REPRESENTATION THEORY FOR FINITE GROUPS

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Abstract. We cover some of the foundational results of representation theory including Maschke's Theorem, Schur's Lemma, and the Schur Orthogonality Relations. We consider character theory, constructions of representations, and conjugacy classes. Finally, we touch upon Pontryagin Duality for abelian groups and Young Tableaux for the Symmetric Group.

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1. What is Representation Theory?

Group representations describe elements of a group in terms of invertible linear transformations. Representation theory, then, allows questions regarding abstract algebra to be reduced to questions regarding linear algebra. One of the notable aspects of these representations is that the general noncommutativity of group multiplication is entirely captured by the analagous general noncommutativity of matrix multiplication. Now, one of the primary goals of representation theory, in general, is to classify all the irreducible representations of a group, up to isomorphism. While it is comparatively simple to do so for finite groups and there are known methods for doing so, it is often very difficult to do so for infinite groups. As this paper is simply an introduction into the simplest forms of representation theory, we deal exclusively with finite groups, in both the abelian and non-abelian case. Besides the kind of group, the study of representation theory can also vary based on the kind of field under study. In this paper, wel exclusively consider representations on complex vector spaces. The complex field is a natural choice since

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it is algebraically closed and of characteristic zero.

Definition 1.1. A representation (ρ, V) of a group G on a vector space V is a group homomorphism $\rho : G \to GL(V)$.

Notation 1.2. We will always refer to a representation by the homomorphism ρ or by the pair (ρ, V). While it is also acceptable and common to refer to V as the representation when the homomorphism is clear, this is not a practice we will adopt in this paper.

A second way to think of representations is through group actions. We say that ρ induces a group action of G on V by linear transformations. Thus we find that the notions of linear action and representation are equivalent.

A final way to think of representations is through modules.

Definition 1.3. The group algebra, $\mathbb{C}[G]$, is the vector space given by the set of linear combinations $\sum_{g_n \in G} c_n g_n$ with coefficients $c_n \in \mathbb{C}$ and multiplication defined as $\sum_{g_n \in G} c_n g_n \sum_{h_n \in G} b_n h_n = \sum_{g_n, h_n \in G} (c_n b_n) g_n h_n$.

We say that V is a $\mathbb{C}[G]$-module where the multiplication vg is defined by $v\rho(g)$. This allows us to convert questions regarding linear transformations into questions regarding vector spaces.

2. Characteristics of Representations

In this section, we go through the basic definitions of dimension, reducibility, and equivalence of representations. We look at a concrete example using the smallest non-abelian group, the symmetric group on a set of three elements.

Example 2.1. Consider S_3 with the following group elements:

$$\{e, (12), (13), (23), (123), (132)\}$$

We define three representations:

- The trivial representation $\rho_1 = [1]$.
- The alternating (sign) representation ρ_2:
$$\sigma \mapsto \begin{cases} [1] & \text{if } \sigma \text{ is even} \\ [\text{-}1] & \text{if } \sigma \text{ is odd} \end{cases}$$
- The permutation representation ρ_3:

$$e \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (13) \mapsto \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad (123) \mapsto \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$(12) \mapsto \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (23) \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad (132) \mapsto \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

We can start to classify these representations with a simple yet important characteristic of representations.

Definition 2.2. The dimension, or degree, of a representation (ρ, V) is defined to be the dimension of V, the representation space of ρ.

Example 2.3.

$$\dim \rho_1 = 1$$
$$\dim \rho_2 = 1$$
$$\dim \rho_3 = 3$$

In the context of representation theory, modules are vector spaces. We can, in fact, understand representations entirely by their accompanying representation spaces. We construct the corresponding modules of our three representations for S_3:

Example 2.4.

$$V_1 = \mathbb{C}$$
$$V_2 = \mathbb{C}$$
$$V_3 = \mathbb{C}^3$$

Another way to classify representations is reducibility.

Definition 2.5. The representation (ρ, W) with $\rho : G \to GL(W)$ is called a subrepresentation of (ρ, V) if $W \subseteq V$ is an invariant subspace under G.

Definition 2.6. A representation (ρ, V) of G is irreducible if the only G-invariant subspaces of V are 0 and V. The representation is reducible otherwise.

Remark 2.7. We say that V is irreducible if the representation (ρ, V) is irreducible.

Definition 2.8. A representation (ρ, V) is fully reducible if we can write $V = W_1 \oplus W_2 \oplus ... \oplus W_k$ where each W_i is irreducible.

For the representations of S_3 that we constructed, we see that ρ_1 and ρ_2 are irreducible since they are one-dimensional. We claim that ρ_3, like all permutation representations, is reducible.

Example 2.9. We consider the following vector spaces:

$$W_1 = \operatorname{span}\{(1, 1, 1)\}$$
$$W_2 = \{(a, b, c) \mid a + b + c = 0\}$$

We see that

$$W_1 \oplus W_2 = V_3 = \mathbb{C}^3$$

We find that both W_1 and W_2 are nonzero proper G-invariant subspaces of V_3. In other words, for all $g \in G$ and all $w_1 \in W_1$ and all $w_2 \in W_2$ we have $\rho_3(g)w_1 \in W_1$ and $\rho_3(g)w_2 \in W_2$. We now write the elements of the image of ρ_3 in terms of a new basis $B = \{b_1, b_2, b_3\}$ given by basis elements chosen from V_1 and V_2.

$$B = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

With basis B we call this permutation representation ρ_3':

$$e \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (13) \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \qquad (123) \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

$$(12) \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \qquad (23) \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad (132) \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

Notably, we see that this basis simultaneously block diagonalizes all the matrices into a 1 by 1 and a 2 by 2 square matrix. This suggests that ρ_3' is in fact reducible; it can be decomposed into two subrepresentations, one of which is the trivial representation. By isolating the 2 by 2 matrix we find a two-dimensional representation. In later sections, we develop the tools to show that this two-dimensional representation is, in fact, irreducible.

Besides the notion of subrepresentation, there is another relation that two representations may have: equivalence.

Definition 2.10. Let (ρ, V) and (ρ', V') be representations of finite group G. Then $L : V \to V'$ is an intertwining map if for all $g \in G$ we have $\rho'(g)L = L\rho(g)$.

