PLANE CONICS IN ALGEBRAIC GEOMETRY

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Abstract. We first examine points and lines within projective spaces. Then we classify affine conics based on the classification of projective conics. Based on the parametrization of conics, we also prove two easy cases of Bézout's Theorem. In the end we turn to the discussion of the space of conics and the notion of pencils of conics.

Contents

1. Introduction

Example 1.1. We describe the parametrization of all integer solutions for $X^2 + Y^2 = Z^2$.

Because the given equation is homogeneous (every term has the same degree), we can use factorization to simplify the task.

When $Z \neq 0$, we parametrize the equation by factorizing out Z^2. We then have the equation $(\frac{X}{Z})^2 + (\frac{Y}{Z})^2 = (\frac{Z}{Z})^2$. Now define x to be $\frac{X}{Z}$ and y to be $\frac{Y}{Z}$. The new equation $x^2 + y^2 = 1$ represents a circle in $\mathbb{R}^2$.

Since X, Y, Z are integers, (x, y) represent all points on the circle whose coordinates are both rational numbers. We choose an arbitrary point $O = (1, 0)$, and from O draw a line $y = \lambda(x - 1)$, for every slope $\lambda \in \mathbb{Q}$. In other words, $\lambda = l/m$, where $l, m \in \mathbb{N}$. There are two variables, x and y, and two equations. Solving the equations, we have:

$$x = \frac{\lambda^2 - 1}{\lambda^2 + 1} = \frac{X}{Z},\ y = \frac{-2\lambda}{\lambda^2 + 1} = \frac{Y}{Z},\ \lambda = \frac{y}{x - 1}.$$
$$X = l^2 - m^2,\ Y = 2lm,\ Z = l^2 + m^2.$$

Here $l, m \in N$ and l is co-prime with m. When $l = m = 0$, the above formula provides the solution $(0, 0, 0)$, where $Z = 0$. So these formulas include all integer solutions for $X^2 + Y^2 = Z^2$.

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This example illustrates the process of solving algebraic equations with geometric methods. Dehomogenization reduces the number of variables in the equation, which enables us to find geometric representations of equations on a plane. These intersections between the set of lines with rational slopes starting from O and the circle generate a set of rational points. Multiplying coordinates of these points by Z results in integer solutions of X, Y for $X^2 + Y^2 = Z^2$.

Similarly, dehomogenization and projection are also used in finding coordinates of points in projective spaces.

2. $\mathbb{P}^2_{\mathbb{R}}, \mathbb{P}^1_{\mathbb{R}}$, and projective transformations

By defining projective spaces, projective planes and projective lines, we can analyze the relationship between projective and affine objects. We will also discover how parallel lines intersect in projective spaces. And we will compare projective transformations with affine transformations.

Definition 2.1. The *projective space* $\mathbb{P}(V)$ is the set of lines passing through the origin of a vector space V. The projection of $V = \mathbb{R}^3$ is the projective plane $\mathbb{P}^2_{\mathbb{R}}$. When $V = \mathbb{R}^2$, the projective space is the projective line $\mathbb{P}^1_{\mathbb{R}}$.

We can understand projective planes based on equivalence classes and homogeneous coordinates. By definition, homogeneous coordinates are coordinates that are invariant up to scaling. For example: $(x, y, z) = (2x, 2y, 2z)$. In $\mathbb{R}^3$, every line passing through origin can be expressed by the equation $aX + bY + cZ = 0$. If one point (x, y, z) lies on a line, then all of multiples of this point, $\alpha(x, y, z)$, also lie on the same line. So all points on one line form an equivalence class represented by (x, y, z). In a projective space, we denote this equivalence class by point (x, y, z). Here (x, y, z) is a homogeneous coordinate. So $\mathbb{P}^2_{\mathbb{R}}$ is the set of homogeneous coordinates that represent all lines passing through the origin in $\mathbb{R}^3$. However, the origin in $\mathbb{R}^3$ is not included in projective space.

$$\mathbb{P}^2_{\mathbb{R}} = (\mathbb{R}^3 \setminus 0)/\sim,$$

and each homogeneous coordinate [a:b:c] is a representative of its equivalence class,

$$(a, b, c) \sim (\lambda a, \lambda b, \lambda c).$$

We analyze the projective space $\mathbb{P}^2_{\mathbb{R}}$ by dividing it into affine components.

Proposition 2.2. $\mathbb{P}^2_{\mathbb{R}} = \mathbb{A}^2 \sqcup \mathbb{P}^1_{\mathbb{R}}$.

Proof. The set of all equivalence classes $[X : Y : Z]$ in $\mathbb{R}^3$ consists of $[x : y : 1]$ (where $x = \frac{X}{Z}, y = \frac{Y}{Z}, Z \neq 0$) and $[X : Y : 0]$. The first case contains a copy of $\mathbb{R}^2$. We call this affine plane $\mathbb{A}^2$ since it is invariant up to scaling (multiply by λ) and shearing (the shifting of the vector space while preserving a line; for projective plane, this is the line at infinity, which we will discuss later).

The equivalence classes represented by $[X : Y : 0]$ do not intersect with any points from $[x : y : 1]$ as they do not share the same third coordinates. Thus these two sets of elements in $\mathbb{P}^2_R$ are disjoint. Because

$$[x : y : 1] = (\mathbb{A}^2 \setminus 0)/\sim \text{ and } [X : Y : 0] = \mathbb{P}^1_{\mathbb{R}},$$

$\mathbb{P}^2_{\mathbb{R}} = \mathbb{A}^2 \sqcup \mathbb{P}^1_{\mathbb{R}}$. □

Proposition 2.3. $\mathbb{P}^1_{\mathbb{R}} = \mathbb{A}^1 \sqcup \mathbb{A}^0$.

Proof. The set of all equivalence classes $[X : Y]$ in $\mathbb{R}^2$ consists of $[\frac{X}{Y} : 1]$ (where $Y \neq 0$) and $[X : 0]$. The first case contains a copy of $\mathbb{R}^1$. Thus it is bijective to $\mathbb{A}^1$.

$[X : 0]$ represents the case of Y=0. Factoring out Y generates $[\frac{X}{0}, \frac{0}{0}]$. Since any number divided by 0 does not generate a number, $[X : 0]$ is the point at infinity that does not intersect with $[X : Y]$. We write it as $\mathbb{A}^0$. □

Proposition 2.4. $\mathbb{P}^n_{\mathbb{R}} = \mathbb{A}^n \sqcup \mathbb{P}^{n-1}$.

