MODES OF CONVERGENCE FOR FOURIER SERIES

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Abstract. Issues of convergence of Fourier series in various modes are explored. In particular, some necessary and/or sufficient conditions for L^p convergence, uniform convergence, and almost everywhere convergence are presented. Properties of the Dirichlet kernel, the Fejer kernel, and the Hilbert transform will be analyzed and used to help provide these conditions. Finally, a few examples of functions with divergent Fourier series are presented.

Contents

1. Introduction

Fourier series are one of the most powerful tools in mathematics, finding mathematical applications in solving differential equations and technological applications in signal processing, image processing, and electrical engineering, among others. When Joseph Fourier first developed them, however, he had no formal definition of a function, let alone of convergence, and thus did not bother to investigate the convergence of these series. As such, for a while they were used heuristically, with the understanding that they "should" equal in some sense the original function they represent. It was not until duBois Reymond showed the existence of a continuous function with a divergent Fourier series that issues of convergence took precedence, leading to many of the results presented in this paper. As such, we will investigate the question of convergence of Fourier series in many senses, such as convergence in the L^p norm for $1 \leq p < \infty$, uniform convergence, and almost everywhere pointwise convergence.

We assume the reader is familiar with Lebesgue integration, normed linear spaces, basic properties of convolution, and basic trigonometric knowledge.

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2. Basic results

Let $\mathbb{T}$ be the unit circle. Throughout this paper, the integral of a function f on $\mathbb{T}$ (denoted $\int_{\mathbb{T}} f(\theta)d\theta$ or simply $\int_{\mathbb{T}} f$) will be defined to be

$$\int_{\mathbb{T}} f = \int_{\mathbb{T}} f dm = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)d\theta$$

with m being the Lebesgue measure on $\mathbb{T}$ such that $dm = \frac{d\theta}{2\pi}$. Note that, under this definition, the integral of the characteristic function on the torus $\mathbb{1}_{\mathbb{T}}$ is 1. Note also that, due to the periodicity of any function on $\mathbb{T}$, any pair of real numbers (a, b) with $b - a = 2\pi$ serve equally well as the limits of integration.

Definitions 2.1. For $1 \leq p < \infty$, define $L^p(\mathbb{T})$ to be the set of all measurable $f : \mathbb{T} \to \mathbb{C}$ such that

$$\|f\|_p := \left(\int_{\mathbb{T}} |f|^p\right)^{\frac{1}{p}} < \infty.$$

$L^p(\mathbb{T})$ becomes a Banach space when equipped with the norm above.

Define $L^\infty(\mathbb{T})$ to be the set of all measurable $f : \mathbb{T} \to \mathbb{C}$ such that

$$\|f\|_\infty := \inf\{C \geq 0 : m(\{\theta : |f(\theta)| > C\}) = 0\} < \infty.$$

Again, $L^\infty(\mathbb{T})$ becomes a Banach space when equipped with the norm above.

Remark 2.2. Technically, as normed linear spaces the L^p spaces should consist of equivalence classes of such functions, where two functions are equivalent if and only if they differ on a set of measure zero. This is because a function that is zero only almost everywhere, under this definition, will still have norm zero. However, we can usually refer to an element of an L^p space as a function instead of an equivalence class of functions without much trouble.

We proceed to define the Fourier coefficients.

Definition 2.3. Let $f \in L^1(\mathbb{T})$. The *nth Fourier coefficient* of f is defined to be

$$\hat{f}(n) = \int_{\mathbb{T}} f(\theta)e^{-in\theta}d\theta.$$

Note that this is well-defined for any $f \in L^1(\mathbb{T})$. By Hölder's inequality on measure spaces, we have

$$\|f\|_1 = \|f\mathbb{1}_{\mathbb{T}}\|_1 \leq \|f\|_p\|\mathbb{1}_{\mathbb{T}}\|_{p'} = \|f\|_p$$

for any $1 \leq p \leq \infty$, where $\frac{1}{p} + \frac{1}{p'} = 1$. Hence $L^p(\mathbb{T}) \subseteq L^1(\mathbb{T})$ for any $1 \leq p \leq \infty$, so the above is well-defined for $f \in L^p(\mathbb{T})$ as well.

Some immediate properties of the Fourier coefficients follow.

Proposition 2.4. *Let $f, g \in L^1(\mathbb{T})$ and $n \in \mathbb{Z}$. Then*

1. $\widehat{(f+g)}(n) = \hat{f}(n) + \hat{g}(n)$
2. $\widehat{(cf)}(n) = c\hat{f}(n)$ *for any* $c \in \mathbb{C}$
3. $\widehat{(f * g)}(n) = \hat{f}(n)\hat{g}(n)$
4. $\widehat{(\tau_\delta f)}(n) = e^{-in\delta}\hat{f}(n)$, *where* $\tau_\delta f(\theta) = f(\theta - \delta)$
5. $\hat{g}(n) = \hat{f}(n - m)$ *if* $g(\theta) = f(\theta)e^{im\theta}$
6. $\widehat{f'}(n) = in\hat{f}(n)$ *if f is continuously differentiable*

(7) $\sup_{n\in\mathbb{Z}} |\hat{f}(n)| \leq \|f\|_1$

(8) $\widehat{f_m}(n) \xrightarrow{m\to\infty} \hat{f}(n)$ *if* $\{f_m\}$ *converge uniformly to* f*.*

Proof. Statements (1), (2), (4), and (5) are straightforward to verify. (3) can be proven using Fubini's theorem and the translational invariance of Lebesgue measure, (6) can be proven using integration by parts, and both (7) and (8) can be proven using the integral inequality $|\int f| \leq \int |f|$. □

Definitions 2.5. For $f \in L^1(\mathbb{T})$, we define the N^{th} *partial sum of the Fourier series* of f to be

$$S_N f(\theta) = \sum_{n=-N}^{N} \hat{f}(n) e^{in\theta}.$$

We associate with f a formal power series called the *Fourier series* of f defined by

$$\mathcal{F}f(\theta) = \sum_{n\in\mathbb{Z}} \hat{f}(n) e^{in\theta}.$$

Definition 2.6. A *trigonometric polynomial* is a function of the form

$$P(\theta) = \sum_{n\in\mathbb{Z}} a_n e^{in\theta}$$

such that there exists an $N \in \mathbb{N}$ for which $|a_n| + |a_{-n}| = 0$ for all $n > N$. The smallest such N is called the *degree* of P.

Note that the sum above involves a finite number of terms, so the trigonometric polynomials are well-defined. Furthermore, it is straightforward to check that $\widehat{P}(n) = a_n$ for all n, so $\mathcal{F}P$ exists and is equal to P. Hence, the Fourier series of any trigonometric polynomial converges back to itself. For other arbitrary functions, it is not clear whether its Fourier series converges, in what sense (i.e. pointwise, uniformly, in L^p norm, etc.), and to what value.

We begin our investigation by noting that

$$\begin{aligned}
S_N f(\theta) &= \sum_{n=-N}^{N} \hat{f}(n) e^{in\theta} \\
&= \sum_{n=-N}^{N} e^{in\theta} \int_{\mathbb{T}} f(\eta) e^{-in\eta} d\eta \\
&= \int_{\mathbb{T}} f(\eta) \left(\sum_{n=-N}^{N} e^{in\theta} e^{-in\eta} \right) d\eta = \int_{\mathbb{T}} f(\eta) \left(\sum_{n=-N}^{N} e^{in(\theta-\eta)} \right) d\eta.
\end{aligned}$$

Hence, if $D_N(\theta) = \sum_{n=-N}^{N} e^{in\theta}$, then we can write $S_N f(\theta) = (D_N * f)(\theta)$.

Definition 2.7. The N^{th} *Dirichlet kernel* is defined to be

$$D_N(\theta) = \sum_{n=-N}^{N} e^{in\theta}.$$

We can also investigate the arithmetic means of the partial sums and its kernel as well.