Definition 2.11. Let $L : V \to V'$ be an intertwining map for the representations ρ and ρ'. If L is invertible, then L^{-1} is also an intertwining map. Then we say that L is an equivalence of G-representations and that ρ and ρ' are equivalent.

Notation 2.12. There are various names for what we call an intertwining map. Among these are G-module homomorphism, G-map, and intertwining operator.

Example 2.13. For our example of S_3, we have permutation representations (ρ_3, V_3) with the standard basis and (ρ_3', V_3') with basis B. We note that $B : V_3 \to V_3'$ is an intertwining map. Now, $\det B \neq 0$ so B is invertible and $\rho_3 \cong \rho_3'$. Since ρ_3' is reducible we see that ρ_3 is also reducible.

3. Construction of Representations

In this section, we develop the tools to construct new representations from known representations. For a group like S_3, it is very easy to construct all the irreducible representations without the use of these tools. For any group, there is always the trivial representation. For any symmetric group, there is also the sign representation. Then for S_3 there is just one more that we need to find. However, for much larger and more complex groups, these constructions are indeed quite useful.

Proposition 3.1. If $\rho_1 : G \to GL(V_1)$ and $\rho_2 : G \to GL(V_2)$ are two representations then the direct sum representation $\rho_1 \oplus \rho_2 : G \to GL(V_1 \oplus V_2)$ is given by $(\rho_1 \oplus \rho_2)(g)(v_1, v_2) = (\rho_1(g)(v_1), \rho_2(g)(v_2))$.

As seen from the example of S_3 the direct sum of two representations is obviously reducible into its known constituents, so it will not be of much use in constructing irreducible representations. We look at four other tools:

Proposition 3.2. If $\rho_1 : G \to GL(V_1)$ and $\rho_2 : G \to GL(V_2)$ are representations of G then the tensor product representation $\rho_1 \otimes \rho_2 : G \to GL(V_1 \otimes V_2)$ is given by $(\rho_1 \otimes \rho_2)(g)(v_1 \otimes v_2) = \rho_1(g)(v_1) \otimes \rho_2(g)(v_2)$.

Proposition 3.3. If $\rho : G \to GL(V)$ is a representation then the dual representation $\rho^* : G \to GL(V^*)$ is given by $\rho^*(g)(v^*) = v^*\rho(g^{-1})$.

Proposition 3.4. If $\rho : G \to GL(V)$ is a representation then the exterior power representation $\rho_{\wedge^2 V} : G \to GL(V \wedge V)$ is given by $\rho_{\wedge^2 V}(g)(v_1 \wedge v_2) = \rho(g)v_1 \wedge \rho(g)v_2$ where $v_1 \wedge v_2 = \frac{1}{2}(v_1 \otimes v_2 - v_2 \otimes v_1)$.

Proposition 3.5. If $\rho : G \to GL(V)$ is a representation then the symmetric power representation $\rho_{\mathrm{Sym}^2 V} : G \to GL(\mathrm{Sym}^2 V)$ is given by $\rho_{\mathrm{Sym}^2 V}(g)(v_1 v_2) = (\rho(g)v_1)(\rho(g)v_2)$ where $v_1 v_2 = \frac{1}{2}(v_1 \otimes v_2 + v_2 \otimes v_1)$.

4. The Building Blocks of Representation Theory

In this section, we look at foundational yet powerful results in representation theory, specifically Maschke's Theorem, Schur's Lemma, and a resultant implication.

The following theorem shows us that when we use one of the above four constructions to find a new representation, we can completely decompose it into irreducible representations.

Theorem 4.1 (Maschke)**.** *Every representation of a finite group over $\mathbb{C}$ is fully reducible.*

Proof. We let (ρ, V) be a representation of a finite group G. We take any Hermitian inner product $\langle v, w\rangle_0$ on V. We define

$$\langle v, w\rangle := \frac{1}{\mid G \mid} \sum_{g \in G} \langle \rho(g)v, \rho(g)w\rangle_0$$

We show that $\langle v, w\rangle$ is an invariant Hermitian inner product. Conjugate symmetry is satisfied since $\langle v, w\rangle_0$ is Hermitian. For positive definiteness, we assume $v \neq 0$. Since $\langle v, v\rangle_0 \geq 0$ for all v we get $\langle v, v\rangle \geq 0$. But $\langle \rho(e)v, \rho(e)v\rangle_0 = \langle v, v\rangle_0 > 0$ so $\langle v, v\rangle > 0$ if $v \neq 0$. Finally for invariance, since the left regular action is transitive we see that for all $g \in G$ and all $v, w \in V$ we have

$$\langle \rho(g)v, \rho(g)w\rangle = \langle v, w\rangle$$

Now, we let W be an invariant subspace of V. We take $v \in W^\perp$. Under our invariant Hermitian inner product we get

$$\langle \rho(g)v, w\rangle = \langle v, \rho(g^{-1})w\rangle = 0$$

Since W is invariant we have $\rho(g^{-1}) \in W$ so $W^\perp$ is also an invariant subspace of V.

Finally, we use induction on the dimension of V. If $\dim(V) = 1$, then ρ is irreducible. We assume that if $\dim V \leq (n-1)$, then ρ is fully reducible. We suppose $\dim V = n$. If ρ is irreducible, then we are done. If ρ is reducible, then there exists a nonzero invariant subspace $W \subset V$ such that $V = W \oplus W^\perp$. Then, from our assumption, (ρ, W) and $(\rho, W^\perp)$ are fully reducible so ρ is fully reducible. □

Theorem 4.2 (Schur)**.** *If (ρ, V) and (π, W) are irreducible representations of G and $\phi : V \to W$ is an intertwining map then:*

- *Either ϕ is an isomorphism or $\phi = 0$.*

- *If $V = W$ then $\phi = \lambda \cdot I$ for some $\lambda \in \mathbb{C}$.*

Proof. We assume $\phi \neq 0$. We take $v \in \operatorname{Ker}\phi$. Then $\phi(v) = 0$ and $\phi(\rho(g)v) = \pi(g)\phi(v) = 0$. This implies that $\rho(g)v \in \operatorname{Ker}\phi$. Then $\operatorname{Ker}\phi$ is an invariant subspace of V and $(\rho, \operatorname{Ker}\phi)$ is a subrepresentation of (ρ, V). Since ρ is irreducible, either $\operatorname{Ker}\phi = V$ or $\operatorname{Ker}\phi = 0$. By our assumption, $\operatorname{Ker}\phi = 0$. So ϕ is one-to-one. Similarly, we show that $(\pi, \operatorname{Im}\phi)$ is a subrepresentation of (π, W). Since π is irreducible, either $\operatorname{Im}\phi = 0$ or $\operatorname{Im}\phi = W$. Since ϕ is one-to-one, then $\operatorname{Im}\phi = W$ and ϕ is onto. So ϕ is bijective, and thus, an isomorphism.