Proof. The subspaces for $\mathbb{P}^n_{\mathbb{R}}$ include $[x_1, x_2 \ldots x_n, 1]$ and $[x_1, x_2 \ldots x_n, 0]$. The former subspace consists of a copy of $\mathbb{R}^{n+1}$, and it is bijective to $\mathbb{A}^n$. The latter space consists of lines passing through the origin in $\mathbb{R}^{n-1}$, so it is $\mathbb{P}^{n-1}$. □

The above propositions explicate the structure for projective spaces. By induction,

$$\mathbb{P}^n_{\mathbb{R}} = \mathbb{A}^n \sqcup \mathbb{A}^{n-1} \sqcup \cdots \sqcup \mathbb{A}^0.$$

Proposition 2.5. *Parallel lines in $\mathbb{P}^2_{\mathbb{R}}$ meet at the line at infinity.*

Proof. We start with the formula of lines in the Euclidean Space $\mathbb{R}^3$,

$$ax + by + c = 0, \text{ with slope k = -a/b.}$$

Since projective spaces only use homogeneous coordinates, we want to homogenize the general equation for lines so that each term has the same degree. To homogenize, we multiply the equation with a variable Z and define $X = xZ$, $Y = yZ$. Thus we have

$$aX + bY + cZ = 0.$$

$[X : Y : Z]$ represents the equivalence classes in $\mathbb{P}^2_{\mathbb{R}}$. So $aX + bY + cZ = 0$ is invariant up to scaling and we can use $[a : b : c]$ to represent the equivalence classes of lines. If two lines l_1, l_2 are parallel in the projective plane, then $[a_1 : b_2] \sim [a_2 : b_2]$. Therefore we can scale the 3-tuples of both lines to make the first two coordinates equal.

$$aX + bY + c_1Z = aX + bY + c_2Z.$$

Since l_1,l_2 are parallel, $c_1 \neq c_2$. The solution is $Z = 0$, which means the intersection lies on the line at infinity in $\mathbb{P}^2_{\mathbb{R}}$. □

This counter-intuitive result actually corresponds to real-life experiences of seeing two train tracks intersecting at the horizon. Every pair of non-identical lines in the projective space has exactly one intersection. Projective space is regarded as affine space with a line at infinity.

After understanding the structure and property of projective spaces, we are able to study transformations within it.

Definition 2.6. An *affine change* of coordinates in $\mathbb{R}^2$ is of the form

$$T(\mathbf{x}) = A\mathbf{x} + B,$$

where $\mathbf{x}$ = (x,y) $\in R^2$ and A is a 2×2 invertible matrix.

Definition 2.7. A *projective transformation* of $\mathbb{P}^2_{\mathbb{R}}$ is of the form T($\mathbf{X}$) = M$\mathbf{X}$, where M is an invertible 3 x 3 matrix. The projective transformation defines an isomorphic mapping from one space to another. It preserves the linearity of the space and it is well-defined.

Definition 2.8. $T([\mathbf{X}])$ is the set of images of projective transformations from elements in the equivalence class $[\mathbf{X}]$. $T([\mathbf{X}]) = \{Mx | x \in [\mathbf{X}]\}$.

Exercise 2.9. $[T(\mathbf{X})] = T([\mathbf{X}])$.

Proof. Take $M = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$.

$$T([\mathbf{X}]) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \left[\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right] = \left[\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} \right].$$

$[T(x : y : z)] = T([x : y : z])$ as $\left[\begin{bmatrix} \lambda x' \\ \lambda y' \\ \lambda z' \end{bmatrix} \right] = \lambda \left[\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} \right]$. Therefore the projective transformation is well defined. □

A projective transformation on an affine space $\mathbb{R}^2 \subset \mathbb{P}^2_{\mathbb{R}}$ is the fractional-linear transformation (a rational function of the form $f(z) = \frac{az+b}{cz+d}$) mapping $\mathbb{R}^2$ to $\mathbb{R}^2$

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto (A \begin{bmatrix} x \\ y \end{bmatrix} + B)/cx + dy + e \ (A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}),$$

where

$$M = \begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ c & d & e \end{bmatrix}.$$

Proposition 2.10. *There is a unique projective transformation in $\mathbb{P}^1$ that maps a triplet $(0, 1, \infty)$ to $(P, Q, R) \in \mathbb{P}^1$.*

Proof. Since $\mathbb{A}^1 \subset \mathbb{P}^1_{\mathbb{R}}$, $(0, 1, \infty) \subset \mathbb{A}^1_{\mathbb{R}}$. Respectively, their coordinates in $\mathbb{P}^1_{\mathbb{R}}$ are

$$0 \longmapsto [0 : 1],$$
$$1 \longmapsto [1 : 1],$$
$$\infty \longmapsto [1 : 0].$$

We want to show there exists a unique $M_{2\times 2}$ such that $M[0, 1, \infty] = [P, R, Q]$ with $P = [p_1 : p_2]$, $Q = [q_1 : q_2]$, $R = [r_1 : r_2]$, where P, Q, R are distinct.

$$M \times P = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda p_1 \\ \lambda p_2 \end{bmatrix}, a_{12} = \lambda p_1, a_{22} = \lambda p_2.$$
$$M \times Q = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \beta q_1 \\ \beta q_2 \end{bmatrix}, a_{11} = \beta q_1, a_{21} = \beta p_2.$$
$$M \times R = \begin{bmatrix} \beta q_1 & \lambda p_1 \\ \beta p_2 & \lambda p_2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha r_1 \\ \alpha r_2 \end{bmatrix}.$$

This gives us two linear equations with three unknown variables β, λ, α,

$$\beta q_1 + \lambda p_1 = \alpha r_1,$$
$$\beta q_2 + \lambda p_2 = \alpha r_2.$$

So we can calculate the ratio among the three variables. This ratio also describes the equivalence class $[\beta : \lambda : \alpha]$. Suppose $\alpha = 1$. The new equations are now in $\mathbb{A}^1$:

$$\beta q_1 + \lambda p_1 = r_1,$$
$$\beta q_2 + \lambda p_2 = r_2.$$

To ensure that this system of linear equations has a unique solution, we require $q_1p_2 - q_2p_1 \neq 0$. The points P and Q satisfy this condition since they are distinct. Otherwise, if $q_1p_2 - q_2p_1 = 0$, then $\frac{q_1}{q_2} = \frac{p_1}{p_1}$. P and Q would be the same point, which is a contradiction.

