S = \emptyset$, so x is not a limit point.

Case 4. $0 < x < 1$. Set $\epsilon = \frac{1}{2}(x - \frac{1}{p})$, where $\frac{1}{p} < x$. Then $B_\epsilon(x) \cap S = \emptyset$ or $B_\epsilon(x) \cap S = n$ so x is not a limit point.

□

Theorem 2.1. *Let $S_i, i = 1, ...n$ be subsets of a metric space X. If X_i is the set of limit points of S_i, and X is the set of limit points of $S = \bigcup_{i=1}^n S_i$, then $\bigcup X_i = X$.*

Proof. We first show that $\bigcup X_i \subset X$. Let $x \in X_i$, so x is a limit point of S_i. Take an ϵ neighborhood $B_\epsilon(x)$ of x. Since x is a limit point of S_i, there exists $s_i \in S_i$ such that $s_i \neq x$ and $s_i \in B_\epsilon(x)$. So $s_i \in \bigcup_{i=1}^n S_i$ implies $s_i \in S$. Thus x is a limit point of S, so $x \in X$, and $\bigcup X_i \subset X$.

We now show that $X \subset \bigcup X_i$. Take x such that $x \notin \bigcup X_i$. Since x is not a limit point of S_i, for each i there exists ϵ_i such that $B_{\epsilon_i}(x) \cap S_i = \emptyset$ or $\{x\}$. Let $\epsilon = \min(\epsilon_1, ..., \epsilon_n)$. So $B_\epsilon(x) \subset B_{\epsilon_1}(x)$ for all i. Assume by contradiction that there exists $s \in S$, $s \neq x$ such that $s \in B_\epsilon(x)$. Since $s \in \bigcup S_i$, there exists some i such that there exists $s \in S_i$. But we know that $B_\epsilon(x) \cap S_i = \emptyset$ or $\{x\}$ because $B_\epsilon(x) \cap S_i$ is contained in $B_{\epsilon_i}(X) \cap S_i$. Because $s \neq x$, such an s does not exist. □

Example 4. Theorem 2.1 can fail for an infinite union of sets: Let $LP(X)$ denote the set of limit points of the set X. Let $\{U_\alpha\}$ be such that $LP(\bigcup_\alpha U_\alpha) \neq \bigcup_\alpha LP(U_\alpha)$.

Proof. Let $U_\alpha = \{\alpha\}, \alpha \in \mathbb{R}$. Then U_α is a finite set and thus has no limit points. So

$$\cup_{\alpha \in \mathbb{R}} LP(\{\alpha\}) = \cup_{\alpha \in \mathbb{R}} \emptyset = \emptyset.$$

However, $\cup_{\alpha \in \mathbb{R}} \{\alpha\} = \mathbb{R}$, so $LP(\cup_{\alpha \in \mathbb{R}} \{\alpha\}) = \mathbb{R}$. Thus $LP(\cup_{\alpha \in \mathbb{R}} \{\alpha\}) \neq \cup_\alpha LP(\{\alpha\})$.

□

Definition 7. $U \subseteq M$, where M is a metric space, is *open* if for all $x \in U$, there exists $\epsilon > 0$ such that $B_\epsilon(x) \subset U$.

Theorem 2.2. *If U and V are open subsets of a metric space, then $U \cap V$ is open.*

Proof. Case 1. If $U \cap V = \emptyset$, then $U \cap V$ is open because the empty set is an open set. (The empty set is an open set because there are no x in the empty set, thus the condition for all $x \in U$, there exists $\epsilon > 0$ such that $B_\epsilon(x) \subset U$ is vacuously satisfied.)

Case 2. $U \cap V \neq \emptyset$. Take $x \in U \cap V$. Because U and V are open sets, there exists $\epsilon_U > 0$ and $\epsilon_V > 0$ such that $B_{\epsilon_U}(x) \subset U$ and $B_{\epsilon_V}(x) \subset V$. Thus we can choose $\epsilon = \min(\epsilon_U, \epsilon_V) > 0$ such that $B_\epsilon(x) \subset U \cap V$. □

Corollary 2.3. *If $U_1, ..., U_n$ are open sets in a metric space, then $\bigcap_{i=1}^n U$ is open.*

Example 5. However, the infinite intersection of open sets is not necessarily open: The set $\bigcap_{n=1}^\infty (\frac{-1}{n}, \frac{1}{n})$ is not open.

Proof. $\bigcap_{n=1}^{\infty}(\frac{-1}{n}, \frac{1}{n}) = \{0\}$ and a set consisting of a single point is not open. We know that 0 is the result of the intersection, because given any other point r, we can choose an n such that $\frac{1}{n} < |r|$. □

Theorem 2.4. *If $\{U_i\}_{i \in I}$ are open, then $\bigcup U_i$ is open.*

Proof. Take $x \in \bigcup_i U_i$. For some i, $x \in U_i$. Because U_i is open, there exists $\epsilon > 0$ such that $B_\epsilon(x) \subset U_i$. Because $B_\epsilon(x) \subset U_i$, $B_\epsilon(x) \subset \bigcup_i U_i$. □

3. Topological Spaces

We begin by defining a topological space. We then discuss examples of different topological spaces and show how they satisfy the basic axioms.