Since $\mathbb{C}$ is algebraically closed, we see that ϕ has an eigenvalue $\lambda \in \mathbb{C}$. Now, ϕ and λI are intertwining maps so $\phi - \lambda I : V \to V$ is also an intertwining map. Since $\lambda \in \ker(\phi - \lambda I)$, we know $\ker(\phi - \lambda I) \neq 0$ and from the previous part, we get $\ker(\phi - \lambda I) = V$, so $\phi - \lambda I = 0$ and $\phi = \lambda I$. □

Theorem 4.3. *For any finite-dimensional representation (ρ, V) of a finite group G there is a unique decomposition $V = V_1^{\oplus a_1} \oplus V_2^{\oplus a_2} \oplus .... \oplus V_i^{\oplus a_i}$ where the V_i are inequivalent and irreducible with unique multiplicities a_i.*

Proof. We suppose $V = W_1^{\oplus b_1} \oplus W_2^{\oplus b_2} \oplus .... \oplus W_j^{\oplus b_j}$. Then we let $\phi : V \to V$ be the identity map. We use Schur's Lemma. For each irreducible $V_i^{\oplus a_i}$, we restrict the domain of ϕ to that component. Then, either $\phi = 0$ or ϕ is an isomorphism. If $j = i$, then $\phi(V_i^{\oplus a_i}) \neq 0$ for any i. For each component, ϕ is an isomorphism such that $V_i^{\oplus a_i}$ maps to $W_j^{\oplus b_j}$ where V_i is isomorphic to W_j. □

5. Character Theory

In this section, we introduce the character of a group representation, a function that associates with every element of the group a number equal to the trace of the corresponding matrix for that element. We see that looking at trace is far more efficient than analyzing the matrices themselves.

Definition 5.1. The character of a representation (ρ, V) is the function $\chi : G \to \mathbb{C}$ given by $\chi_\rho(g) = \operatorname{Tr}(\rho(g))$.

Remark 5.2. We say that a character χ_V is irreducible if (ρ, V) is irreducible.

We look at a character table for S_3, a graphical object that tells, for each representation listed for that group, the value of the character evaluated at each element of the group.

S_3	e	(12)	(13)	(23)	(123)	(132)
Trivial representation	1	1	1	1	1	1
Sign representation	1	-1	-1	-1	1	1
Standard representation	2	0	0	0	-1	- 1

Table 1. Character table for S_3

6. Characters of Constructions

In this section, we look at formulas for the characters of our constructed representations. While we don't provide rigorous proof for each construction, it should be noted that each proposition is easily provable using the properties of trace.

Proposition 6.1. If (ρ, V) and (π, W) are representations of G then the character of the direct sum representation is the sum of the characters of the representations, i.e. $\chi_{V \oplus W} = \chi_V + \chi_W$.

Proposition 6.2. If (ρ, V) and (π, W) are representations of G then the character of the tensor product representation is the product of the characters of the representations, i.e. $\chi_{V \otimes W} = \chi_V \cdot \chi_W$.

Proposition 6.3. If (ρ, V) is a representation of G then the character of the dual representations is the complex conjugate of the character of the representation, i.e. $\chi_{V^*} = \overline{\chi_V}$.

Proposition 6.4. If (ρ, V) is a representation of G then the character of the wedge product representation is $\chi_{\wedge^2 V}(g) = \frac{1}{2}(\chi_V(g)^2 - \chi_V(g^2))$.

Proposition 6.5. If (ρ, V) is a representation of G then the character of the symmetric product representation is $\chi_{\mathrm{Sym}^2 V}(g) = \frac{1}{2}(\chi_V(g)^2 + \chi_V(g^2))$.

7. Schur Orthogonality Relations

In this section, we look at the Schur Orthogonality Relations, a result that can be used to determine both equivalence and reducibility of representations.

We first define an inner product for characters:

Definition 7.1. The Hermitian inner product for two characters χ and ψ of G is defined to be

$$< \chi, \psi >= \frac{1}{|G|} \sum_{g \in G} \overline{\psi(g)} \chi(g)$$

Theorem 7.2. *Let χ and ψ be two respective characters of irreducible representations (ρ, V) and (π, W) of G. Then:*

$$< \chi, \psi >= \begin{cases} 1 & \rho \cong \pi \\ 0 & \rho \ncong \pi \end{cases}$$

Proof. We define the stabilizer of V by G as

$$V^G = \{v \in V \mid gv = v, \ \forall g \in G\}$$

We define $\psi : V \to V^G$ with

$$\psi(v) = \frac{1}{|G|} \sum_{g \in G} gv$$

For each $h \in G$ we have

$$h\psi(v) = \frac{1}{|G|} \sum_{g \in G} hgv = \frac{1}{|G|} \sum_{g \in G} gv = \psi(v).$$

This implies that $\operatorname{Im}\psi \subset V^G$. Similarly, for all $v \in V^G$ we have

$$\psi(v) = \frac{1}{|G|}\sum_{g\in G} v = v$$

This implies that $V^G \subset \operatorname{Im}\psi$, so we get $V^G = \operatorname{Im}\psi$. Then $\psi \circ \psi = \psi$, which means that ψ is a projection.