Returning from $\mathbb{A}^1$to $\mathbb{P}^1$, the unique solution of λ, β still applies and it defines the unique transformation mapping [0,1,∞] to [P,Q,R]. □

Proposition 2.11. *There is a unique projective transformation in $\mathbb{P}^1$ that maps three distinctive points $P_1, Q_1, R_1 \in \mathbb{P}^1$ to $P_2, Q_2, R_2 \in \mathbb{P}^1$.*

Proof. Based on Proposition 2.9, there exists a unique projective transformation T mapping $(0, 1, \infty)$ to $(P, Q, R) \subset \mathbb{P}^1$. So define M_1 as transformation matrix for $(P_1, Q_1, R_1) \subset \mathbb{P}^1$. Similarly, we define M_2 for the triplet (P_2, Q_2, R_2).

Suppose

$$X \in (0, 1, \infty)\,, M_1X = Y_1 \in (P_1, Q_1, R_1),$$

and

$$M_2X = Y_2 \in (P_2, Q_2, R_2).$$

So

$$M_2M_1^{-1}Y_1 = Y_2.$$

Define

$$M = M_2M_1^{-1},\ Y_2 = MY_1.$$

M is the matrix for the unique transformation between two triplets in $\mathbb{P}^1_{\mathbb{R}}$. □

Definition 2.12. Equations of the conics in the Euclidean space correspond to the inhomogeneous quadratic polynomial

$$q(x, y) = ax^2 + bxy + cy^2 + dx + ey + f.$$

This quadratic polynomial also corresponds to the homogeneous quadratic equation in a projective space

$$Q(X, Y, Z) = aX^2 + bXY + cY^2 + dXZ + eYZ + fZ^2,$$

as $q(x, y) = Q(X/Z, Y/Z)$ and $Q(X, Y, Z)$ is well-defined.

Proposition 2.13. *The projective curve $C : (Q(X, Y, Z) = 0)$ meets the affine piece $\mathbb{R}^2$ in the conic given by $q = 0$.*

Proof. Dehomogenize $Q(X, Y, Z)$ by dividing all coordinates by Z.

$$Q(X/Z, Y/Z, 1) = a(X/Z)^2 + b(XY/Z^2) + c(Y/Z)^2 + d(X/Z) + e(Y/Z) + f.$$

Let $X/Z = x$ and $Y/Z = y$. Then we have

$$Q(X/Z, Y/Z) = q(x, y) = ax^2 + bxy + cy^2 + dx + ey + f = 0.$$

C meets the affine piece in the affine conic given by q = 0.

□

3. Classification of Conics in $\mathbb{P}^2$ and $\mathbb{A}^2$

Our experience with conics in the Euclidean space inspires us to define and classify conics in projective spaces. Since projective spaces use homogeneous coordinates, the formulas of projective conics are simpler than those of Euclidean or affine conics. Moreover, we will discover how the classification of projective conics leads to the classification of affine conics. Linear algebra is heavily used in proofs in this section.

Definition 3.1. The *dot product* between two vectors v_1, v_2 is the sum of the products of each pair of coordinates in v_1 and v_2. It is expressed as $v_1 \cdot v_2$.

Definition 3.2. The *characteristic* of a ring R, denoted $\mathrm{char}(R)$, is the smallest n such that $n \cdot 1 = 0$, where 1 is the identity element.

Proposition 3.3. *If k is a field of characteristic $\neq 2$, there is a natural bijection between quadratic forms $k^3 \to k$ and symmetric bilinear forms on k^3.*

Proof. Suppose there exist variables v, w in k^3 and that the quadratic form $Q(v) = v*v$. The symmetric bilinear forms can be written as the dot product $b(v, w) = v \cdot w$. To prove there exists a bijection from $b(v, w)$ to $Q(v)$, we let $w = v$. Then

$$Q(v) = b(v, v).$$

The function f mapping $b(v, v)$ to $Q(v)$ is an identity function. So it is a bijective map from b to Q.

To prove there exists a bijection from $Q(v)$ to $b(v, w)$, replace the v i $Q(v)$ with $v + w$. The equation for Q becomes

$$\begin{aligned} Q(v + w) = (v + w) * (v + w) &= v^2 + w^2 + 2v * w \\ &= Q(v) + Q(w) + 2v * w. \end{aligned}$$

So the bilinear form $b(v, w) = \frac{1}{2}(Q(v + w) - Q(v) - Q(w))$. Both sides of the equation include the same variables v, w and have degree 2. This demonstrates a bijective relation. We prove this by showing the map from Q to b is mutually inverse.

$$b(Q^{-1}(v), Q^{-1}(v)) = v * v$$

$Q(v)$ is a bijection with $v * v$, so $Q(v, v)^{-1}$ exists for every $v * v$:

$$b^{-1}(b(Q(v), Q(v)) = b^{-1}(Q(v) * Q(v)) = Q(v).$$

The function b^{-1} is a two-sided inverse function. Therefore there is a bijective map from Q to b. We have thus proven there is a bijection between quadratic forms and symmetric bilinear forms. □

Proposition 3.4. *If V is a vector space over k and Q: $V \to k$ is a quadratic form, then there exists a set of basis vectors $\{x_1, x_2 \ldots x_n\}$ of V such that*

$$Q = \varepsilon_1 x_1^2 + \varepsilon_2 x_2^2 + \ldots + \varepsilon_n x_n^2 \text{ , with } \varepsilon_i \in k.$$

Proof. For any $v \in V$, pick any basis $\{a_1, a_2 \ldots a_n\}$ of V, we have

$$v = \beta_1 a_1 + \beta_2 a_2 + \ldots + \beta_n a_n.$$

So $Q = v * v = (\beta_1 a_1 + \beta_2 a_2 + \ldots + \beta_n a_n)^2$.