Definitions 2.8. The N^{th} *Fejer kernel* is defined to be

$$F_N(\theta) = \frac{1}{N} \sum_{n=0}^{N-1} D_n(\theta).$$

For $f \in L^1(\mathbb{T})$, the N^{th} *Fejer mean* of f is defined to be

$$\sigma_N f(\theta) = F_N * f(\theta) = \frac{1}{N}\sum_{n=0}^{N-1} S_n f(\theta).$$

The Dirichlet and Fejer kernels are closely related to what is known as an approximation to the identity, a family of functions that, when put in convolution with any suitably well-behaved function, will return another function that approximates the original in some way.

Definition 2.9. A family of functions $\{K_n\}_{n\in\mathbb{N}}$ is called an *approximation to the identity* if the following three properties hold:

(1) $\int\limits_{\mathbb{T}} K_n(\theta)d\theta = 1$ for all n,
(2) $\sup\limits_{n\in\mathbb{N}} \int\limits_{\mathbb{T}} |K_n(\theta)|d\theta < \infty$ for all n,
(3) $\int\limits_{\delta\le|\theta|\le\pi} |K_n(\theta)|d\theta \to 0$ as $n\to\infty$ for all $\delta > 0$.

Theorem 2.10. *Let $\{K_n\}_{n\in\mathbb{N}}$ be an approximation to the identity. Then, for any continuous f, we have*

$$\|K_n * f - f\|_\infty \xrightarrow{n\to\infty} 0.$$

Furthermore, for any $f \in L^p(\mathbb{T})$ with $1 \le p < \infty$, we have

$$\|K_n * f - f\|_p \xrightarrow{n\to\infty} 0.$$

Proof. Suppose f is continuous. Since $\mathbb{T}$ is compact, f is uniformly continuous on $\mathbb{T}$, so let $\delta > 0$ satisfy $|x-y| < \delta \implies |f(x) - f(y)| < \epsilon$. Furthermore, $|f|$ is bounded, say by M. Then

$$\begin{aligned}
|K_n * f(\theta) - f(\theta)| &= \left| \int\limits_{\mathbb{T}} f(\eta)K_n(\theta-\eta)d\eta - f(\theta)\int\limits_{\mathbb{T}} K_n(\theta-\eta)d\eta \right| \\
&= \left| \int\limits_{\mathbb{T}} (f(\eta) - f(\theta))K_n(\theta-\eta)d\eta \right| \\
&\le \left| \int\limits_{|\theta-\eta|<\delta} (f(\eta) - f(\theta))K_n(\theta-\eta)d\eta \right| + \left| \int\limits_{|\theta-\eta|\ge\delta} (f(\eta) - f(\theta))K_n(\theta-\eta)d\eta \right| \\
&\le \int\limits_{|\theta-\eta|<\delta} \epsilon|K_n(\theta-\eta)|d\eta + \int\limits_{|\theta-\eta|\ge\delta} 2M|K_n(\theta-\eta)|d\eta \\
&\le \epsilon + 2M \int\limits_{|\theta-\eta|\ge\delta} |K_n(\theta-\eta)|d\eta \xrightarrow{n\to\infty} \epsilon.
\end{aligned}$$

Hence $\|K_n * f - f\|_\infty \xrightarrow{n\to\infty} 0$. For $f \in L^p(\mathbb{T})$, there exists $g \in C(\mathbb{T})$ such that $\|f-g\|_p < \epsilon$, by the density of continuous functions in $L^p(\mathbb{T})$. By Minkowski's integral inequality, we have $\|K_n * (f-g)\|_p \le \|K_n\|_1\|f-g\|_p < \epsilon \sup_{n\in\mathbb{N}} \|K_n\|_1$. Hence,

$$\begin{aligned}
\|K_n * f - f\|_p &\le \|K_n * f - K_n * g\|_p + \|K_n * g - g\|_p + \|g - f\|_p \\
&\le \epsilon \sup_{n\in\mathbb{N}} \|K_n\|_1 + \|K_n * g - g\|_\infty + \epsilon \xrightarrow{n\to\infty} (\sup_{n\in\mathbb{N}} \|K_n\|_1 + 1)\epsilon.
\end{aligned}$$

$\square$

The above result suggests that if either the Dirichlet or Fejer kernels form an approximation to the identity, then we can draw some powerful conclusions regarding the L^p convergence of Fourier series. It remains to determine if either of the kernels do form an approximation to the identity. To do this, we need explicit formulas for the two kernels.

Proposition 2.11. *For all $N \in \mathbb{Z}$ and $\theta \in \mathbb{T}$, we have the following formulas:*

$$\begin{aligned} D_N(\theta) &= \frac{\sin(\frac{2N+1}{2}\theta)}{\sin(\frac{\theta}{2})}, \\ F_N(\theta) &= \frac{1}{N}\left(\frac{\sin(\frac{N\theta}{2})}{\sin(\frac{\theta}{2})}\right)^2. \end{aligned}$$

Proof. We have

$$\begin{aligned} D_N(\theta) = \sum_{n=-N}^{N} e^{in\theta} &= e^{-iN\theta}\sum_{n=0}^{2N} e^{in\theta} \\ &= e^{-iN\theta}\frac{e^{(2N+1)i\theta}-1}{e^{i\theta}-1} \\ &= \frac{e^{-(N+\frac{1}{2})i\theta}(e^{(2N+1)i\theta}-1)}{e^{\frac{i\theta}{2}}-e^{-\frac{i\theta}{2}}} \\ &= \frac{e^{\frac{2N+1}{2}i\theta}-e^{-\frac{2N+1}{2}i\theta}}{e^{\frac{i\theta}{2}}-e^{-\frac{i\theta}{2}}} \\ &= \frac{2i\sin(\frac{2N+1}{2}\theta)}{2i\sin(\frac{\theta}{2})} = \frac{\sin(\frac{2N+1}{2}\theta)}{\sin(\frac{\theta}{2})}. \end{aligned}$$

In addition, we have

$$\begin{aligned} F_N(\theta) = \frac{1}{N}\sum_{n=0}^{N-1} D_n(\theta) &= \frac{1}{N}\sum_{n=0}^{N-1}\frac{\sin(\frac{2n+1}{2}\theta)}{\sin(\frac{\theta}{2})} \\ &= \frac{1}{N\sin(\frac{\theta}{2})}\sum_{n=0}^{N-1}\operatorname{Im}(e^{\frac{2n+1}{2}i\theta}) \\ &= \frac{1}{N\sin(\frac{\theta}{2})}\operatorname{Im}\left(e^{\frac{i\theta}{2}}\frac{e^{iN\theta}-1}{e^{i\theta}-1}\right) \\ &= \frac{1}{N\sin(\frac{\theta}{2})}\operatorname{Im}\left(\frac{e^{iN\theta}-1}{e^{\frac{i\theta}{2}}-e^{-\frac{i\theta}{2}}}\right) \\ &= \frac{1}{N\sin^2(\frac{\theta}{2})}\operatorname{Im}\left(\frac{e^{iN\theta}-1}{2i}\right) \\ &= \frac{1}{N\sin^2(\frac{\theta}{2})}\operatorname{Re}\left(\frac{e^{iN\theta}-1}{-2}\right) \\ &= \frac{1}{N}\frac{1-\cos(N\theta)}{2\sin^2(\frac{\theta}{2})} = \frac{1}{N}\left(\frac{\sin(\frac{N\theta}{2})}{\sin(\frac{\theta}{2})}\right)^2. \end{aligned}$$

□

From the above formula for D_N, we find in the following proposition that $\int_{\mathbb{T}} |D_N| = \|D_N\|_1$ grows without bound, leading us to conclude that the Dirichlet kernels do not form an approximation to the identity.

Proposition 2.12. $\|D_N\|_1 \geq \frac{2}{\pi^2}\log(2N+2)$. *In particular* $\|D_N\|_1 \to \infty$ *as* $N \to \infty$.