Now, we note that the eigenvalues of an idempotent matrix are 0 and 1. We then write

$$\dim V^G = \operatorname{Tr}\psi = \frac{1}{|G|}\sum_{g\in G} \operatorname{Tr} g$$

We note that $\dim V^G$ is equal to the multiplicity of the trivial representation in the decomposition of V.

We recall by assumption that V and W are irreducible. Now we let $\operatorname{Hom}_G(V, W)$ represent the vector space of intertwining maps $\phi : V \to W$. We take $\phi \in \operatorname{Hom}_G(V, W)$. Then for all $g \in G$ and $v \in V$, we have $\pi(g)\phi(v) = \phi(\rho(g)v)$. This implies that $\operatorname{Hom}_G(V, W)$ is a G-invariant subspace of $\operatorname{Hom}(V, W)$. We apply Schur's Lemma.

$$\dim \operatorname{Hom}_G(V, W) = \begin{cases} 1 & \rho \cong \pi \\ 0 & \rho \ncong \pi \end{cases}$$

We see that if $\rho \ncong \pi$ then neither irreducible representation can be decomposed into the other. This implies that no homomorphisms $\phi : V \to W$ exist, and so $\dim \operatorname{Hom}_G(V, W) = 0$. On the other hand we see that if $\rho \cong \pi$, then, by Schur's Lemma, all the nonzero intertwining map ϕ are isomorphisms.

Now, since V is finite-dimensional, we note that $\operatorname{Hom}_G(V, W) \cong V^* \otimes W$. Then, we find that $\omega : G \to \operatorname{Hom}(V, W)$ is a representation under the definition $\omega(g)(f) = \pi(g)f\rho(g)^{-1}$ for $f \in \operatorname{Hom}(V, W)$. We use our character formulas to get $\chi_{\operatorname{Hom}_G(V,W)}(g) = \overline{\chi_V(g)}\chi_W(g)$. By substitution we have

$$\frac{1}{|G|}\sum_{g\in G} \overline{\chi_V(g)}\chi_W(g) = \begin{cases} 1 & \rho \cong \pi \\ 0 & \rho \ncong \pi \end{cases}$$

We use the definition of inner product to get our result. □

Corollary 7.3. *The characters of the irreducible representations of a group G are orthonormal.*

Proof. We let (ρ, V) and (π, W) be irreducible representations of G with respective characters χ_V and χ_W. Assuming the representations are inequivalent, we find that the characters χ_V and χ_W are orthogonal. We note that $\langle \chi_V, \chi_W \rangle = 0$ iff the representations are equivalent, so we establish normality, and thus, orthonormality. □

We use the following theorem as a test for equivalence, so that we do not double-count a single representation in our count of irreducibles.

Theorem 7.4. *The representations (ρ, V) and (π, W) of G are equivalent iff their characters are equal, that is, $\chi_V = \chi_W$.*

Proof. We assume $\rho \cong \pi$. Then, using the definition of trace, the traces of action by $g \in G$ are equal by inspection so $\chi_V = \chi_W$.

We assume $\chi_V = \chi_W$. We see here that there exists a unique decomposition such that $V = V_1^{\oplus a_1} \oplus V_2^{\oplus a_2} \oplus ... \oplus V_k^{\oplus a_k}$ and $W = W_1^{\oplus b_1} \oplus W_2^{\oplus b_2} \oplus ... \oplus W_k^{\oplus b_k}$. Using our assumption, we find $\sum_{i=1}^k a_i \chi_{V_i} = \sum_{i=1}^k b_i \chi_{W_i}$. We now note that since the $\{\chi_{V_i}\}$ form an orthonormal set, they are linearly independent. It thus follows that $a_i = b_i$ for all i and $\rho \cong \pi$. □

We use the following theorem to verify that a representation is irreducible.

Theorem 7.5. *(Irreducibility criterion) If χ is the character of a representation (ρ, V), then $\langle \chi, \chi \rangle = 1$ iff ρ is irreducible.*

Proof. We assume $\langle \chi, \chi \rangle = 1$. Then, we have a unique decomposition such that $V \cong V_1^{\oplus a_1} \oplus V_2^{\oplus a_2} \oplus ... \oplus V_k^{\oplus a_k}$. We note that $\chi_V = \sum_{i=1}^k a_i \chi_{V_i}$ and

$$\langle \chi_V, \chi_V \rangle = \langle \sum_{i=1}^{k} a_i \chi_{V_i}, \sum_{i=1}^{k} a_i \chi_{V_i} \rangle = \sum_{i=1}^{k} a_i^2 = 1$$

We see that there must exist some i such that $a_i = 1$ and such that for all j if $j \neq i$, then $a_j = 0$. Then we find that $V = V_i$, which implies ρ is irreducible.

We assume ρ is irreducible. We easily get

$$\langle \chi, \chi \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\chi(g)} \chi(g) = 1$$

□

We use the following theorem as a check to see when we have found all the irreducible representations of a group, and that no more exist.

Theorem 7.6. *The sum of the squares of the irreducible representations of a group G is equal to the order of G.*

Proof. We let the group G act on a finite set X. We let (ρ, V) be the regular representation (this means that in essence we set $X = G$). We easily see that the elements of X form a basis for V. We note that the action of G is by permutation of basis vectors. Then, with the basis $\{b_x\}_{x \in X}$, the action of g corresponds with a matrix where each entry is either 1 or 0. We see that for a given matrix, a diagonal entry is 1 iff $b_{gx} = b_x$ and 0 otherwise. We thus find that $\chi_V(g)$ is equal to the number of elements of X fixed by g.

We see that for all $g \in G$ such that $g \neq e$ that g fixes no element of G. Then we note

$$\operatorname{Tr} g = \begin{cases} |G| & g = e \\ 0 & g \neq e \end{cases}$$

We let (α, A) be an irreducible representation of G, and we let (β, B) be another representation of G. We then have a unique decomposition such that $B \cong B_1 \oplus B_2 \oplus ... \oplus B_k$, where the B_i are not necessarily distinct. We write $\langle \chi_A, \chi_B \rangle =$

$\langle \chi_A, \chi_{B_1}\rangle + \langle \chi_A, \chi_{B_2}\rangle + ... + \langle \chi_A, \chi_{B_k}\rangle$. We see that, for each individual inner product

$$\langle \chi_A, \chi_{B_i}\rangle = \begin{cases} 1 & A \cong B_i \\ 0 & A \ncong B_i \end{cases}$$

We find that $\langle \chi_A, \chi_{B_i}\rangle$ is thus equal to the number of times A occurs in the decomposition of B. We equally say that $\langle \chi_A, \chi_{B_i}\rangle$ is equal to the number of times α occurs in the decomposition of β. We call this value the multiplicity of α in β.