Then transform the chosen basis $\{a_1, a_2 \ldots a_n\}$ into the orthonormal basis

$\{u_1, u_2 \dots\ u_n\}$. The transformation is done by Gram-Schmidt Orthonormalization. Take the first element of the basis $\{a_1, a_2 \dots\ a_n\}$ as the first element $u_1$of the new basis, then we have

$$u_i = \sum_{k=1}^{i-1} \alpha_k u_k + a_i \text{ with } \alpha_k = -\frac{\langle a_i, u_k \rangle}{\|u_k\|^2}.$$

When $u_k = \overrightarrow{0}$, α_i can be any value in k. Finally we scale the new vectors so their lengths are all 1, and we let $x_i = \frac{u_i}{||u_i||}$. The new basis satisfies

$$\langle\, x_i, x_i \rangle = 1, \langle\, x_i, x_j \rangle = 0 (i \neq\ j),$$

and x_j, a_j satisfy that

$$x_i - a_i \in\ U_{i-1} = span(a_1\ \dots\ a_{i-1}).$$

Lemma 3.5. *$span(u_1, u_2 \dots\ u_i) = U_i$.*

Proof. First we want to show $span(u_1, u_2 \dots\ u_n) \subseteq\ U_i$.

We know $u_i \in\ U_i$, because $u_i - a_i \in\ U_{i-1} = span(a_1\ \dots\ a_{i-1})$ and

$$u_i = (u_i - a_i) + a_i \in span(a_1\ \dots\ a_i) = U_i.$$

Therefore $span(u_1, u_2\ \dots\ u_i) \subseteq U_i$.

Now we want to show $U_i \subseteq\ span(u_1, u_2 \dots\ u_n)$. The proof is done by induction.

If $i = 1$, then $u_1 = a_1$ and $U_1 \subseteq span(u_1) = span(a_1)$.

Suppose $U_{i-1} \subseteq span(u_1, u_2 \dots\ u_{i-1})$. Since $u_i - a_i \in\ U_{i-1} \subseteq span(u_1, u_2 \dots\ u_{i-1})$,

$$a_i \in span(u_1, u_2 \dots\ u_i).$$

Therefore $span(a_1, a_2 \dots\ a_{i-1}, a_i) \subseteq span(u_1, u_2 \dots\ u_{i-1}, u_i)$, which means $U_i \subseteq span(u_1, u_2 \dots\ u_n)$. □

Based on Lemma 3.3, $u_i - a_i = \alpha_1 u_1 + \dots + \alpha_{i-1} u_{i-1}$. We will now find the unique α_ks according to the Gram-Schmidt process.

$$u_i = \sum_{k}^{i-1} \alpha_k u_k + a_i.$$

$$\begin{aligned} 0 = \langle\, u_i, u_j \rangle_{(i \neq j)} &= \langle\, a_i, u_j \rangle + \alpha_j \langle u_j, u_j \rangle \\ &= \alpha_j \|u_j\|^2 + \langle\, a_i, u_j \rangle. \end{aligned}$$

When $u_j \neq \overrightarrow{0}$, $\alpha_j = \frac{-\langle a_i, u_j \rangle}{|u_j\|^2}$. To ensure $\langle\, x_i, x_i \rangle = 1$, we let $x_i = \frac{u_i}{||u_i||}$ so that $\|x_i\| = 1$.

Now we have transformed the original basis of $\{a_1, a_2 \dots\ a_n\}$ into the orthonormal basis of $\{x_1, x_2 \dots\ x_n\}$.

$$Q(v) = Q(\beta_1 x_1 + \beta_2 x_2 + \dots + \beta_n x_n) = (\beta_1 x_1)^2 + (\beta_2 x_2)^2 + \dots + (\beta_n x_n)^2.$$

Replace $(\beta_1)^2, (\beta_2)^2 \dots (\beta_n)^2$ with $\epsilon_1, \epsilon_2 \dots\ \epsilon_n$. We have

$$Q = \varepsilon_1 x_1^2 + \varepsilon_2 x_2^2 + \dots + \varepsilon_n x_n^2, \text{ with } \varepsilon_i\ \in\ k.$$

□

Corollary 3.6. *In a suitable system of coordinates, any conic in $\mathbb{P}^2_{\mathbb{R}}$ is one of the following:*
(a) non-degenerate conic, $C : (X^2 + Y^2 - Z^2 = 0)$;
(b) empty set, given by $(X^2 + Y^2 + Z^2 = 0)$;
(c) line pair, given by $(X^2 - Y^2 = 0)$;
(d) one point (0,0,1), given by $(X^2 + Y^2 = 0)$;
(e) double line, given by $(X^2 = 0)$.

Proof. The equations of conics in $\mathbb{P}^2_{\mathbb{R}}$ are given by $C : (Q(X, Y, Z) = 0)$. Q is a quadratic form. So we can find an orthonormal basis (e_1, e_2, e_3) of (X, Y, Z) and

$$Q = \varepsilon_1 X^2 e_1^2 + \varepsilon_2 Y^2 e_2^2 + \varepsilon_3 Z^2 e_3^2 = \varepsilon_1 X^2 + \varepsilon_2 Y^2 + \varepsilon_3 Z^2.$$

The coefficient for each x_i^2 term is scaled to either 0 or ± 1. Now we can identify all classes of conics in $\mathbb{P}^2_{\mathbb{R}}$ as mentioned in Corollary 3.4. □

After classifying projective conics, we can classify affine conics by paying attention to the line at infinity in projective spaces. Examining the transformations of ellipses, parabolas and hyperbolas from $\mathbb{A}^2$ into conics in $\mathbb{P}^2$, we find that the new conics have different numbers of intersections with the line at infinity.

Example 3.7. An ellipse in $\mathbb{A}^2$ of the form $q(x, y) = \frac{4x^2}{9} + \frac{y^2}{9} = 1$ is transformed into a conic $Q(X, Y, Z) = 4X^2 + Y^2 - 9Z^2 = 0 \in \mathbb{P}^2_{\mathbb{R}}$ with $X = xZ, Y = yZ, Z = 1$. It has no intersection with the line at infinity Z=0.

A parabola in $\mathbb{A}^2$ of the form $q(x, y) = y - x^2 = 0$ is transformed into $Q(X, Y, Z) = -X^2 + YZ = 0$ with $X = xZ, Y = yZ$. It has one intersection $[0 : 1 : 0]$ on the line at infinity Z=0.

A hyperbola in $\mathbb{A}^2$ of the form $q(x, y) = \frac{4x^2}{9} - \frac{y^2}{9} = 1$ is transformed into $Q(X, Y, Z) = 4X^2 - Y^2 - 9Z^2 = -4X^2 + Y^2 + 9Z^2 = 0$. It has two intersections, $[1 : 2 : 0]$ and $[-1 : 2 : 0]$ on the line at infinity.