Proof. Note that $|\sin(\theta)| \leq |\theta|$ for $\theta > 0$. As such, for $\frac{k\pi}{2N+1} \leq \theta \leq \frac{(k+1)\pi}{2N+1}$ $(k \geq 0)$, we have

$$|D_N(\theta)| = \left|\frac{\sin(\frac{2N+1}{2}\theta)}{\sin(\frac{\theta}{2})}\right| \geq \frac{|\sin(\frac{2N+1}{2}\theta)|}{\frac{\theta}{2}} \geq \frac{4N+2}{(k+1)\pi}\left|\sin\left(\frac{2N+1}{2}\theta\right)\right|$$

Since $\int\limits_{\frac{k\pi}{2N+1}}^{\frac{(k+1)\pi}{2N+1}} |\sin\left(\frac{2N+1}{2}\theta\right)|d\theta = \frac{1}{2N+1}$ for all k, we have

$$\begin{aligned}
\|D_N\|_1 = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi}|D_N(\theta)|d\theta = \frac{1}{\pi}\int\limits_{0}^{\pi}|D_N(\theta)|d\theta &= \frac{1}{\pi}\sum_{k=0}^{2N}\int\limits_{\frac{k\pi}{2N+1}}^{\frac{(k+1)\pi}{2N+1}}|D_N(\theta)|d\theta \\
&\geq \frac{1}{\pi}\sum_{k=0}^{2N}\int\limits_{\frac{k\pi}{2N+1}}^{\frac{(k+1)\pi}{2N+1}}\frac{4N+2}{(k+1)\pi}\left|\sin\left(\frac{2N+1}{2}\theta\right)\right|d\theta \\
&= \frac{1}{\pi}\sum_{k=0}^{2N}\frac{2}{(k+1)\pi} \\
&= \frac{2}{\pi^2}\sum_{k=0}^{2N}\frac{1}{k+1} \\
&\geq \frac{2}{\pi^2}\log(2N+2)
\end{aligned}$$

where we used the fact that $\frac{2}{\pi^2}\sum\limits_{k=0}^{2N}\frac{1}{k+1} \geq \frac{2}{\pi^2}\sum\limits_{k=0}^{2N}\int\limits_{k+1}^{k+2}\frac{1}{t}dt = \frac{2}{\pi^2}\int\limits_{1}^{2N+2}\frac{1}{t}dt = \frac{2}{\pi^2}\log(2N+2)$. $\square$

Remark 2.13. A similar argument shows that $\|D_N\|_1 \leq C'\log N$ for some constant C'.

For the Fejer kernels, however, we find the opposite result:

Proposition 2.14. *The Fejer kernels form an approximation to the identity.*

Proof. Property (1) is straightforward to verify directly from the definition. Property (2) follows from property (1) since $F_N(\theta) \geq 0$ for all θ. For property (3), we have $\sin^2(\frac{\theta}{2}) \geq (\frac{\delta}{2})^2$ for $\delta \leq |\theta| \leq \pi$, so

$$\int\limits_{\delta\leq|\theta|\leq\pi}|F_N(\theta)|d\theta = \int\limits_{\delta\leq|\theta|\leq\pi}\frac{1}{N}\left(\frac{\sin(\frac{N\theta}{2})}{\sin(\frac{\theta}{2})}\right)^2 d\theta \leq \frac{1}{N}\int\limits_{\delta\leq|\theta|\leq\pi}\left(\frac{2}{\delta}\right)^2 d\theta \xrightarrow{N\to\infty} 0$$

$\square$

We thus obtain the following corollaries:

Corollary 2.15. *Trigonometric polynomials are dense in* $L^p(\mathbb{T})$.

Proof. Note that $\sigma_N f = F_N * f$ is a trigonometric polynomial for any N. Since $\{F_N\}_{N\in\mathbb{N}}$ is an approximation to the identity, by Theorem 2.10 we have $\|F_N * f - f\|_p \xrightarrow{N\to\infty} 0$. $\square$

Corollary 2.16. *(Riemann-Lebesgue Lemma) For any $f \in L^1(\mathbb{T})$, we have $\hat{f}(n) \xrightarrow{|n|\to\infty} 0$.*

Proof. Let P be a trigonometric polynomial such that $\|f-P\|_1 < \epsilon$, and let N be its degree. Then $\hat{P}(n) = 0$ for any $|n| > N$. Hence, for any $|n| > N$, we have

$$|\hat{f}(n)| = |\hat{f}(n) - \hat{P}(n)| = |(\widehat{f-P})(n)| \leq \|f-P\|_1 < \epsilon.$$

$\square$

Theorem 2.17. *(Uniqueness theorem) If $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$, then $f = 0$ almost everywhere. Hence, if $\hat{f}(n) = \hat{g}(n)$ for every $n \in \mathbb{Z}$, then $f = g$ almost everywhere by linearity.*

Proof. Suppose $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$. Then $\sigma_N f = 0$ for all $N \in \mathbb{N}$, so

$$\|f\|_1 = \|\sigma_N f - f\|_1 \xrightarrow{N\to\infty} 0.$$

Hence, $\|f\|_1 = 0$, so $f = 0$ almost everywhere. $\square$

We can now obtain one sufficient condition for almost everywhere pointwise convergence:

Theorem 2.18. *Suppose $\sum_{n\in\mathbb{Z}} |\hat{f}(n)| < \infty$. Then $S_N f$ converges to f almost everywhere. In particular, f is equal to a continuous function almost everywhere.*

Proof. Since $|S_N f(\theta)| \leq \sum_{n=-N}^{N} |\hat{f}(n)|$ for every θ, it follows that $S_N f$ converges pointwise to some function g. Furthermore, this convergence is uniform, since $|S_N f(\theta) - S_M f(\theta)| \leq \sum_{n=M+1}^{N} |\hat{f}(n)|$, which does not depend on θ. Hence, g is continuous, and $\hat{g}(n) = \lim_{N\to\infty} \widehat{S_N f}(n) = \hat{f}(n)$ for every $n \in \mathbb{Z}$. Hence, $g = f$ almost everywhere. Thus, $S_N f$ converges to f almost everywhere, and f is equal to a continuous function almost everywhere. $\square$

Finally, there are two theorems that will be used in this paper which will not be proven. The first is the Riesz-Thorin Interpolation Theorem, whose proof can be found in [1]:

Theorem 2.19. *Let $1 \leq p_1, p_2, q_1, q_2 \leq \infty$. Suppose T is an operator which is bounded mapping $L^{p_1}(\mathbb{T})$ to $L^{q_1}(\mathbb{T})$ and is bounded mapping $L^{p_2}(\mathbb{T})$ to $L^{q_2}(\mathbb{T})$. Let $A = \|T\|_{L^{p_1}(\mathbb{T})\to L^{q_1}(\mathbb{T})}$ and $B = \|T\|_{L^{p_2}(\mathbb{T})\to L^{q_2}(\mathbb{T})}$. Then, for any $0 \leq \alpha \leq 1$, we have*

$$\|Tf\|_{q_\alpha} \leq A^{1-\alpha} B^{\alpha} \|f\|_{p_\alpha}$$

where $\frac{1}{p_\alpha} = \frac{1-\alpha}{p_1} + \frac{\alpha}{p_2}$ and $\frac{1}{q_\alpha} = \frac{1-\alpha}{q_1} + \frac{\alpha}{q_2}$.

The other theorem is the Uniform Boundedness Principle, also known as the Banach-Steinhaus Theorem:

Theorem 2.20. *Let X be a Banach space and Y be a normed linear space. Let $\{T_i\}_{i\in I}$ be a family of operators from X to Y. If, for every $x \in X$, there exists M_x such that $\|T_i(x)\|_Y < M_x$ for all $i \in I$, then there exists M such that $\|T_i\|_{X\to Y} \leq M$ for all $i \in I$.*

3. L^2 CONVERGENCE

One space where we can obtain significant convergence results is $L^2(\mathbb{T})$. Thanks to its Hilbert space structure, $L^2(\mathbb{T})$ is one of the most well-behaved (and hence most well-understood) Banach spaces.

Proposition 3.1. *$L^2(\mathbb{T})$ is a Hilbert space, with inner product*

$$\langle f, g\rangle = \int_{\mathbb{T}} f(\theta)\overline{g(\theta)}d\theta.$$

Hölder's inequality shows that $\langle ., .\rangle$ is well-defined; the fact that $\langle ., .\rangle$ satisfies the properties of an inner product is straightforward to verify. In particular, $\langle f, f\rangle = \|f\|_2^2$.