We now see that for the regular representation (ρ, V), we have a unique decomposition such that $V \cong V_1^{\oplus a_1} \oplus V_2^{\oplus a_2} \oplus .... \oplus V_k^{\oplus a_k}$. We write

$$a_i = \langle \chi_{V_i}, \chi_V\rangle = \frac{1}{|G|}\sum_{g\in G}\overline{\chi_{V_i}}\chi_V(g) = \frac{1}{|G|}\overline{\chi_{V_i}(e)}|G| = \overline{\chi_{V_i}(e)} = \dim V_i$$

We thus find that each irreducible representation appears in the regular representation a number of times equal to the dimension of the representation. (As a side note, we see that this also shows that all the irreducible representations are contained in the regular representation.) Now we have

$$\chi_V(g) = \sum_{i=1}^{k} a_i\chi_{V_i}(g) = \sum_{i=1}^{k}\dim V_i\chi_{V_i}(g)$$

and

$$\chi_V(e) = \sum_{i=1}^{k}\dim V_i^2 = |G|$$

□

8. Finite Abelian Groups

In this section, we establish basic results of representation theory of finite abelian groups, including Pontryagin Duality. For the sake of brevity, we assume a basic understanding of group theory, including the Fundamental Theorem of Finite Abelian Groups:

Theorem 8.1. *Every finite abelian group G is isomorphic to the direct sum of cyclic groups:*

$$G \cong \mathbb{Z}/n_1 \oplus \mathbb{Z}/n_2 \oplus ... \oplus \mathbb{Z}/n_k$$

where $n_1 \geq 2$ and $n_{i+1}|n_i$ for all i.

We find that the irreducible representations of finite abelian groups are, in general, not very interesting. In fact, they are all one-dimensional.

Theorem 8.2. *Every irreducible representation of a finite abelian group is one-dimensional.*

Proof. We let (ρ, V) be an irreducible representation of finite abelian G. We write

$$\rho(g)\rho(h) = \rho(gh) = \rho(hg) = \rho(h)\rho(g)$$

Now we use Schur's Lemma. We see that $\rho(g) = \lambda_g I$ for some $\lambda_g \in \mathbb{C}$. Then we get $\rho(g)v = \lambda_g v$ for all $g \in G$. This implies that all subspaces $W \subset V$ are G-invariant so (ρ, W) is a subrepresentation of (ρ, V). Since ρ is irreducible, we see that either $W = 0$ or $W = V$. Now, we assume that $\dim V > 1$. But we note that if we take the span of one basis element, then we get a subspace of V which must be invariant by our previous statement. Since ρ is irreducible, this cannot happen. We thus find that $\dim V = 1$, and ρ is one-dimensional. $\square$

As a direct result, we note that all the individual matrices of an irreducible representation commute, which parallels the commutavitiy of group elements.

Definition 8.3. The character of a representation of a finite abelian group G is a homomorphism $\chi : G \to S^1$, where $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$, the circle group.

Example 8.4. Let $G = \langle a \rangle / (a^4)$ such that $|G| = 4$. Then $a^4 = 1$ and thus a has four possible values, and there are four one-dimensional representations. Since G is finite abelian, we note that $\chi(a^k) = \chi(a)^k$.

	1	a	a^2	a^3
χ_1	1	1	1	1
χ_2	1	-1	1	-1
χ_3	1	i	-1	$-i$
χ_4	1	$-i$	-1	i

Table 2. Character Table for a Finite Abelian Group of Order 4

We now come to Pontryagin Duality, which deals with dual and double dual groups.

Definition 8.5. The dual group $\hat{G}$ of finite abelian G is the set of homomorphisms $\phi : G \to S^1$ with the multiplication of pointwise multiplication of characters, that is, $(\chi\psi)(g) = \chi(g)\psi(g)$.

As an analogy, we note that a dual group is to a group as a dual vector space is to a vector space. Furthermore, we note that the identity element of $\hat{G}$ is defined as the trivial character $\chi(g) = 1$. We also note that $\hat{G}$ is abelian, since $G \subset C^{\times}$, the multiplicative group of all complex units.

Theorem 8.6. *Let $G = \langle a \rangle$ be cyclic. Then $G \cong \hat{G}$.*

Proof. We set $\chi : G \to S^1$ for all natural numbers $i \leq n$:

$$\chi(a^k) = e^{\frac{2\pi ik}{n}}$$

We take $\psi \in \hat{G}$ so $\psi(a) = e^{\frac{2\pi ik}{n}}$ for some $k \in \mathbb{N}$. We have $\psi(a) = \chi(a^k) = \chi(a)^k$ so

$$\psi(a^j) = \psi(a)^j = \chi(a)^{jk} = \chi(a^j)^k$$

This implies that $\psi = \chi^k$. Then we say that χ generates $\hat{G}$ so $\hat{G}$ is cyclic. Finally, we see that since every representation is one-dimensional, then $|G| = |\hat{G}| = n$. We thus find $G \cong \hat{G}$. $\square$

Theorem 8.7. *If G is a finite abelian group, then $G \cong \hat{G}$.*

Proof. We let χ be a character of $A \times B$. We consider the subgroups $A \times \{1\}$ and $\{1\} \times B$. We define the restricted characters χ_A and χ_B of A and B such that $\chi_A(a) = (a, 1)$ and $\chi_B(b) = (1, b)$ Then we have

$$\chi(a, b) = \chi((a, 1), (1, b)) = \chi(a, 1)\chi(1, b) = \chi_A(a)\chi_B(b)$$

We construct the map $\phi : A \hat{\times} B \to \hat{A} \times \hat{B}$. If χ_A and χ_B are trivial characters, then we see that since $\chi(a,b) = \chi_A(a)\chi_B(b) = 1$, that χ must also be the trivial character. We let ψ be another character of $A \times B$ with the analgous restricted ψ_A and ψ_B. Then

$$\psi(a,b)\chi(a,b) = \psi((a,1)(1,b))\chi((a,1)(1,b)) = \psi_A(a)\psi_B(b)\chi_A(a)\chi_B(b) =$$
$$(\chi_A\psi_A)(a)(\chi_A\psi_A)(b) = (\chi\psi)(a,1)(\chi\psi)(1,b) = (\chi\psi)((a,1)(1,b)) = (\chi\psi)(a,b)$$

We now see that the properties of homomorphism are satisfied. Since $|G| = |\hat{G}|$, we get that ϕ is an isomorphism.