We will classify the smooth conics in affine spaces with consideration of the line at infinity in projective spaces. Below is a diagram that displays the relations between objects of interests in the classification.

$$\begin{tikzcd}
Smooth\,Affine\,Conic \arrow[r, "classification"] \arrow[d, "homogenization"'] & Ellipse, parabola, hyperbola \\
Smooth\,Projective\,Conic\,with\,line\,at\,\infty \arrow[r, "projective\,linear\,transformation"'] & X^2 + Y^2 - Z^2 = 0\,with\,a\,line \arrow[u, "remove\,a\,line\,relative\,to\,chosen\,A^2"']
\end{tikzcd}$$

Proposition 3.8. *Classify affine ellipses based on the classification of projective conics.*

Removing a line that does not intersect the conic C, we get ellipses in an affine space. For example, $C \cap (L : Z = 0) = \varnothing$. Dehomogenizing the equation of C produces $x^2 + y^2 = 1$, which represents an ellipse in $\mathbb{A}^2$.

Since $Z = 0$ is the arbitrarily chosen line, we need show the same result applies when we remove any line L_i that does not intersect with the projective conic. In other words, we need prove the existence of a projective transformation mapping L_i to L and C to C.

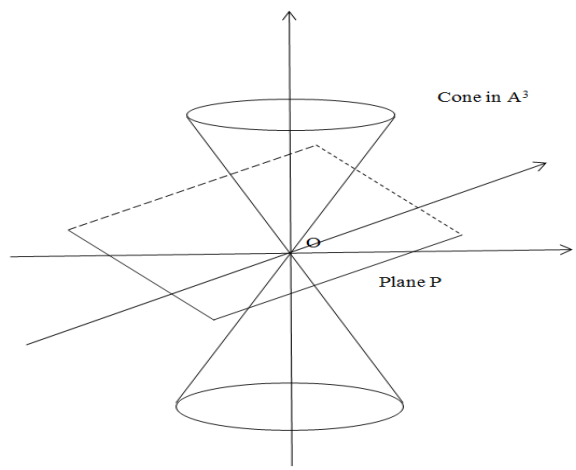

FIGURE 1. A plane P and a cone in $\mathbb{A}^3$

Suppose $M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$. The line transformation requires

$$\begin{bmatrix} m_{31} & m_{32} & m_{33} \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = 0.$$

Based on the bijection between symmetric bilinear forms and quadratic forms (Proposition 3.1), coordinate tuples of points x on C satisfy $x^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x = 0$. To preserve the conic, M satisfies

$$(Mx)^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} Mx = 0.$$

Thus we have the following lemma.

Lemma 3.9. *Given any line $L_i : AX + BY + CZ = 0 \subset \mathbb{P}^2_{\mathbb{R}}$ such that $L_i \cap C = \varnothing$, there exists a matrix M such that*

$$\begin{bmatrix} m_{31} & m_{32} & m_{33} \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = AX + BY + CZ$$

and $M^t \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

We also visualize projective conics and lines in the $\mathbb{R}^3$ space in Figure 1. Points $[X : Y : Z]$ in $\mathbb{P}^2$ represent lines in $\mathbb{R}^3$, so lines in $\mathbb{P}^2$ represent planes in $\mathbb{R}^3$ and conics in $\mathbb{P}^2$ represent cones in $\mathbb{R}^3$. In Figure 1 the plane P has one intersection with the double cone at the origin. In addition, the points inside of the cone satisfy

$X^2+Y^2-Z^2<0$ and points outside of the cone satisfy $X^2+Y^2-Z^2>0$. Based on this geometric model of a plane and a cone, we can rewrite Lemma 3.7 as follows.

Lemma 3.10. *Given any plane $P \subset A^2$ passing through the origin O and satisfying that there is no intersection between plane P and the cone, there exist v_1, v_2, v_3 such that*

(1) $v_1 \perp v_2 \perp v_3$,
(2) $P = span(v_1, v_2)$,
(3) $v_1^2 = 1, v_2^2 = 1, v_3^2 = -1$.

Proof. Choose 3 vectors $(u_1, u_2, u_3) \subset \mathbb{A}^2$. u_1, u_2, u_3 are linearly independent, $P \subset span(u_1, u_2)$ and $\langle u_i, u_i \rangle \neq 0$. Now we follow the Gram-Schmidt process to orthogonalize the chosen basis.

$$v_1 = u_1, v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\|v_1\|^2} v_1, v_3 = u_3 - \frac{\langle u_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle u_3, v_2 \rangle}{\|v_2\|^2} v_2.$$

Thus

$$\begin{aligned} v_1 v_2 &= \langle u_2, v_1 \rangle - \langle u_2, v_1 \rangle = 0, \\ v_1 v_3 &= \langle u_3, v_1 \rangle - 0 - \langle u_3, v_1 \rangle = 0, \\ v_2 v_3 &= \langle u_3, v_2 \rangle - \langle u_3, v_2 \rangle = 0. \end{aligned}$$

We have shown that v_1, v_2, v_3 are orthogonal to each other. Since $P \subset span(u_1, u_2)$, we need to show $span(u_1, u_2) = span(v_1, v_2)$.

Set $\frac{\langle u_2, v_1 \rangle}{\|v_1\|^2} = \alpha \in \mathbb{R}$, $v_1 = u_1, v_2 = u_2 - \alpha u_1$. Thus $span(u_1, u_2) = span(v_1, v_2)$.

To find values for $\langle v_1, v_1 \rangle, \langle v_2, v_2 \rangle, \langle v_3, v_3 \rangle$, we need to define the inner product in $\mathbb{A}_2$. Since v_1, v_2 are not standard bases in P, we need an inner product matrix M that calculates the inner product with respect to the standard basis. For example, choose any $x = \alpha v_1 + \beta v_2 = (\alpha, \beta)$. Then

$$\langle x, x \rangle = \alpha^2 \langle v_1, v_1 \rangle + \beta^2 \langle v_2, v_2 \rangle.$$