Definition 8. A *topological space* X is a set X with a collection T of subsets of X called *open sets* satisfying the following:
1. $\emptyset, X \in T$. (Thus $\emptyset$ and X are both open.)
2. If $\{U_\alpha\}$ is a collection of open sets ($U_\alpha \in T$), then $\cup_\alpha U_\alpha$ is open.
3. If $U_1, ..., U_n$ are open, so is $U_1 \cap U_2 ... \cap U_n$.

Example 6. If X is any set, there are two easily defined topologies on X :
1. The *indiscrete topology*, where the only open sets are $\emptyset$ and X.
2. The *discrete topology*, where everything is open: $T = \mathcal{P}X$.

Example 7. A metric space M is a topological space with open sets defined as in the previous section.

We must show that the three axioms are satisfied.

1. Noted in the proof of Theorem 2.2.
2. Proved in Theorem 2.4.
3. Proved in Corollary 2.3.

Example 8. Let X be a set. The open sets of the *cofinite topology* on U are $T = \{\emptyset\} \cup \{U \mid X - U \text{ is finite}\}$. This is a topology.

Proof. We must show the three axioms are satisfied.

1. $\emptyset$ is open in X by definition of the cofinite topology. X is open in X because $X - X = \emptyset$ and $\emptyset$ is finite.

2. *Arbitrary Union.*

Case $1 : U_\beta \neq \emptyset$ for some β. Because U_β is open, we know that $X - U_\beta$ is finite. Since $X - \bigcup_\alpha U_\alpha \subset X - U_\beta$, we know that $X - \bigcup_\alpha U_\alpha$ is finite because a subset of a finite set is necessarily finite.

Case $2 : U_\alpha = \emptyset$ for all α. Then $\bigcup U_\alpha = \emptyset$. By definition, $\emptyset$ is open, so this case of the infinite union is also open.

3. *Finite Intersection. Case* $1 : U_i \neq \emptyset$ for all $i \leq n$.

$$X - \cap U_1 \cap \cdots \cap U_n = (\cap U_1 \cap \cdots \cap U_n)^c = \bigcup_{i=1}^{n}(X - U_i) = \bigcup_{i=1}^{n} {U_i}^c.$$

Because U_i is open, ${U_i}^c$ is finite by definition, so $X - \bigcap_{i=1}^{n} U_i$ is open. Thus the finite intersection of open sets is open.

Case $2 : U_n = \emptyset$ for all n. Then $\bigcap_n U_n = \emptyset$. By definition $\emptyset$ is open, thus this case of the finite intersection is also open.

□

Remark 1. If X is finite, the cofinite topology is the discrete topology.

4. Continuity

We examine two definitions of continuity and prove that they are equivalent.

Definition 9 (The $\epsilon - \delta$ definition of continuity)**.** If f is an arbitrary metric space we say f is *continuous* if for all $x \in \mathbb{R}$ and for all $\epsilon > 0$ there exists some $\delta = \delta(x, \epsilon)$ so that

$$d(x, y) < \delta \implies d(f(x), f(y)) < \epsilon.$$

Definition 10 (topological definition of continuity)**.** Let X and Y be topological spaces and let $f : X \to Y$. Then f is continuous if for all open $U \subset Y$, $f^{-1}(U)$ is open in X.

Theorem 4.1. *The topological definition of continuity is equivalent to the $\epsilon - \delta$ definition of continuity if $f : A \to \mathbb{R}$ is a function on an arbitrary metric space.*

Proof. Topological definition $\implies$ $\epsilon - \delta$ definition. Suppose $f : A \to \mathbb{R}$ is a function on an arbitrary metric space and is continuous in the topological sense. Let $x \in \mathbb{R}, \epsilon > 0$. Then $f^{-1}(B_\epsilon(f(x)))$ is open. Because $f^{-1}(B_\epsilon(f(x)))$ is open and $x \in f^{-1}(B_\epsilon(f(x)))$, there exists $\delta > 0$ so that $B_\delta(x) \subset f^{-1}(B_\epsilon(f(x)))$. Therefore $f(B_\delta(x)) \subset B_\epsilon(f(x))$. This shows that if

$$d(x, y) < \delta, \text{then } d(f(x), f(y)) < \epsilon.$$

$\epsilon - \delta$ definition $\implies$ topological definition. We want to show that if U is open, then $f^{-1}(U)$ is open. Let x be any element of $f^{-1}(U)$. Because U is open, there exists $\epsilon > 0$ so that $B_\epsilon(f(x)) \subset U$. We know there exists $\delta > 0$ such that if $d(x, y) < \delta$, then $d(f(x), f(y)) < \epsilon$. In particular, this says that $B_\delta(x) \subset f^{-1}(U)$. □

5. Orbits of the Rotation Map

Definition 11. A *dynamical system* is a set S and a map $f : S \to S$.

Definition 12. Take a point $p \in S$. Then $\{f(p), f(f(p)), ..., f^n(p)...\}$ is the *orbit* of p under f.