Since every cyclic group is finite abelian (we carefully note that the converse is not true), we extend this isomorphism further such that

$$(A_1 \times A_2 \hat{\times} ... \times A_k) \cong \hat{A}_1 \times \hat{A}_2 \times ... \times \hat{A}_k \cong A_1 \times A_2 \times ... \times A_k$$

where A_i is cyclic. Using the Fundamental Theorem of Finite Abelian Groups, we get our result. $\square$

As a side note, we say that the dual group of a finite abelian group is isormorphic to the group itself but that it is not a canonical isormorphism, that is, the isomorphism is dependent on the choice of generator. However, we show that the double dual group of a finite abelian group is canonically isomorphic to the group. As an analogy, the dual vector space is isomorphic to the vector space itself, but it is not a canonical isomorphism, that is, the isomorphism is dependent on the choice of basis. Similarly, the double dual vector space is canonically isomorphic to the vector space.

Theorem 8.8. *Let G be a finite abelian group. Then $\hat{\hat{G}}$ is canonically isomorphic to G. (Pontryagin Duality)*

Proof. We construct the natural mapping $\kappa : G \to (\hat{G} \to S^1)$ such that $g \mapsto (\chi \mapsto \chi(g))$. We let $g, h \in G$. We let $\kappa(g) = \phi_g$ and $\kappa(h) = \phi_h$. Then

$$\kappa(gh) = \phi_{gh} = \phi_g\phi_h = \kappa(g)\kappa(h)$$

We also see that $\kappa(e) = \dim \chi = 1$. We get that κ is a homomorphism. We let $g \in \operatorname{Ker} G$. Then for all $\chi \in \hat{G}$, we have $\chi(g) = e$, which implies that $g = e$. Then, we find κ is injective. Since $|G| = |\hat{\hat{G}}|$, we get surjective, and thus, our result. $\square$

9. Conjugacy Classes

In this section, we look at conjugacy classes. In particular, we look at conjugacy classes for the symmetric group and construct an upper bound for the number of irreducible representations.

Definition 9.1. For a given element x in a group G, the conjugacy class of x in G is defined as $x_G = \{g^{-1}xg \mid g \in G\}$.

Since the trace of a matrix is independent of the choice of basis, we see that $\chi(hgh^{-1}) = \chi(g)$ for all $g, h \in G$. As such, we see that χ is constant on the conjugacy classes of G, and we call χ a class function.

We look at the conjugacy classes of S_3 and S_4.

Example 9.2. The following are the conjugacy classes of S_3:

$$C_1 = \{e\} \qquad C_2 = \{(12),(13),(23)\} \qquad C_3 = \{(123),(132)\}$$

Conjugacy class	e	(12)	(123)	(1234)	(12)(34)
Number of elements	1	6	8	6	3

TABLE 3. The Conjugacy Classes of S_4

We use the following theorem to construct an upper bound for the number of irreducible representations. We note that these values are actually equal, but an upper bound is sufficient for our purposes.

Theorem 9.3. *The number of irreducible representations of finite group G is at most equal to the number of conjugacy classes of G.*

Proof. We let χ and ψ be characters of G and we let $C_1, C_2, ..., C_n$ represent the conjugacy classes of G. We write

$$\delta_{ij} = \langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\chi}\psi = \frac{1}{|G|} \sum_{i=1}^{n} |C_i|\overline{\chi(C_i)}\psi(C_i)$$

We consider χ and ψ as n-dimensional vectors, where the ith coordinate represents the character of the ith conjugacy class. Under this consideration, we see that since these vectors are orthonormal, there can be at most n of them. □

We see that for finite abelian groups, each conjugacy class is a singleton. There are thus as many irreducible representations of a finite abelian group as there are elements of the group. For the non-abelian group, S_n, we find that each conjugacy class is determined by cycle type.

Theorem 9.4. *In S_n cycle type entirely determines conjugacy class.*

Proof. We let $\sigma = (\alpha_1, \alpha_2, ..., \alpha_k)(\beta_1, \beta_2, ...\beta_j)...(\omega_1, \omega_2, ...\omega_i)$ be an element of S_n. Then we let τ be another element of S_n. We write $\tau\sigma\tau = \tau^{-1}(\alpha_1, \alpha_2, ..., \alpha_k)(\beta_1, \beta_2, ...\beta_j)$ $...(\omega_1, \omega_2, ...\omega_i)\tau^{-1} = \tau^(\alpha_1, \alpha_2, ..., \alpha_k)\tau{-}1\tau(\beta_1, \beta_2, ...\beta_j)\tau^{-1}...\tau(\omega_1, \omega_2, ...\omega_i)\tau^{-1}$. Then for each individual cycle $(a_1, a_2, ..., a_n)$ we have $\tau(a_1, a_2, ..., a_n)\tau^{-1} =$ $(\tau(a_1), \tau(a_2), ..., \tau(a_n))$. We thus find that conjugation for each individual cycle does not change cycle length, so cycle type must be constant for all conjugates.

We let σ and τ be two elements of S_n such that they have the same cycle type. We construct a bijection γ that maps a cycle of length k of σ with a cycle of length k of τ. For each pair of cycles $(\sigma_1, \sigma_2, ..., \sigma_n)$ and $(\tau_1, \tau_2, ..., \tau_n)$, we define $\psi(\sigma_1) = \tau_1$. Then, we find $\psi \in S_n$ and $\tau = \psi\sigma\psi^{-1}$, which implies σ and τ are conjugate. □

10. The Symmetric Group

In this section, we look at representation theory of the symmetric group. We introduce Young Tableaux, a combinatorial object that we use to construct irreducible representations.