Here $\langle v_1, v_1 \rangle, \langle v_2, v_2 \rangle$ are inner products of v_1, v_2 with respect to the standard basis. In addition, $M = \begin{bmatrix} Mv_1 & Mv_2 \end{bmatrix}$. For any $x, y \in \mathbb{A}^2$

$$\langle x, y \rangle = x^T M y \text{ with } v_i v_j = v_i^T M v_j = m_{ij}.$$

Therefore we get $M = \begin{bmatrix} \langle v_1, v_1 \rangle & \langle v_1 v_2 \rangle \\ \langle v_2, v_1 \rangle & \langle v_2, v_2 \rangle \end{bmatrix}$. M also represents the bilinear form that determines the equation of the affine conic. Recall the chosen smooth affine conics have no intersection with the line at infinity, so the affine conic $Q(x, y)$ has no zero. Therefore $Q(x, y) = x^2 + y^2$. Correspondingly the symmetric bilinear form for this equation is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. We normalize the orthogonal bases v_1, v_2 so that $\langle v_1, v_1 \rangle = 1$ and $\langle v_2, v_2 \rangle = 1$. We can do this because (v_1, v_2, v_3) is also a orthogonal basis for conic $C : X^2 + Y2 - Z^2 = 0$, $\langle v_3, v_3 \rangle$ = -1. We have identified the class of ellipses in smooth affine conics. □

Compared with Lemma 3.7, Lemma 3.8 defines the symmetric bilinear form A with $A_{ij} = b(v_i, v_j)$, which corresponds to the quadratic form $x^2 + y^2 - z^2 = 0$. x, y, z are coefficients with respect to the basis. So the affine conic within the projective conic stays the same. And since $P \subset span(v_1, v_2)$, the coefficient z for v_3 equals 0. The plane P can be transformed to the plane $Z = 0$ by changing basis.

Proposition 3.11. *Classify affine hyperbolas based on the classification of projective conics.*

Removing a line that has two intersections with the conic C, we get hyperbolas in an affine space. For example, $C \cap (L : X = 0) = [0 : 1 : 1] \cup [0 : 1 : -1]$. Dehomogenizing the equation of C produces $y^2 - z^2 = 1$, which represents the class of hyperbolas in $\mathbb{A}^2$.

Similar to the classification of affine ellipses, the classification of affine hyperbolas focuses on proving the lemma about finding a set of bases for $\mathbb{A}^2$

Lemma 3.12. *Given any plane $P \subset A^2$ passing through the origin O satisfying that there exist two intersection lines between the plane P and the cone, there exist v_1, v_2, v_3 such that*

(1) $v_1 \perp v_2 \perp v_3$,
(2) $P = span(v_2, v_3)$,
(3) $v_1^2 = 1, v_2^2 = 1, v_3^2 = -1$.

Proof. Choose 3 vectors $u_1, u_2, u_3 \subset \mathbb{A}^2$ that are linearly independent. Let $P \subset span(u_2, u_3)$ and $\langle u_i, u_i \rangle \neq 0$. Now we follow the Gram-Schmidt process to orthogonalize the chosen basis. Similar to the proof of Lemma 3.8, we show v_1, v_2, v_3 are orthogonal to each other and $P \subset span(v_2, v_3)$.

To find values of inner products, we need to determine the inner product matrix, which is also the symmetric bilinear form corresponding to the affine conics.

$$M = \begin{bmatrix} \langle v_2, v_2 \rangle & \langle v_2 v_3 \rangle \\ \langle v_3, v_2 \rangle & \langle v_3, v_3 \rangle \end{bmatrix} = \begin{bmatrix} \langle v_2, v_2 \rangle & 0 \\ 0 & \langle v_3, v_3 \rangle \end{bmatrix}.$$

Recall the chosen affine conic has two intersections with the chosen line, so $Q(x, y)$ has two zeros. Therefore $Q(x, y) = x^2 - y^2$. Correspondingly the symmetric bilinear form for this conic is $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. After normalization we have $\langle v_2, v_2 \rangle = 1$ and $\langle v_3, v_3 \rangle = -1$. Naturally the normalized $\langle v_1, v_1 \rangle$ equals 1. We have thus classified the affine hyperbolas. □

Proposition 3.13. *Classify affine parabolas based on the classification of projective conics.*

Removing a line that has one intersection with the chosen projective conic, we get parabolas in an affine space. For example, $(C : X^2 + YZ = 0) \cap (L : Z = 0) = [0 : 1 : 0]$. Notice that $X^2 + YZ = 0$ is projectively equal to $X^2 + Y^2 - Z^2 = 0$ as changes of basis will transform the latter equation into the former. Also, we can let $Y = W + V$ and $Z = W - V$, where W and V are coordinates in $\mathbb{P}^2_{\mathbb{R}}$. Then $X^2 + YZ = 0 = X^2 + W^2 - V^2$.

We then dehomogenize the equation of C by dividing V^2 over the equation. The formula in $\mathbb{A}^2$ is $x^2 + y = 0$, which represents the class of parabolas in $\mathbb{A}^2$.

Lemma 3.14. *Given any plane $P \subset A^2$ passing through the origin O satisfying that the plane P has one intersection line with the cone, there exist bases v_1, v_2, v_3 such that*

(1) $v_1 \perp v_2$ *and* $v_1 \perp v_3$ *and* $\langle v_2, v_3 \rangle = 1/2$,

(2) $P = span(v_1, v_2, v_3)$,
(3) $\langle v_1, v_1\rangle = 1, \langle v_2, v_2\rangle = 0, \langle v_3, v_3\rangle = 0$.

Proof. Arbitrarily choose 3 vectors $(u_1, u_2, u_3) \subset \mathbb{A}^2$. Then $\langle u_2, u_3\rangle \neq 0$, $P \subset span(u_1, u_2, u_3)$, and $\langle u_1, u_1\rangle \neq 0$. We can follow Gram-Schmidt process within vectors u_1, u_2 and u_1, u_3 while letting $v_1 = u_1$. So $v_1 \perp v_2$ and $v_1 \perp v_3$. To find the values for inner products between vectors, we need to determine the inner product matrix for this case.

$$M = \begin{bmatrix} \langle v_1, v_1\rangle & \langle v_1, v_2\rangle \\ \langle v_2, v_1\rangle & \langle v_2, v_2\rangle \end{bmatrix} = \begin{bmatrix} \langle v_1, v_1\rangle & 0 \\ 0 & \langle v_2, v_2\rangle \end{bmatrix}.$$

Since $Q(x, y)$ only has only one zero, the quadratic form $Q(x, y) = x^2$. Therefore $\langle v_1, v_1\rangle = 1$ and $\langle v_2, v_2\rangle = 0$. Returning to the bilinear form for the conic in the projective space, we determine that $\langle v_2, v_3\rangle = 1/2$ and $\langle v_3, v_3\rangle = 0$.

□

Therefore we have classified affine conics based on projective conics and removal of lines at infinity in projective space.