Proposition 3.2. *Let $e_n(\theta) = e^{in\theta}$. Then $\{e_n\}_{n\in\mathbb{Z}}$ is an orthonormal set in $L^2(\mathbb{T})$.*

Proof. By direct computation,

$$\langle e_m, e_n\rangle = \int_{\mathbb{T}} e^{i(m-n)\theta}d\theta = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}.$$

□

Proposition 3.3. *Let $f \in L^2(\mathbb{T})$. Then*

(1) $S_N f$ is the best degree N trigonometric L^2 approximation to f, in the sense that given any trigonometric polynomial P of degree at most N, we have $\|f - S_N f\|_2 \leq \|f - P\|_2$.
(2) $\|f - S_N f\|_2 \xrightarrow{N\to\infty} 0$.
(3) (Parseval's identity) $\|f\|_2^2 = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2$.

Proof. Let P be a trigonometric polynomial of degree at most N, and let $Q = S_N f - P$. Then Q is a trigonometric polynomial of degree at most N, so write $Q = \sum_{n=-N}^{N} a_n e_n$. Then we have $\langle f - S_N f, Q\rangle = \langle f, \sum_{n=-N}^{N} a_n e_n\rangle - \langle \sum_{n=-N}^{N} \hat{f}(n)e_n, \sum_{n=-N}^{N} a_n e_n\rangle = \sum_{n=-N}^{N} \hat{f}(n)\overline{a_n} - \sum_{n=-N}^{N} \hat{f}(n)\overline{a_n} = 0$. Similarly, $\langle Q, f - S_N f\rangle = 0$. We thus have

$$\begin{aligned}\|f - P\|_2^2 = \langle f - P, f - P\rangle &= \langle (f - S_N f) + Q, (f - S_N f) + Q\rangle \\ &= \langle f - S_N f, f - S_N f\rangle + \langle f - S_N f, Q\rangle + \langle Q, f - S_N f\rangle + \langle Q, Q\rangle \\ &= \|f - S_N f\|_2^2 + \|Q\|_2^2 \geq \|f - S_N f\|_2^2\end{aligned}$$

with equality if and only if $S_N f - P = Q = 0$.

Since $\sigma_N f$ is a trigonometric polynomial of degree less than N, and $\|f - \sigma_N f\|_2 \xrightarrow{N\to\infty} 0$, we have

$$\|f - S_N f\|_2 \leq \|f - \sigma_N f\|_2 \xrightarrow{N\to\infty} 0.$$

Hence $\|f - S_N f\|_2 \xrightarrow{N\to\infty} 0$.

Finally, since $S_N f$ is a trigonometric polynomial of degree at most N, we have $\langle f - S_N f, S_N f\rangle = \langle S_N f, f - S_N f\rangle = 0$. As a result,

$$\begin{aligned}\|f\|_2^2 = \langle f, f\rangle &= \langle S_N f, S_N f\rangle + \langle S_N f, f - S_N f\rangle + \langle f - S_N f, S_N f\rangle + \langle f - S_N f, f - S_N f\rangle \\ &= \langle \sum_{n=-N}^{N} \hat{f}(n)e_n, \sum_{n=-N}^{N} \hat{f}(n)e_n\rangle + \|f - S_N f\|_2^2 = \sum_{n=-N}^{N} |\hat{f}(n)|^2 + \|f - S_N f\|_2^2.\end{aligned}$$

Letting $N \to \infty$ yields $\|f\|_2^2 = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2$. □

4. L^p convergence and the Hilbert transform

The Hilbert space structure of $L^2(\mathbb{T})$ allowed us to establish results such as L^2 convergence and Parseval's identity. With other L^p spaces, more work needs to be done to establish simply convergence in the L^p norm. We will see that L^1 and L^∞ convergence does not always occur, but L^p convergence for all other p does occur.

Theorem 4.1. *Let $1 \leq p < \infty$. The following are equivalent:*

(1) $\|S_n f - f\|_p \xrightarrow{n\to\infty} 0$ for all $f \in L^p(\mathbb{T})$,
(2) $\sup\limits_{n\in\mathbb{N}} \|S_n\|_p < \infty$, where $\|S_n\|_p = \|S_n\|_{L^p(\mathbb{T})\to L^p(\mathbb{T})}$ is the operator norm when viewing S_n as an operator from $L^p(\mathbb{T})$ to itself.

Proof. Suppose (2) is true. For all $n \in \mathbb{N}$, we have that $\sigma_n f$ is a trigonometric polynomial of degree at most $n-1$, so $S_n(\sigma_n f) = \sigma_n f$. Let N satisfy $\|\sigma_n f - f\|_p < \epsilon$ for $n > N$. Then, for $n > N$, we have

$$\begin{aligned}\|S_n f - f\|_p = \|S_n f - S_n(\sigma_n f) + \sigma_n f - f\|_p &\leq \|S_n(f - \sigma_n f)\|_p + \|\sigma_n f - f\|_p \\ &\leq (\sup_{n\in\mathbb{N}} \|S_n\|_p + 1)\|\sigma_n f - f\|_p \\ &\leq (\sup_{n\in\mathbb{N}} \|S_n\|_p + 1)\epsilon.\end{aligned}$$

Conversely, suppose (1) is true. Given $f \in L^p(\mathbb{T})$, for all $n \in \mathbb{N}$, we have $\|S_n f - f\|_p \leq M_f$ for some M_f (since $\|S_n f - f\|_p \xrightarrow{n\to\infty} 0$). Hence, $\|S_n f\|_p \leq \|f\|_p + \|S_n f - f\|_p \leq \|f\|_p + M_f$ for all $n \in \mathbb{N}$. By the uniform boundedness principle, there exists M such that $\|S_n\|_p < M$ for all $n \in \mathbb{N}$. Hence, $\sup\limits_{n\in\mathbb{N}} \|S_n\|_p \leq M < \infty$. ∎

It would be unreasonable to expect the Fourier series of an arbitrary L^∞ function to converge to itself in the L^∞ norm, i.e. uniformly, given that the uniform limit of continuous functions must be continuous. Hence, it is helpful to think of $L^\infty(\mathbb{T})$ as $C(\mathbb{T})$, the space of all continuous functions, when discussing issues of convergence. The above theorem still holds if we replace $L^p(\mathbb{T})$ with $C(\mathbb{T})$. Nonetheless, we can still easily obtain two negative consequences regarding convergence in $L^1(\mathbb{T})$ and $C(\mathbb{T})$.

Proposition 4.2. *$\sup\limits_{N\in\mathbb{N}} \|S_N\|_1 = \infty$ and $\sup\limits_{N\in\mathbb{N}} \|S_N\|_{C(\mathbb{T})\to C(\mathbb{T})} = \infty$. Consequently, there exists $f \in L^1(\mathbb{T})$ and $g \in C(\mathbb{T})$ such that $S_N f$ and $S_N g$ do not converge to f and g in the L^1 and L^∞ norms, respectively.*

Proof. The Fejer kernels F_M have L^1 norm 1; hence

$$\|S_N\|_1 \geq \|S_N(F_M)\|_1 = \|D_N * F_M\|_1 \xrightarrow{M\to\infty} \|D_N\|_1 \geq C\log N.$$

The function $f = \operatorname{sgn}(D_N)$ has L^∞ norm 1; hence

$$\|S_N\|_\infty \geq \|S_N f\|_\infty \geq |S_N f(0)| = |\int_{\mathbb{T}} \operatorname{sgn}(D_N)(D_N)| = \|D_N\|_1 \geq C\log N.$$

While $f \notin C(\mathbb{T})$, for any $\epsilon > 0$ there exists $g \in C(\mathbb{T})$ such that $\|g - f\|_1 < \epsilon$ and $\|g\|_\infty = 1$. We have

$$\|S_N(g-f)\|_\infty \leq \sum_{n=-N}^{N} |\widehat{g-f}(n)| < (2N+1)\epsilon.$$

As such, for continuous functions, we have

$$\|S_N\|_{C(\mathbb{T})\to C(\mathbb{T})} \geq \|S_N g\|_\infty \geq \|S_N f\|_\infty - \|S_N(g-f)\|_\infty \geq C\log N - (2N+1)\epsilon.$$

Letting $\epsilon \to 0$ yields $\|S_N\|_{C(\mathbb{T})\to C(\mathbb{T})} \geq C\log N$. ∎

In Section 5, we will construct a continuous function f whose Fourier series diverges at some point, providing an explicit example for the above proposition.