Example 10.1. We construct the irreducible representations of S_4. We first construct the trivial, sign, and standard irreducible representations. We tensor the sign and the standard representations for another irreducible representation. We use the Schur Orthogonality Relations to find the last representation.

S_4	e	(12)	(123)	(1234)	(12)(34)
Trivial representation	1	1	1	1	1
Sign representation	1	-1	1	-1	1
Standard representation	3	1	0	-1	-1
Sign $\otimes$ standard representation	3	-1	0	1	-1
Fifth representation	2	0	-1	0	2

TABLE 4. Character table for S_4

We show that each representation is irreducible and inequivalent to all the others using our inner product for characters. We also see that

$$1^2 + 1^2 + 3^2 + 3^2 + 2^2 = 24 = |S_4|$$

as well as

$$|\{\text{Conjugacy Classes of } S_4\}| = |\{\text{Irreducible Representations of } S_4\}| = 5.$$

We define a partition, which naturally corresponds to a conjugacy class of S_n.

Definition 10.2. A partition $\lambda = (\lambda_1, \lambda_2, ..., \lambda_k)$ of a number $n \in \mathbb{N}$ is a sequence of numbers $\lambda_i \in \mathbb{N}$ such that $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_k$ and $\lambda_1 + \lambda_2 + ... + \lambda_k = n$.

Definition 10.3. A Young diagram of shape $\lambda = (\lambda_1, \lambda_2, ..., \lambda_k)$ is a collection of left justified boxes where the ith row is of length λ_i.

Definition 10.4. A Young tableau of shape λ with n boxes is a Young diagram filled in with the numbers $1...n$ with each number occurring exactly once.

Definition 10.5. A Young tableau where the numbers are increasing across each row and column is called a standard Young tableau.

An element of S_n acts on a Young tableau by permutation. For example

$$(123)\ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} = \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 & 4 \\ \hline \end{array}$$

In addition to considering standard Young tableaux, we also look at tabloids, equivalence classes of Young tableaux.

Definition 10.6. We say that two Young tableaux are equivalent if each of their rows contain the same numbers. We call an equivalence class of Young tableaux a tabloid.

Example 10.7. We let a Young tableaux with values increasing across each row be a representative Young tableaux for each tabloid. For the partition $\lambda = (2, 1)$ we have the following tabloids:

$$\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$$

We note, however, that only the last two are standard Young tableaux.

We use tabloids to construct representations. In essence, what we construct are permutation representations. We say that the tabloids form a basis for the representation space of the representation. Then the dimension of the representation given by a partition is equal to the number of tabloids for that partition.

Theorem 10.8. *Given a partition $\lambda = (\lambda_1, \lambda_2, ..., \lambda_k)$ of n, the number of tabloids is given by the equation*

$$N = \frac{n!}{\lambda_1!\lambda_2!...\lambda_k!}$$

Proof. We see that there are $n!$ ways to assign n numbers to n boxes. We claim that this is equal to the total number of distinct Young tableaux. For each row λ_i, we have $\lambda_i!$ ways to arrange the λ_i numbers in that row. We divide by these numbers to eliminate all the equivalent Young tableaux. □

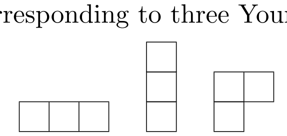
We use the following theorem to find the character of our representation.

Theorem 10.9. *We let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_k)$ be a partition of n, and we let $\sigma \in S_n$. Then we let $c_1, c_2, ..., c_j$ represent the individual cycles of σ. Then, the character of the representation given by the partition at σ is given by the coefficient $x_1^{\lambda_1} x_2^{\lambda_2} ... x_k^{\lambda_k}$ in the product*

$$P_\sigma = \prod_{i=1}^{j} (x_1^{c_i} + x_2^{c_i} + ... + x_k^{c_i})$$

Proof. For each individual cycle c_i, we let $(x_1^{c_i} + x_2^{c_i} + ... + x_k^{c_i})$ represent the assignment of c_i numbers to the x_i row. The product of these polynomials, iterating from i to j (the total number of cycles), then gives us all of the ways to assign the cycles to rows. For example, the term $x_1^2 x_2$ represents the assignment of 2 elements to row 1 and 1 element to row 2. We see that the coefficient on any term then represents the number of ways to achieve a specific assignment. We thus find that the coefficient of $x_1^{\lambda_1} x_2^{\lambda_2} ... x_k^{\lambda_k}$ gives us the number of ways that σ fixes a given tabloid. This means that each cycle c_i only permutes the elements of one row. We find that the number of fixed tabloids is equal to the trace, and thus the character evaluated at σ. □

Example 10.10. We consider our example of S_3. We have three conjugacy classes, and thus three partitions corresponding to three Young diagrams.



We consider the first diagram with partition (3). By our first equation, we have 1 tabloid, and thus the dimension of the representation is 1. By our second equation, we get the polynomials

$$P_e = P_{(12)} = P_{(123)} = x_1^3$$

Since we are looking at the coefficient of x_1^3, the character of the representation is $\chi_1 = (1,1,1)$.

For the second diagram with partition $(1,1,1)$, we get dimension 3. By our second equation, we get the polynomials

$$P_e = (x_1 + x_2 + x_3)^3$$
$$P_{(12)} = (x_1^2 + x_2^2 + x_3^2)(x_1 + x_2 + x_3)$$
$$P_{(123)} = x_1^3 + x_2^3 + x_3^3$$

Since we are looking at the coefficient of $x_1x_2x_3$, the character of the representation is $\chi_2 = (6,0,0)$.

For the last diagram with partition $(2,1)$, we get dimension 2. By our second equation, we get the polynomials

$$P_e = (x_1 + x_2)^3$$
$$P_{(12)} = (x_1^2 + x_2^2)(x_1 + x_2)$$
$$P_{(123)} = (x_1^3 + x_2^3)$$

Since we are looking at the coefficient of $x_1^2x_2$, the character of the representation is $\chi_2 = (3,1,0)$.