4. Parametrization of Conics in $\mathbb{P}^2_{\mathbb{R}}$

Based on the classification of conics, we can parametrize a conic. For a non-degenerate, non-empty conic C of $\mathbb{P}^2_{\mathbb{R}}$, take the coordinates (X+Y,X-Y,Z). This is projectively the same as $XZ = Y^2$ (or $x = y^2$ in $\mathbb{R}^2$). And there is an injective map from a projective line to a projective conic,

$$\mathbb{P}^1_{\mathbb{R}} \longrightarrow C \subset \mathbb{P}^2_{\mathbb{R}} \text{ defined by } P(U : V) = (U^2 : UV : V^2).$$

Exercise 4.1. Suppose k is a field with at least 4 elements and $C : (XZ = Y^2) \subset \mathbb{P}^2_k$. If $Q(X, Y, Z)$ is a quadratic form which vanishes on C, then $Q = \lambda(XZ - Y^2)$.

Proof. There exists a unique expression $Q(x, y, z) = \lambda q + A(X, Z) + YB(X, Z)$ where $A(X, Z)$ is a quadratic form in X, Z and $B(X, Z)$ is a linear form in X, Z.

Set $q = XZ - Y^2$ and replace Y^2 in Q with $(XZ - q)$. The remaining part in Q has degree less or equal to 1 in Y. Therefore, it can be expressed as $+A(X, Z) + YB(X, Z)$. C is the parametrized conic given by

$$X = U^2, Y = UV, Z = V^2,$$

so

$$Q(U^2, UV, V^2) = A(U^2, V^2) + UVB(U^2, V^2).$$

We thus have

$$Q \equiv 0 \text{ on } C \text{ if and only if } A(U^2, V^2) + UVB(U^2, V^2) \equiv 0 \text{ on } C.$$

The above relation implies that

$$A(U^2, V^2) + UVB(U^2, V^2) = 0 \text{ on } C,$$

and

$$[U : V] \Longleftrightarrow A(X, Z) = B(X, Z) = 0.$$

Since $V \neq 0$, set $u = U/V$. Then we have

$$A(U^2, V^2) + UVB(U^2, V^2) = A(u^2, 1) + uB(u^2, 1) = 0.$$

As the first part of the equation includes terms with even degrees and the second includes terms with odd degrees, terms from different parts of the equation will not cancel out. In addition, $u \neq 0$, so $A(X, Z) = 0 = B(X, Z)$.

□

Definition 4.2. A *form* is a polynomial whose nonzero terms all have the same degree. The *degree* of a term is the sum of all the exponents of its variables.

Proposition 4.3. *Let F(U,V) be a non-zero form of degree d in (U,V). Then F has at most d zeros on $\mathbb{P}^1$. Furthermore, if k is algebraically closed, then F has exactly d zeros on $\mathbb{P}^1$.*

Proof. Suppose $F(U, V) = a_d U^d + a_{d-1} U^{d-1} V + \cdots + a_i U^i V^{d-i} + \cdots + a_0 V^d$.

F is associated with an inhomogeneous polynomial in 1 variable u = U/V,

$$f(u) = a_d u^d + a_{d-1} u^{d-1} + \cdots + a_i u^i + \cdots + a_0$$

There exists $\alpha \in k$ such that $f(\alpha) = 0$ and $(u-\alpha)|f(u)$. So there is a polynomial $p(u)$ satisfying $p(u)(u-\alpha) = f(u)$. We then homogenize $f(u)$ by adding the variable V into the equation. We thus have

(1) $(U - \alpha V)p(u)V^{d-1} = F(U, V)$,
(2) $deg(U - \alpha V) = 1$,
(3) $deg(F(U, V)) = d$.

Thus $\deg(p(u)V^{d-1}) = d - 1$. There exists $P(U, V)$ of degree d-1 such that $P(U, V)(U - \alpha V) = F(U, V)$.

On the other hand, as $F(U, V)$ is a multiple of $(U - \alpha V)$, there exists $P(U, V)$ of degree d-1. Dehomogenization of $P(U, V)$ generates $f(\alpha) = 0$.

We have shown that $f(\alpha) = 0$ if and only if $F(\alpha, 1) = 0$. So the zeros of f correspond to zeros of F away from from the point (1,0). When F has a zero at infinity,

$$F(1, 0) = 0, \text{ which requires } a_d = 0.$$

Therefore, $deg(f) < d$.

Definition 4.4. The *multiplicity* of a zero of F in $\mathbb{P}^1$ is
(i) the multiplicity of f at the corresponding $\alpha \in k$;
(ii) d-deg(f) if (1,0) is the zero.

For a non-zero form of deg d in P^1, let the multiplicity of zero at (1,0) be x. The degree of the inhomogeneous polynomial f is only $d - x$. There are at most d roots for the polynomial f.

Over an algebraically closed field k, F can factorize as a product of d first degree polynomials. Since each polynomial has a zero, F naturally has d zeros. □

Now we have the foundation for proving the easy case of Bézout's Theorem.

Theorem 4.5. *Let $L \subset \mathbb{P}^2_k$ be a line and let $D \subset \mathbb{P}^2_k$ be a curve defined by D : $(G_d(X, Y, Z) = 0)$, where G is a form of degree d in X,Y,Z. Assume that $L \nsubseteq D$. Then $|L \bigcap D| \leq d$. The equality holds if we count multiplicities and k is algebraically closed.*

Proof. Let $L \subset P^2_k$ be given by an equation with $\alpha = 0$, where α as a linear form with 3 variables X, Y and Z. We parametrize X, Y, Z to be $(a(U, V), b(U, V), c(U, V))$ with a, b, c being the three linear forms. For example, if $Y = U, Z = V, \alpha =$

$\beta X + \gamma Y + \delta Z = 0$, then $X = -\gamma/\beta U - \delta/\beta V$.

To find the number of intersections between L and D, we need the number of solutions for $G_d(a(U,V), b(U,V), c(U,V)) = 0$. Based on Proposition 4.1, there are at most d solutions, and the equality holds for algebraically closed k.

□

Respectively, the non-degenerate conic $C \subset \mathbb{P}^2_{\mathbb{R}}$ is a quadratic form with X, Y, Z parametrized by U, V. We thus have $X = a(U,V)$, $Y = b(U,V)$, $Z = c(U,V)$. This is a result of C being a projective transformation of $XY = Z^2$.