In addition to the negative results for $L^1(\mathbb{T})$ and $C(\mathbb{T})$, we have a positive result for $L^2(\mathbb{T})$. For $1 < p < \infty$, we can investigate the issue of convergence in L^p by considering the Hilbert transform.

Definition 4.3. For trigonometric polynomials f, define the *Hilbert transform* of f by

$$Hf(\theta) = \sum_{n\in\mathbb{Z}} -i\text{sgn}(n)\hat{f}(n)e^{in\theta}.$$

To help us understand the Hilbert transform, it will be helpful to define a few other operators as well. The Riesz projections P_+ and P_- are defined by

$$P_+f(\theta) = \sum_{n=1}^{\infty} \hat{f}(n)e^{in\theta},$$
$$P_-f(\theta) = \sum_{n=-\infty}^{-1} \hat{f}(n)e^{in\theta}.$$

Finally, define the operators A and S_N^+ by

$$Af(\theta) = P_+f(\theta) + \hat{f}(0) = \sum_{n=0}^{\infty} \hat{f}(n)e^{in\theta},$$
$$S_N^+f(\theta) = \sum_{n=0}^{2N} \hat{f}(n)e^{in\theta}.$$

Note that all of these operators are well-defined for trigonometric polynomials, and S_N^+ is well-defined for all L^p functions. Furthermore, these operators are also well-defined for C^∞ functions, by the following fact:

Proposition 4.4. *Let $f \in C^\infty(\mathbb{T})$. Then $\hat{f}(n) = O(n^{-m})$ for all $m > 0$. In particular, $\hat{f}(n) = O(n^{-2})$, and $\sum_{n\in\mathbb{Z}} |\hat{f}(n)| < \infty$.*

Proof. By Proposition (2.4.6), we have

$$\widehat{f'}(n) = in\hat{f}(n)$$

for any continuously differentiable f. It follows that

$$\widehat{f^{(m)}}(n) = (in)^m\hat{f}(n)$$

for all $m \in \mathbb{N}$. By the Riemann-Lebesgue lemma, $\widehat{f^{(m)}}(n) \xrightarrow{|n|\to\infty} 0$, so $\{\widehat{f^{(m)}}(n)\}_{n\in\mathbb{Z}}$ is bounded. Thus, we have

$$|\hat{f}(n)| = \frac{|\widehat{f^{(m)}}(n)|}{n^m} \leq \frac{C}{n^m}.$$

□

Proposition 4.5. *Let $1 < p < \infty$. The following are equivalent:*

(1) $\|H\|_p < \infty$
(2) $\|P_+\|_p < \infty$
(3) $\sup_{N\in\mathbb{N}} \|S_N\|_p < \infty$
(4) $\sup_{N\in\mathbb{N}} \|S_N^+\|_p < \infty$
(5) $\|A\|_p < \infty$

Proof. Notice that $f = P_+f + P_-f + \hat{f}(0)$ and $Hf = -iP_+f + iP_-f$. Consequently,

$$P_+f = \frac{1}{2}(f + iHf - \hat{f}(0)).$$

Since $|\hat{f}(0)| \leq \|f\|_1 \leq \|f\|_p$, the boundedness of P_+ and H are equivalent, showing (1) $\iff$ (2). Next, notice that if $g(\theta) = f(\theta)e^{iN\theta}$, then

$$\sum_{n=-N}^{N} \hat{f}(n)e^{in\theta} = e^{-iN\theta}\sum_{n=0}^{2N} \hat{f}(n-N)e^{in\theta} = e^{-iN\theta}\sum_{n=0}^{2N} \hat{g}(n)e^{in\theta}.$$

Hence, $\|S_Nf\|_p = \|S_N^+g\|_p$. Since $\|f\|_p = \|g\|_p$, it follows that $\|S_N\|_p = \|S_N^+\|_p$, showing (3) $\iff$ (4). Suppose (4) holds. Then by Fatou's Lemma we have

$$\|Af\|_p = \|\lim_{N\to\infty} S_N^+f\|_p \leq \liminf_{N\to\infty} \|S_N^+f\|_p \leq (\sup_{N\in\mathbb{N}} \|S_N^+\|_p)\|f\|_p.$$

Hence, (4) $\implies$ (5). Furthermore, notice that

$$\begin{aligned}
S_N^+f(\theta) &= \sum_{n=0}^{\infty} \hat{f}(n)e^{in\theta} - \sum_{n=2N+1}^{\infty} \hat{f}(n)e^{in\theta} \\
&= Af(\theta) - e^{i(2N+1)\theta}\sum_{n=0}^{\infty} \hat{f}(n+2N+1)e^{in\theta} \\
&= Af(\theta) - e^{i(2N+1)\theta}Ag(\theta)
\end{aligned}$$

where $g(\theta) = f(\theta)e^{-i(2N+1)\theta}$. As such, if (5) holds, then

$$\sup_{N\in\mathbb{N}} \|S_N^+f\|_p \leq \|A\|_p\|f\|_p + \|A\|_p\|g\|_p = 2\|A\|_p\|f\|_p$$

for all smooth f, and hence by density for all $f \in L^p(\mathbb{T})$ as well. Hence, (5) $\implies$ (4). Finally, since $Af = P_+f + \hat{f}(0)$, (5) $\iff$ (1). $\square$

Corollary 4.6. *Let $1 < p < \infty$. Then S_Nf converges to f in the L^p norm for all $f \in L^p(\mathbb{T})$ if and only if $\|H\|_p < \infty$.*

The following proof, due to Salomon Bochner, shows that $\|H\|_p < \infty$ for $1 < p < \infty$.

Theorem 4.7. *Let $1 < p < \infty$. Then there exists a constant A_p such that*

$$\|Hf\|_p \leq A_p\|f\|_p$$

for all trigonometric polynomials f, and H extends to a bounded operator from $L^p(\mathbb{T})$ to $L^p(\mathbb{T})$. Hence, S_Nf converges to f in the L^p norm for all $f \in L^p(\mathbb{T})$ for $1 < p < \infty$.

Proof. Let f be a nonzero real-valued trigonometric polynomial with $\hat{f}(0) = 0$. Then it is easy to check that $\hat{f}(-n) = \overline{\hat{f}(n)}$ for all $n \in \mathbb{Z}$. Hence, we have

$$\begin{aligned}
Hf(\theta) &= -i\sum_{n>0} \hat{f}(n)e^{in\theta} + i\sum_{n<0} \hat{f}(n)e^{in\theta} \\
&= \sum_{n>0} (-i\hat{f}(n)e^{in\theta}) + (i\hat{f}(-n)e^{-in\theta}) \\
&= \sum_{n>0} (-i\hat{f}(n)e^{in\theta}) + \overline{(-i\hat{f}(n)e^{in\theta})} = 2\mathrm{Re}\left(\sum_{n>0} -i\hat{f}(n)e^{in\theta}\right).
\end{aligned}$$

Hence, Hf is also real-valued. Since $\hat{f}(0) = 0$, we have

$$(f + iHf)(\theta) = \sum_{n\in\mathbb{Z}} (1 + i(-i\mathrm{sgn}(n)))\hat{f}(n)e^{in\theta} = 2\sum_{n>0} \hat{f}(n)e^{in\theta}.$$

As such, $(f+iHf)^{2k}$ will only have positive frequencies for any $k \in \mathbb{N}$, so $\int\limits_{\mathbb{T}} (f+iHf)^{2k} = 0$. Expanding, we obtain

$$\sum_{j=0}^{2k} \int\limits_{\mathbb{T}} i^j \binom{2k}{j} (Hf)^j f^{2k-j} = 0.$$