We construct a preliminary character table.

S_3	e	(12)	(123)
χ_1	1	1	1
χ_2	6	0	0
χ_3	3	1	0

Using the inner products $\langle \chi_2, \chi_2 \rangle = 6$ and $\langle \chi_3, \chi_3 \rangle = 2$, we see that χ_2 and χ_3 are reducible. We also see that χ_1 is irreducible. Using the inner products $\langle \chi_1, \chi_3 \rangle = 1$, we reduce χ_3 to $(2,0,-1)$. By the inner product $\langle \chi_3, \chi_3 \rangle = 1$, we see that this representation is irreducible. Using the inner products $\langle \chi_1, \chi_2 \rangle = 1$ and $\langle \chi_3, \chi_2 \rangle = 2$, we reduce χ_2 to $(5,-1,-1)$ and then $(1,-1,1)$. Then, by $\langle \chi_2, \chi_2 \rangle = 1$, we see that this representation is irreducible.

We now look at the process of constructing representations from partitions and standard Young tableaux. We essentially follow what is outlined in Chapter 4 of Fulton and Harris. We first note that we can exclusively consider standard Young tableaux to construct all the irreducible representations. We see that for a given partition λ, two distinct Young tableau yield different yet isomorphic representations. For our given standard Young tableau, we define

$$P_\lambda = \{g \in S_n \mid g \text{ preserves each row of } \lambda\}$$
$$Q_\lambda = \{g \in S_n \mid g \text{ preserves each column of } \lambda\}$$

We define two vectors in the group algebra $\mathbb{C}[S_n]$. For each $g \in S_n$, we let e_g be the corresponding unit vector for that group element.

$$a_\lambda = \sum_{g \in P_\lambda} e_g$$
$$b_\lambda = \sum_{g \in Q_\lambda} \operatorname{sgn}(g) e_g$$

We define a Young symmetrizer c_λ, an object that will correspond to each irreducible representation of S_n.

$$c_\lambda = a_\lambda b_\lambda = \sum_{g \in P_\lambda h \in Q_\lambda} \operatorname{sgn}(h) e_{gh}$$

We consider the tensor product vector space $V^{\otimes n}$. We let an element of S_n act on $V^{\otimes n}$ by permutation of indices. We define a representation $\phi : \mathbb{C}[S_n] \to \operatorname{End}(V^{\otimes n})$. For each partition λ of S_n, we claim that $\operatorname{Im} c_\lambda = V_\lambda$, where S_n acts by left mulitplication, is a representation space of an irreducible representation.

We look at an example using S_3.

Example 10.11. For S_3, we have three partitions and the following four standard Young tableaux

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array} \qquad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}$$

For the first partition, we have $P_{(3)} = S_3$ and $Q_{(3)} = \{e\}$. Then

$$c_{(3)} = \sum_{g \in S_3} \operatorname{sgn}(e) e_{ge} = \sum_{g \in S_3} e_g$$

For the second partition, we have $P_{(1,1,1)} = \{e\}$ and $Q_{(1,1,1)} = S_3$. Then

$$c_{(1,1,1)} = \sum_{h \in S_3} \operatorname{sgn}(h) e_{eh} = \sum_{h \in S_3} \operatorname{sgn}(h) e_h$$

For the third partition, we have $P_{(2,1)} = \{e, (12)\}$ and $Q_{(2,1)} = \{e, (13)\}$. Then

$$c_{(2,1)} = (e_e + e_{(12)})(e_e - e_{(13)}) = e + e_{(12)} - e_{(13)} - e_{(132)}$$

We find

$$V_{(3)} = [\mathbb{C}]S_n \sum_{g \in S_3} e_g = \mathbb{C} \sum_{g \in S_3} e_g$$
$$V_{(1,1,1)} = [\mathbb{C}]S_n \sum_{h \in S_3} \operatorname{sgn}(h) e_h = \mathbb{C} \sum_{h \in S_3} \operatorname{sgn}(h) e_h$$

We claim that $V_{(3)}$ is the representation space of the one-dimensional trivial representation. We claim that $V_{(1,1,1)}$ is the representation space of the one-dimensional sign representation. We see that the two standard tableau form a basis for $V_{(2,1)}$. We thus claim that $V_{(2,1)}$ is the representation space for the two-dimensional standard representation.

We also construct a formula to calculate the dimension of these representations.

Definition 10.12. The hook length of a box in a Young diagram is the number of boxes directly below or directly to the right of the box, including the box itself.

Proposition 10.13. The dimension of the representation given by a partition λ of length n is equal to $n!$ divided by the product of the hook lengths in the Young diagram of shape λ. In other words,

$$\dim V_\lambda = \frac{n!}{\prod h_\lambda}$$

where h_λ represents the hook length of a given box.

Example 10.14. We build off our previous example of S_3. Below, we show the three Young diagrams with the hook-lengths of each box.

$$\begin{array}{|c|c|c|}\hline 3 & 2 & 1 \\ \hline \end{array} \qquad \begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 1 \\ \cline{1-1} \end{array}$$

Then, we find the dimensions of the corresponding representations.

$$\dim V_{(3)} = \frac{3!}{3 \cdot 2 \cdot 1} = 1$$
$$\dim V_{(1,1,1)} = \frac{3!}{3 \cdot 2 \cdot 1} = 1$$
$$\dim V_{(2,1)} = \frac{3!}{3 \cdot 1 \cdot 1} = 2$$

We thus end this paper with a practical overview of the process of constructing the irreducible representations of the symmetric group. For further reading, it is recommended that one examine more rigorously the construction of the irreducible representations of the symmetric group, including the Specht modules.

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References

[1] Keith Conrad. Chracters of Finite Abelian Groups. http://www.math.uconn.edu/ kconrad/blurbs/grouptheory/charthy.pdf.
[2] William Fulton and Joe Harris. Representation Theory: A First Course. Springer-Verlag New York Inc. 1991.
[3] Yufei Zhao. Young Tableaux and the Representations of the Symmetric Group. http://www.thehcmr.org/issue2_2/tableaux.pdf