THE UNCERTAINTY PRINCIPLE IN HARMONIC ANALYSIS

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ABSTRACT. We present several uncertainty principle results from Fourier analysis. The results we present are formally unrelated to one another, but are united by the heuristic principle that one cannot localize a function and its Fourier transform simultaneously.

CONTENTS

1. INTRODUCTION

The uncertainty principle is a collection of related results which give sense to the following proposition: one cannot concentrate the mass of a function and its Fourier transform simultaneously. This principle is one of the fundamental mechanisms by which one obtains information about a function from its Fourier transform and vice-versa, and understanding it both heuristically and in its various formalizations is vital in intuiting what the Fourier transform "looks like." In this paper, we will consider a variety of ways of making this principle precise, beginning with the eponymous result of Heisenberg in physics. Along the way, we will need to develop a theory of the complex Fourier transform (Paley-Wiener theory). We will attempt to draw links between uncertainty principle results and the theory of PDE; we will use the Logvinenko-Sereda theorem to prove the local solvability of constant-coefficient linear PDE and we will conclude with a discussion of a recent uncertainty principle result which relies on the clever use of the Schrödinger equation.

We assume familiarity with the Fourier transform and some foundational results—in particular, the Fourier inversion theorem and Plancherel's theorem. A treatment of this material can be found in any number of texts, for example [7]. All integrals that appear in this paper are in the sense of Lebesgue.

For convenience, we recall the following definitions.

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Definition 1.1. Let $f : \mathbb{R}^d \to \mathbb{C}$. We define the *Fourier transform* $\hat{f}$ of f by

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i \xi \cdot x} f(x) dx$$

and the *inverse Fourier transform* $\check{f}$ of f by

$$\check{f}(\xi) = \int_{\mathbb{R}^d} e^{2\pi i \xi \cdot x} f(x) dx$$

whenever these integrals make sense.

Definition 1.2. We define the *Schwarz space* to be

$$\mathcal{S}(\mathbb{R}^d) = \{ f \in C^\infty(\mathbb{R}^d) \mid x^\alpha \partial^\beta f \in L^\infty \mathbb{R}^d \ \ \forall \alpha, \beta \}$$

where $\alpha, \beta \in \mathbb{N}^d$ above are arbitrary multi-indices. We call a function $f \in \mathcal{S}(\mathbb{R}^d)$ a *Schwarz function.*

2. Heisenberg's Uncertainty Principle

The following result tells us that the masses of a function and it Fourier transform cannot simultaneously be concentrated around points. This pointwise consideration is the crudest form of the uncertainty principle we will discuss and was the first to be formulated. Although it is a fundamental result in physics, we will not discuss the physical significance of this theorem, except to point out that the operators X and D below have important physical interpretations.

Theorem 2.1 (Heisenberg's uncertainty principle)**.** *Let $f \in \mathcal{S}(\mathbb{R})$ and $x_0, \xi_0 \in \mathbb{R}$. Then we have:*

$$\|f\|_2^2 \leqslant 4\pi \|(x - x_0) f(x)\|_2 \|(\xi - \xi_0)\hat{f}(\xi)\|_2.$$

Proof. By taking a translation of f we may assume $x_0 = 0$, and likewise by multiplying f by a suitable complex number of unit length we may assume $\xi_0 = 0$. In this case, a physicist might rewrite the Heisenberg inequality as

$$\|f\|_2^2 \leqslant 4\pi \|Xf\|_2 \, \|Df\|_2$$

where X and D above are self-adjoint linear operators on $\mathcal{S}(\mathbb{R})$ defined by

$$(Xf)(x) = xf(x), \qquad Df = \frac{1}{2\pi i}\frac{d}{dx}f.$$

(The self-adjointness of D may be seen by using integration by parts.) We note that

$$\begin{aligned} ([D, X]f)(x) &= \frac{x}{2\pi i} f'(x) + \frac{1}{2\pi i} f(x) - \frac{x}{2\pi i} f'(x) \\ &= \frac{1}{2\pi i} f(x) \end{aligned}$$

so, by substitution, we obtain

$$\begin{aligned} \|f\|_2^2 &= \langle f, f \rangle \\ &= 2\pi i \langle [D, X] f, f \rangle \\ &= 2\pi i \left(\langle Xf, Df \rangle - \langle Df, Xf \rangle \right) \\ &= 4\pi \operatorname{Im}(\langle Xf, Df \rangle) \\ &\leqslant 4\pi \|Xf\|_2 \|Df\|_2 \end{aligned}$$

as desired. □

3. Complex Prerequisites and Paley-Wiener Theory

We begin by stating without proof three complex-analytic results that we will have use of later. Two of these are multidimensional variants of familiar results on $\mathbb{C}$; one is a less familiar single-variable result. The proofs of the first two results can be found in [5], while the proof of the third can be found in [6].

Theorem 3.1. *Let $\Omega \subset \mathbb{C}^d$, $f : \Omega \to \mathbb{C}$ be analytic in each variable. If f vanishes on a set of positive measure in $\Omega \cap \mathbb{R}^d$, then $f = 0$.*

Theorem 3.2 (Montel)**.** *If $\mathcal{F}$ is a locally bounded family of analytic functions on $\Omega \subset \mathbb{C}^d$, then every sequence in $\mathcal{F}$ has a subsequence that converges locally uniformly on Ω to an analytic function.*

The following theorem is best understood as a generalization of the fact that a holomorphic function on a bounded domain attains its maximum on the boundary of its domain.