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = M \begin{bmatrix} U^2 \\ UV \\ V^2 \end{bmatrix}.$$

Similarly, we find the number of solutions for $G_d(a(U,V), b(U,V), c(U,V)) = 0$ to be at most 2d. So there are at most $2d$ intersections between a conic and a curve of degree d in P^2_R.

Corollary 4.6. *If $P_1, ..., P_5 \in \mathbb{P}^2_{\mathbb{R}}$ are distinct points such that no four of them are collinear, then there is at most one conic through $P_1, ..., P_5$.*

Proof. Suppose there are two distinct non-degenerate conics $C_1, C_2 \in P^2_R$ passing through $P_1, ..., P_5$. C_1, C_2 are quadratic forms in the projective space. However, this contradicts Theorem 4.2, which states the maximum possible number of intersections between two non-degenerate distinct curves is 4.

Suppose one of the two conics is degenerate. This case also permits at most 4 intersections (As in Corollary 3.4, (c) is a line pair and each line has at most 2 intersections with a conic).

Now suppose both conics are degenerate. The maximum number of intersections between them is achieved by two line pairs represented by $C_1 = L_{11} \cup L_{12}$, $C_2 = L_{21} \cup L_{22}$. Suppose $L_{11} \cap L_{21} = P_1$. Then $P_2, ..., P_5 \subset L_{12}$, this fails the requirement of no 4 points are collinear.

□

5. Spaces of Conics

Definition 5.1. The *space of all conics*, denoted S_2, is the set of quadratic forms on $\mathbb{R}^3$, which is essentially the set of 3 x 3 symmetric matrices. If $Q \in S_2$, then $Q = aX^2 + bXY + ... + fZ^2$.

Pick a point $P_0 = (X_0, Y_0, Z_0) \in \mathbb{P}^2_{\mathbb{R}}$. If $P_0 \in Q \subset S_2$, then

$$Q(X_0, Y_0, Z_0) = aX_0^2 + bX_0Y_0 + ... + fZ_0^2 = 0.$$

This is a linear equation and the set of all conics that contain the point P_0 is

$$S_2(P_0) = \{Q \subset S_2 \,|\, Q(P_0) = 0\}.$$

Since $S_2(P_0)$ can be parametrized as a 2 degree form with 5 variables, it is a 5-dimensional hyperplane that is congruent to $\mathbb{R}^5$, meaning there is an equivalence relation between $S_2(P_0)$ and $\mathbb{R}^5$. For $P_1, ..., P_n \in \mathbb{P}^2_{\mathbb{R}}$, the definition of space of conics is $S_2(P_1, ..., P_n) = \{Q \subset S_2 \,|\, Q(P_i) = 0 \text{ for } i = 1, ...n\}$

Proposition 5.2. $dim S_2(P_1, ...P_n) \geq 6 - n$.

Proof. (i) When $n \geq 6$, clearly $dimS_2(P_1, ...P_n) \geq 6 - n = 0$.
(ii) When $n = 5$, $dimS_2(P_1, ...P_5) = |\{C_i | C_i \supset P_1, ...P_5\}|$.

Corollary 4.4 shows $dimS_2(P_1, ...P_5) \leq 1$ when any four Ps are not collinear; when there are 4 Ps on the same line L, Cs can be degenerated conics (line pairs) that both include L. In this case a pair of Cs can span the space of conics and $dimS_2(P_1, ...P_n) = 2$. When all points are collinear, these conics can be double lines or line pairs, and the $dimS_2(P_1, ...P_5) = 3$.
(iii) When $n \leq 4$, we add 5-n points into the set. Each additional point adds a linear equation to the system of equations. Each new equation at most diminishes the dimension of the conic space by 1, so that

$$dimS_2(P_1, ., P_n, P_{n+1}..P_5) \geq 1 \leq dimS_2(P_1, ., P_n) - (5 - n),$$

thus $dimS_2(P_1, ., P_n) \geq 6 - n$. □

Corollary 5.3. *If $n \leq 5$ and no 4 of $P_1, ., P_n$ are collinear, then*

$$dimS_2(P_1, ., P_n) = 6 - n.$$

Definition 5.4. A *pencil* of conics is a family of the form $C_{(\lambda,\mu)} : (\lambda Q_1 + \mu Q_2) = 0$, and each element in $C_{(\lambda,\mu)}$is a plane curve.

Proposition 5.5. *$C_{(\lambda,\mu)}$ is degenerate if and only if $det(\lambda Q_1 + \mu Q_2) = 0$.*

Proof. Proposition 3.1 states that there is a bijection between quadratic forms and symmetric bilinear forms:

$$aX^2 + 2bXY + +cY^2 + 2dXZ + 2eYZ + fZ^2 \longleftrightarrow \begin{bmatrix} a & b & d \\ b & c & e \\ d & e & f \end{bmatrix}.$$

Suppose $C_{(\lambda,\mu)}$ is degenerate. Then by applying a projective transformation, C can be changed into one of the cases (c), (d) or (e) in Corollary 3.4. Those forms indicate that at least one of the a, c, f is 0 while $b = d = e = 0$.

So $\det \begin{bmatrix} a & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & f \end{bmatrix} = 0$.

All projective conics are of the form $aX^2 + cY^2 + f^Z 2 = 0$.

Suppose $\det \begin{bmatrix} a & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & f \end{bmatrix} = 0$. Then $0 \in [a, c, f]$. Suppose Z =0. Then C is either $X^2 \pm Y^2 = 0$ or $X^2 = 0$ or 0=0, all of which are degenerate. □

So we can write out the formulas for degenerate cases as

$$F(\lambda, \mu) \;=\; det \left| \lambda \begin{bmatrix} a & b & d \\ b & c & e \\ d & e & f \end{bmatrix} + \mu \begin{bmatrix} a' & b' & d' \\ b' & c' & e' \\ d' & e' & f' \end{bmatrix} \right| = 0.$$

$F(\lambda, \mu)$ is a homogeneous cubic form in λ, μ.

Proposition 5.6. *For $C_{(\lambda,\mu)} \subset \mathbb{P}^2_k$ with at least one non-degenerate conic, the pencil has at most 3 degenerate conics. If $k = \mathbb{R}$ then the pencil has at least one degenerate conic.*

Proof. Since $F(\lambda, \mu)$ is a cubic form in P_1, by conclusions from Proposition 4.1, F has at most 3 zeros. Since $\mathbb{R}$ is an algebraically closed field, F over $\mathbb{R}$ has at least one zero. $\square$

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