Let the notation $\{\alpha\}$ denote the fractional portion of α. Let the notation $\lfloor\alpha\rfloor$ denote the largest integer less than or equal to α. So $\{\alpha\} = \alpha - \lfloor\alpha\rfloor$, and $\{\alpha\} \in [0, 1)$.

Theorem 5.1 (Dirichlet approximation theorem)**.** *If $\alpha \in \mathbb{R}, n \in \mathbb{N}$ then there exist $p, q \in \mathbb{Z}$ such that $|p\alpha - q| < \frac{1}{n}$.*

Proof. Split up $[0, 1]$ into n pieces, so each piece is size $\frac{1}{n}$.

Now consider the real numbers $\{\alpha\}, \{2\alpha\}, ..., \{n\alpha\}, \{(n+1)\alpha\}$. By the Pigeonhole Principle, there exist some i, j, r such that $\{i\alpha\}$ and $\{j\alpha\}$ lie in the same interval $\left[\frac{r}{n}, \frac{r+1}{n}\right)$. Thus

$$|\{i\alpha\} - \{j\alpha\}| < \frac{1}{n}.$$

$$|i\alpha - \lfloor i\alpha\rfloor - (j\alpha - \lfloor j\alpha\rfloor)| < \frac{1}{n}.$$

$$|(i - j)\alpha - \lfloor i\alpha\rfloor + \lfloor j\alpha\rfloor| < \frac{1}{n}.$$

Let $q = \lfloor j\alpha\rfloor - \lfloor i\alpha\rfloor$ and $p = i - j$. Thus we have $|p\alpha - q| < \frac{1}{n}$ and we are done. □

An important example of a dynamical system is the following problem regarding the orbits of the rotation map.

Theorem 5.2. *For $\theta \in [0, 2\pi]$, let R_θ be the map from the circle to itself given by counterclockwise rotation by angle θ. Consider the orbit of 1 under R_θ. The orbit of 1 is finite if $\theta \in 2\pi \cdot \mathbb{Q}$ and dense in S^1 otherwise.*

Proof. Suppose the orbit, $I_\theta = \{1, R_\theta(1), ..., {R_\theta}^n(1)\}$, is finite. We know that $R_\theta(1) = e^{i\theta}$. By the definition of orbit, $R_\theta^{n+1}(1) \in I_\theta$. Consider $R_\theta^{n+1}(1) = R_\theta^m(1)$ for some $1 \le m \le n$. Then we have that

$$e^{(n+1)i\theta} = e^{im\theta}$$

$$1 = \frac{e^{(n+1)i\theta}}{e^{im\theta}} = e^{i(n+1-m)\theta}$$

$$2k\pi = (n+1-m)\theta$$

$$\theta = \frac{2k\pi}{n+1-m}$$

Therefore if I_θ is finite, θ must be a rational multiple of 2π.

Conversely, if θ is a rational multiple of 2π then the orbit is finite. Given that $\theta = \frac{2\pi p}{q}$ where p, q are relatively prime, we can define

$$I_\theta = \{R_\theta(1) = e^{i2\pi\frac{p}{q}}, R_\theta^2(1) = e^{i2\pi\frac{2p}{q}} ... R_\theta^q(1) = e^{i2\pi\frac{qp}{q}} = e^{2\pi ip} = 1\}.$$

Thus the orbit is finite and contains q elements.

If θ is not a rational multiple of π, then we want to prove that $I_\theta = \{e^{2\pi in\theta} \mid n \in \mathbb{N}\}$ is dense in the unit circle.

Let $\theta \in [0, 2\pi)$ be irrational. Define $r = \frac{\theta}{2\pi} \in [0, 1)$. Let $p \in S^1$, $p = e^{i\phi}$ with $\phi \in [0, 2\pi]$. Define $\beta = \frac{\phi}{2\pi}$ so $\beta \in [0, 1]$. Pick M such that $\frac{2\pi}{M} < \epsilon$. Dirichlet's Approximation Theorem implies that there exist p, q such that $|q\frac{\theta}{2\pi} - p| < \frac{1}{M}$. This implies that

$$|q\theta - 2\pi p| < \frac{2\pi}{M} < \epsilon.$$

So we know that $R_\theta^q(1)$ is within ϵ of 1. Thus the points $R_\theta^q(1), R_\theta^{2q}(1), ..., R_\theta^M(1)$ break S^1 into M pieces of length less than ϵ. Then the distance from β to any point in I_θ is less than ϵ, so every point in the unit circle is a limit point of the irrational orbit, so I_θ is dense in the unit circle.

□

Remark 2. If $\theta = \frac{p\pi}{q}$, where $p, q \in \mathbb{Q}$, then the set I has exactly q elements if $gcd(p, q) = 1$ and $\frac{q}{gcd(p,q)}$ if $gcd(p, q) \neq 1$.

Remark 3. If $e^{2\pi i\alpha}$ is another point on S^1, then the orbit of $e^{2\pi i\alpha}$ under R_θ is just the orbit of 1 rotated by $2\pi\alpha$.

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