Combining real parts, we obtain

$$\sum_{j=0}^{k} \int\limits_{\mathbb{T}} (-1)^j \binom{2k}{2j} (Hf)^{2j} f^{2k-2j} = 0.$$

As such,

$$\|Hf\|_{2k}^{2k} = \left| \int\limits_{\mathbb{T}} (Hf)^{2k} \right| = \left| -(-1)^k \sum_{j=0}^{k-1} \int\limits_{\mathbb{T}} (-1)^j \binom{2k}{2j} (Hf)^{2j} f^{2k-2j} \right| \leq \sum_{j=0}^{k-1} \binom{2k}{2j} \|(Hf)^{2j} f^{2k-2j}\|_1.$$

For each j we have

$$\|(Hf)^{2j} f^{2k-2j}\|_1 \leq \|(Hf)^{2j}\|_{\frac{k}{j}} \|f^{2k-2j}\|_{\frac{k}{k-j}} = \|Hf\|_{2k}^{2j} \|f\|_{2k}^{2k-2j}$$

by applying Hölder's inequality with coefficients $\frac{k}{j}$ and $\frac{k}{k-j}$, respectively. Hence we have

$$\|Hf\|_{2k}^{2k} \leq \sum_{j=0}^{k-1} \binom{2k}{2j} \|Hf\|_{2k}^{2j} \|f\|_{2k}^{2k-2j}.$$

If we let $R = \frac{\|Hf\|_{2k}}{\|f\|_{2k}}$, then dividing the above expression by $\|f\|_{2k}^{2k}$, we have

$$R^{2k} \leq \sum_{j=0}^{k-1} \binom{2k}{2j} R^{2j}.$$

Since the polynomial $x^{2k} - \sum\limits_{j=0}^{k-1} \binom{2k}{2j} x^{2j}$ has even degree, it follows that $x^{2k} - \sum\limits_{j=0}^{k-1} \binom{2k}{2j} x^{2j} \to \infty$ as $|x| \to \infty$. Hence, there exists a constant C_{2k} such that $x^{2k} - \sum\limits_{j=0}^{k-1} \binom{2k}{2j} x^{2j} \leq 0 \implies |x| \leq C_{2k}$. As a result, we have $R \leq C_{2k}$, so we have $\|Hf\|_{2k} \leq C_{2k} \|f\|_{2k}$ for all real-valued trigonometric polynomials with $\hat{f}(0) = 0$.

Suppose we remove the restriction that $\hat{f}(0) = 0$. Since the Hilbert transform of a constant function is zero, we have

$$\|Hf\|_{2k} = \|H(f - \hat{f}(0))\|_{2k} \leq C_{2k} \|f - \hat{f}(0)\|_{2k} \leq C_{2k}(\|f\|_{2k} + \|\hat{f}(0)\|_{2k}) \leq 2C_{2k} \|f\|_{2k}.$$

Now suppose f is a (not necessarily real-valued) trigonometric polynomial, and let $f(\theta) = \sum_{n=-N}^{N} a_n e^{in\theta}$. Then we have

$$
\begin{aligned}
f(\theta) = \sum_{n=-N}^{N} a_n e^{in\theta} &= \sum_{n=-N}^{N} \frac{a_n + \overline{a_{-n}}}{2} e^{in\theta} + \sum_{n=-N}^{N} \frac{a_n - \overline{a_{-n}}}{2} e^{in\theta} \\
&= \left(\frac{a_0 + \overline{a_0}}{2} + \sum_{n=1}^{N} (a_n + \overline{a_{-n}} + a_{-n} + \overline{a_n}) \frac{e^{in\theta} + e^{-in\theta}}{2} \right) \\
&\quad + i \left(\frac{a_0 - \overline{a_0}}{2i} + \sum_{n=1}^{N} (a_n - \overline{a_{-n}} - a_{-n} + \overline{a_n}) \frac{e^{in\theta} - e^{-in\theta}}{2i} \right) \\
&= \left(\mathrm{Re}(a_0) + \sum_{n=1}^{N} 2\mathrm{Re}(a_n + a_{-n}) \cos(n\theta) \right) \\
&\quad + i \left(\mathrm{Im}(a_0) + \sum_{n=1}^{N} 2\mathrm{Re}(a_n - a_{-n}) \sin(n\theta) \right).
\end{aligned}
$$

Hence, if $P = \mathrm{Re}(a_0) + \sum_{n=1}^{N} 2\mathrm{Re}(a_n + a_{-n}) \cos(n\theta)$ and $Q = \mathrm{Im}(a_0) + \sum_{n=1}^{N} 2\mathrm{Re}(a_n - a_{-n}) \sin(n\theta)$, then P and Q are real-valued trigonometric polynomials, and $f = P + iQ$. Then

$$\|Hf\|_{2k} \leq \|HP\|_{2k} + \|iHQ\|_{2k} \leq 2C_{2k}(\|P\|_{2k} + \|Q\|_{2k}) \leq 2C_{2k}(\|f\|_{2k} + \|f\|_{2k}) = 4C_{2k}\|f\|_{2k}.$$

Hence, there exists a constant A_p such that

$$\|Hf\|_p \leq A_p \|f\|_p$$

for all trigonometric polynomials f when $p = 2k$, with $k \in \mathbb{N}$. Since any $p \geq 2$ is in an interval of the form $[2k, 2(k+1)]$, the Riesz-Thorin interpolation theorem extends this result for all $p \geq 2$. Since trigonometric polynomials are dense in $L^p(\mathbb{T})$, H extends to a bounded operator on $L^p(\mathbb{T})$ as well.

For $1 < p < 2$, we use the fact that H is skew-adjoint; i.e. for $f, g \in L^2(\mathbb{T})$ we have

$$
\begin{aligned}
\langle Hf, g \rangle = \int_{\mathbb{T}} Hf(\theta)\overline{g(\theta)} d\theta &= \int_{\mathbb{T}} \sum_{n\in\mathbb{Z}} -i\mathrm{sgn}(n) \hat{f}(n) e^{in\theta} \overline{g(\theta)} d\theta \\
&= \sum_{n\in\mathbb{Z}} -i\mathrm{sgn}(n) \hat{f}(n) \int_{\mathbb{T}} e^{in\theta} \overline{g(\theta)} d\theta \\
&= \sum_{n\in\mathbb{Z}} -i\mathrm{sgn}(n) \hat{f}(n) \overline{\hat{g}(n)} \\
&= \sum_{n\in\mathbb{Z}} -i\mathrm{sgn}(n) \overline{\hat{g}(n)} \int_{\mathbb{T}} f(\theta) e^{-in\theta} d\theta \\
&= \int_{\mathbb{T}} -f(\theta) \sum_{n\in\mathbb{Z}} \overline{i\mathrm{sgn}(n) \hat{g}(n) e^{in\theta}} d\theta = \int_{\mathbb{T}} -f(\theta) \overline{Hg(\theta)} d\theta = \langle -f, Hg \rangle.
\end{aligned}
$$

As such, if $\frac{1}{p}+\frac{1}{p'}=1$, with $p'\geq 2$, then $|\langle Hf,g\rangle| = |\langle -f,Hg\rangle| \leq \|f\|_p\|Hg\|_{p'} \leq C_{p'}\|f\|_p\|g\|_{p'}$. If f is a trigonometric polynomial, and $g=\overline{\left(\frac{|Hf|^p}{Hf}\right)}$, then $g\in L^{p'}(\mathbb{T})$, since

$$\|g\|_{p'} = \left(\int\limits_{\mathbb{T}} \left|\overline{\left(\frac{|Hf|^p}{Hf}\right)}\right|^{p'}\right)^{\frac{1}{p'}} = \left(\int\limits_{\mathbb{T}} \left(|Hf|^{p-1}\right)^{\frac{p}{p-1}}\right)^{1-\frac{1}{p}} = \|Hf\|_p^{p-1}.$$

Thus, we have

$$\|Hf\|_p^p = |\int\limits_{\mathbb{T}} Hf\frac{|Hf|^p}{Hf}| = |\langle Hf,g\rangle| \leq C_{p'}\|f\|_p\|g\|_{p'} = C_{p'}\|f\|_p\|Hf\|_p^{p-1}.$$

Hence, $\|Hf\|_p \leq C_{p'}\|f\|_p$, and H is L^p bounded on the space of trigonometric polynomials for $1<p<2$ as well. Density of trigonometric polynomials allows us to extend H to a bounded operator on $L^p(\mathbb{T})$, concluding the proof. □

5. Uniform convergence and divergence of certain Fourier series

We now turn our attention to convergence in the L^∞ norm, or uniform convergence, and the rate of convergence of Fourier series, as well as examples of divergent Fourier series.

Definitions 5.1. Let $0<\alpha\leq 1$. Then $f:\mathbb{T}\to\mathbb{C}$ satisfies a *Lip-α* condition if there exists a constant C such that

$$|f(\theta)-f(\eta)| \leq C|\theta-\eta|^\alpha$$

for all $\theta,\eta\in\mathbb{T}$. The minimum such C is called the *Lip-α constant* of f and is denoted $[f]_\alpha$.

Define $C^\alpha(\mathbb{T})$ to be the set of functions $f:\mathbb{T}\to\mathbb{C}$ which satisfy a Lip-α condition.

Remark 5.2. It can be shown that the minimum value of C above does exist by the compactness of $\mathbb{T}$; hence $[f]_\alpha$ is well-defined for each f. In addition, it can be shown that $C^\beta(\mathbb{T})\subseteq C^\alpha(\mathbb{T})$ if $\beta\geq\alpha$, also by the compactness of $\mathbb{T}$.

Theorem 5.3. *Suppose* $0<\alpha<1$. *Then there exists a constant* K_α *such that for all* $f\in C^\alpha(\mathbb{T})$, *we have*

$$\|S_Nf-f\|_\infty < K_\alpha[f]_\alpha N^{-\alpha}\log N.$$

In particular, $\|S_Nf-f\|_\infty \xrightarrow{N\to\infty} 0$.

Proof. From Proposition (2.11), we obtain the estimates $F_N(\theta)\leq\min(N,\frac{\pi^2}{N\theta^2})$. Thus, we have

$$\begin{aligned}
|\sigma_Nf(0)-f(0)| &= \left|\int\limits_{\mathbb{T}} f(\theta)F_N(-\theta)d\theta - \int\limits_{\mathbb{T}} f(0)F_N(-\theta)d\theta\right| \\
&\leq \int\limits_{\mathbb{T}} |f(\theta)-f(0)||F_N(-\theta)|d\theta \\
&\leq \frac{1}{2\pi}\left(\int\limits_{|\theta|<\frac{\pi}{N}} [f]_\alpha|\theta|^\alpha N d\theta + \int\limits_{\frac{\pi}{N}<|\theta|<\pi} C_f|\theta|^\alpha\left(\frac{\pi^2}{N|\theta|^2}\right)d\theta\right) \\
&= \frac{[f]_\alpha}{2\pi}\left(\frac{2N(\frac{\pi}{N})^{\alpha+1}}{\alpha+1} + \frac{2\pi^2}{N(\alpha-1)}\left(\pi^{\alpha-1}-(\frac{\pi}{N})^{\alpha-1}\right)\right) \leq C_\alpha[f]_\alpha N^{-\alpha}
\end{aligned}$$

for some constant C_α. By applying the same argument to $\tau_\theta f$, we see that $|\sigma_N f(\theta) - f(\theta)| \leq C_\alpha [f]_\alpha N^{-\alpha}$ for all θ. Since $S_N(\sigma_N f) = \sigma_N f$, we have

$$\begin{aligned} \|S_N f - f\|_\infty &= \|S_N f - S_N(\sigma_N f) + \sigma_N f - f\|_\infty \\ &\leq \|D_N * (\sigma_N f - f)\|_\infty + \|\sigma_N f - f\|_\infty \\ &\leq (\|D_N\|_1 + 1)(\|\sigma_N f - f\|_\infty) \\ &\leq (1 + C\log N) C_\alpha [f]_\alpha N^{-\alpha} < K_\alpha [f]_\alpha N^{-\alpha} \log N. \end{aligned}$$

□

Since $C^1(\mathbb{T}) \subseteq C^\alpha(\mathbb{T})$ for any $0 < \alpha < 1$, uniform convergence for $C^1(\mathbb{T})$ follows as well, albeit without the same rate of convergence as shown in the theorem.

It is interesting to note that some subspaces of $L^2(\mathbb{T})$, such as $C^\alpha(\mathbb{T})$, have formulas for rates of uniform convergence based solely on N and a seminorm. This is obviously not the case for uniform convergence in $L^2(\mathbb{T})$, since not all functions in $L^2(\mathbb{T})$ are even continuous, but this is also not the case in the L^2 norm either, since any function can be multiplied by exponentials of the form $e^{in\theta}$ without changing the L^2 norm while shifting Fourier coefficient to the right. One family of subspaces of $L^2(\mathbb{T})$ which does provide a rate of L^2 convergence is the family of Sobolev spaces.

Definition 5.4. Let $s > 0$. The Sobolev space $H^s(\mathbb{T})$ is defined to be the set of functions f such that

$$\|f\|_{H^s} := \left(|\hat{f}(0)|^2 + \sum_{n\in\mathbb{Z}} |n|^{2s} |\hat{f}(n)|^2 \right)^{\frac{1}{2}} < \infty.$$

Proposition 5.5. $\|S_N f - f\|_2 \leq \frac{1}{N^{2s}} \|f\|_{H^s}$

Proof. We estimate $\|S_N f - f\|_2 = \sum_{|n|>N} |\hat{f}(n)|^2 \leq \frac{1}{N^{2s}} \sum_{|n|>N} |n|^{2s} |\hat{f}(n)|^2 \leq \frac{1}{N^{2s}} \|f\|_{H^s}$ □

As shown before earlier in Section 4, a function requires more than mere continuity for its Fourier series to converge uniformly. We now present an example, due to Fejer, of a continuous function whose Fourier series diverges at a point, and a continuous function whose Fourier series diverges in an everywhere dense set of points.

Theorem 5.6. *There exists $f \in C(\mathbb{T})$ such that $S_N f(\theta)$ diverges for some θ.*

Proof. Let μ and n be positive integers, and let

$$\begin{aligned} Q_{\mu,n}(\theta) = &\frac{\cos(\mu\theta)}{n} + \frac{\cos((\mu+1)\theta)}{n-1} + \cdots + \frac{\cos((\mu+n-2)\theta)}{2} + \frac{\cos((\mu+n-1)\theta)}{1} \\ &- \frac{\cos((\mu+n+1)\theta)}{1} - \cdots - \frac{\cos((\mu+2n)\theta)}{n}. \end{aligned}$$

Since $\cos(\theta) = \frac{e^{i\theta}+e^{-i\theta}}{2}$, we have

$$\widehat{Q_{\mu,n}}(m) = \begin{cases} \frac{1}{2(n-k)} & \text{if } |m| = \mu + k, \quad 0 \leq k \leq n-1 \\ -\frac{1}{2k} & \text{if } |m| = \mu + n + k, \quad 1 \leq k \leq n \\ 0 & \text{otherwise} \end{cases}.$$

Using the identity $\cos(a-b) - \cos(a+b) = 2\sin a \sin b$, we also have

$$Q_{\mu,n}(\theta) = \sum_{k=1}^{n} \frac{\cos((\mu-k)\theta) - \cos((\mu+k)\theta)}{k} = 2\sin(\mu\theta) \sum_{k=1}^{n} \frac{\sin(k\theta)}{k}.$$

The partial sums of $\sum_{k=1}^{\infty} \frac{\sin(k\theta)}{k}$ are uniformly bounded (see [6]), so there exists C such that $|Q_{\mu,n}(\theta)| < C$ for all μ, n, and θ. Thus, if $\sum_{k=1}^{\infty} |\alpha_k| < \infty$, then $\sum_{k=1}^{\infty} \alpha_k Q_{\mu_k,n_k}$ converges uniformly to a continuous function $f_{\{\alpha_k\},\{\mu_k\},\{n_k\}}$ for any sequences $\{\mu_k\}$, $\{n_k\}$. In particular, if $\mu_{k+1} > \mu_k + 2n_k$ for all k, then Q_{μ_k,n_k} and $Q_{\mu_{k+1},n_{k+1}}$ have disjoint frequencies for all k, so

$$\hat{f}(m) = \begin{cases} \frac{\alpha_k}{2(n_k - j)} & \text{if } |m| = \mu_k + j, \qquad 0 \leq j \leq n_k - 1 \\ -\frac{\alpha_k}{2j} & \text{if } |m| = \mu_k + n_k + j, \qquad 1 \leq j \leq n_k \\ 0 & \text{otherwise} \end{cases}.$$

Let $\alpha_k = \frac{1}{k^2}$, $\mu_k = n_k = 2^{k^2}$, and let $f := f_{\{\alpha_k\},\{\mu_k\},\{n_k\}}$ be defined as above. Then f is continuous. However, for all k we have

$$\begin{aligned} |S_{\mu_k+n_k-1}f(0) - S_{\mu_k-1}f(0)| &= |\sum_{\mu_k \leq |j| \leq \mu_k+n_k-1} \hat{f}(m)| \\ &= |2 \sum_{j=0}^{n_k-1} \frac{1}{2k^2(n_k - j)}| \\ &= \frac{1}{k^2} \sum_{j=1}^{n_k} \frac{1}{j} \geq \frac{1}{k^2} \log(n_k) = \log(2). \end{aligned}$$

Hence, by the Cauchy criterion, the Fourier series of f at 0 does not converge. □

Theorem 5.7. *There exists a function $f^* \in C(\mathbb{T})$ whose Fourier series diverges at an everywhere dense set of points.*

Proof. Let f be the function above. Let $\{\theta_i\}_{i\in\mathbb{N}}$ be an everywhere dense subset of $\mathbb{T}$ (say the rationals ordered in some way), and let $\{\epsilon_i\}_{i\in\mathbb{N}}$ satisfy $\epsilon_i > 0$ for all i and $\sum_{i\in\mathbb{N}} \epsilon_i < \infty$. Then the sum $\sum_{i=1}^{\infty} \epsilon_i f(\theta - \theta_i)$ converges uniformly to a continuous function, which we denote by f^*. The Fourier series of f^*, whose coefficients are the sums of the coefficients of $\epsilon_i f(\theta - \theta_i)$, diverges at every θ_i, and hence in an everywhere dense set of points. □

In fact, Katznelson extended this result further, to the existence of a continuous function whose Fourier series diverges at every point of a measure zero set for any such specified measure zero subset of $\mathbb{T}$. The result (which can be found in [3]) will not be proven here, but it does call into question whether the Fourier series of any arbitrary continuous function converges for a large set of points, or even for any points at all. This issue will be resolved in the next section.

Theorem 5.8. *(Katznelson) Let E be a measure zero subset of $\mathbb{T}$. Then there exists $f \in C(\mathbb{T})$ such that the Fourier series of f diverges at every point of E (and possibly others).*

Finally, Kolmogorov showed in [4] that if we relaxed the condition of continuity to mere integrability, then there exists an integrable function whose Fourier series diverges everywhere. This result exacerbates the issue of determining the convergence of arbitrary Fourier series and opens up the possibility of the Fourier series of a continuous function possibly diverging everywhere.

6. Almost everywhere convergence of Fourier series of L^p functions

In 1966, Lennart Carleson resolved the issue of whether Fourier series of continuous functions were guaranteed to converge almost everywhere, by proving a remarkable result: that the Fourier series of an L^p function converges almost everywhere back to the original function. So far, no simple proof of Carleson's theorem is known, so an attempt at proving this theorem is outside the scope of this paper, although a brief sketch involving some of the easier details is presented.

Definition 6.1. Let $f \in L^1(\mathbb{T})$. The maximal operator M is defined by

$$Mf(\theta) = \sup_{N\in\mathbb{N}} |S_N f(\theta)|$$

The crux of this theorem depends on the following fact, which will not be proven here:

Theorem 6.2. *Let $1 < p < \infty$. Then there exists a constant C_p for which*

$$\|Mf\|_p \leq C_p \|f\|_p$$

for all $f \in L^p(\mathbb{T})$.

The proof of this fact can be found in [2].

To prove Carleson's theorem, we will also need the following variant of Markov's inequality:

Theorem 6.3. *Let $f \in L^p(\mathbb{T})$, and for $\lambda > 0$, let $E_\lambda = \{\theta : |f(\theta)| \geq \lambda\}$. Then*

$$m(E_\lambda) \leq \frac{\|f\|_p^p}{\lambda^p}.$$

Proof. We estimate

$$\|f\|_p^p = \int\limits_{\mathbb{T}} |f|^p \geq \int\limits_{E_\lambda} |f|^p \geq \int\limits_{E_\lambda} \lambda^p = \lambda^p m(E_\lambda).$$

$\square$

Now we are ready to present Carleson's Theorem:

Theorem 6.4. *(Carleson) Let $f \in L^p(\mathbb{T})$, $1 < p < \infty$. Then $S_N f$ converges to f almost everywhere.*

Proof. Let g be a trigonometric polynomial. Then $S_N g = g$ for large enough N. As such, for any θ, we have

$$\begin{aligned}
\limsup_N |S_N f(\theta) - f(\theta)| &= \limsup_N |S_N f(\theta) - S_N g(\theta) + g(\theta) - f(\theta)\\
&\leq \limsup_N |S_N(f-g)(\theta)| + |g(\theta) - f(\theta)|\\
&\leq M(f-g)(\theta) + |g(\theta) - f(\theta)|.
\end{aligned}$$

For any $\epsilon > 0$, let

$$\begin{aligned}
D_\epsilon &= \{\theta : \limsup_N |S_N f(\theta) - f(\theta)| \geq \epsilon\},\\
E_\epsilon &= \{\theta : M(f-g)(\theta) \geq \frac{\epsilon}{2}\},\\
F_\epsilon &= \{\theta : |(f-g)(\theta)| \geq \frac{\epsilon}{2}\}.
\end{aligned}$$

Since $\limsup_N |S_N f(\theta) - f(\theta)| \leq M(f-g)(\theta) + |(f-g)(\theta)|$, it follows that if $\limsup_N |S_N f(\theta) - f(\theta)| \geq \epsilon$, then $M(f-g)(\theta) \geq \frac{\epsilon}{2}$ or $|(f-g)(\theta)| \geq \frac{\epsilon}{2}$. Hence, $D_\epsilon \subseteq E_\epsilon \cup F_\epsilon$, and $m(D_\epsilon) \leq m(E_\epsilon) + m(F_\epsilon)$. By Markov's inequality, $m(E_\epsilon) \leq \frac{2^p}{\epsilon^p}\|M(f-g)\|_p^p \leq \frac{2^p}{\epsilon^p} C_p^p \|f-g\|_p^p$, and $m(F_\epsilon) \le \frac{2^p}{\epsilon^p}\|f-g\|_p^p$. Hence, $m(D_\epsilon) \le \frac{2^p}{\epsilon^p}(C_p^p+1)\|f-g\|_p^p$. Since trigonometric polynomials are dense in $L^p(\mathbb{T})$, it follows that $m(D_\epsilon) = 0$ for all ϵ. Thus,

$$\begin{aligned} m(\{\theta : \limsup_N |S_N f(\theta) - f(\theta)| > 0\}) &= m\left(\bigcup_{k=1}^{\infty} \{\theta : \limsup_N |S_N f(\theta) - f(\theta)| \ge \frac{1}{k}\}\right) \\ &= m\left(\bigcup_{k=1}^{\infty} D_{\frac{1}{k}}\right) = 0, \end{aligned}$$

i.e. $S_N f(\theta) \to f(\theta)$ for almost every θ. □

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