Theorem 3.3 (Phragmén-Lindelóf)**.** *Let S be a sector of the complex plane with angle π/α, $\alpha > 1/2$. Let f be holomorphic in S and continuous in ∂S. If f is bounded on ∂S by some constant bound M and satisfies the bound $|f(z)| \leqslant Ce^{\tau|z|^\rho}$ for some $0 < \rho < \alpha$, $C, \tau > 0$, then f is in fact bounded by M on all of S.*

We now develop a theory of the complex Fourier transform, i.e. the Fourier transform as a function of a complex variable. Our goal shall be to arrive at insights regarding functions of a real variable by considering them as restrictions of complex functions, a tactic familiar to the reader who has studied certain real-variable integrals. This theory is rich and interesting in its own right, but we present only what simple results we need to develop our uncertainty principle results. For a fuller treatment, see [6].

Definition 3.4. Let $f \in L^1(\mathbb{R}^d) \cap C_c(\mathbb{R}^d)$. We define the complex *Fourier transform* $\hat{f}$ of f to be

$$\hat{f}(z) = \int_{\mathbb{R}^d} e^{-2\pi i x \cdot z} f(x) dx$$

defined for all $z \in \mathbb{C}^d$.

Proposition 3.5. *If $f \in L^1(\mathbb{R}^d)$ satisfies*

$$\int_{\mathbb{R}^d} |f(x)| e^{x \cdot y} dx < \infty$$

for all $y \in \mathbb{R}^d$, then $\hat{f}$ is entire.

Proof. We may differentiate under the integral sign to obtain a derivative of $\hat{f}$. □

Proposition 3.6 (Paley-Wiener)**.** *If $f \in L^2(\mathbb{R}^d)$ is supported in the ball of radius R, then the complex Fourier transform $\hat{f}$ of f is an entire function that satisfies the bound*

$$|\hat{f}(z)| \leqslant Ce^{2\pi R|z|}. \quad (3.7)$$

Proof. Since compactly supported L^2 functions are L^1, it follows from Proposition 3.5 that $\hat{f}$ is entire. Moreover, applying the Cauchy-Schwarz inequality, we have

$$\begin{aligned} |\hat{f}(z)| &= \left| \int_{(-R,R)} e^{-2\pi i z\xi} \hat{f}(\xi) d\xi \right| \\ &\leqslant e^{2\pi R|z|} \|f\|_2 \end{aligned}$$

as desired. □

The converse of this theorem is also true: if the complex Fourier transform of a function f is entire and satisfies the bound (3.7), then f is supported in the ball of radius R. It is usually this "if and only if" statement that is called the Paley-Wiener theorem, but the converse is not necessary for our purposes.

4. Amrein-Berthier Theorem

In Section 1, the Heisenberg uncertainty principle gave us a bound on how much we can simultaneously localize a function and its Fourier transform around points. That is, we showed that if the mass of a function is concentrated around a point, then the mass of its Fourier transform must be appropriately diffuse. We will now consider the question of simultaneous localization on *sets* and show that, for appropriate notions of "smallness," if a function vanishes outside a small set then its Fourier transform cannot. If we take "small" to mean "compact," then this result falls out of our consideration of the complex Fourier transform.

Proposition 4.1. *If $f \in L^2(\mathbb{R}^d)$ is compactly supported and $\hat{f}$ is compactly supported, then $f = 0$.*

Proof. If $\hat{f}$ is compactly supported, then, by Proposition 3.6, f is the restriction to $\mathbb{R}$ of an analytic function. If f is compactly supported, then its zeroes are clearly not isolated, so the analyticity of f implies $f = 0$. □

We now consider the case when "small" is taken to mean "of finite measure." Every real compact set has finite measure, of course, but sets of finite measure need not be bounded.

Theorem 4.2 (Amrein-Berthier)**.** *Let $f \in L^2(\mathbb{R}^d)$, $E, F \subset \mathbb{R}$ have finite measure. Then*

$$\|f\|_{L^2(\mathbb{R}^d)} \leqslant C(\|f\|_{L^2(E^c)} \|\hat{f}\|_{L^2(F^c)}) \tag{4.3}$$

where C depends only on E, F, d. In particular, if both f and $\hat{f}$ are supported on sets of finite measure, then $f = 0$.

Before we prove this result, we require a lemma.

Lemma 4.4. *Let $E, F \subset \mathbb{R}^d$ have finite measure. If there exists a constant C' such that* $\mathrm{supp}(\hat{f}) \subset F \implies \|f\|_2 \leqslant C' \|f\|_{L^2(E^c)}$, *then there exists a constant C such that, for all $f \in L^2(\mathbb{R}^d)$, $\|f\|_2 \leqslant C(\|f\|_{L^2(E^c)} + \|\hat{f}\|_{L^2(F^c)})$.*

Proof. Define the operator P by $P_F f = \overset{\smile}{(\chi_F \hat{f})}$. Then

$$\begin{aligned} \|f\|_2 &\leqslant \|P_F f\|_2 + \|P_{F^c} f\|_2 \\ &\leqslant C' \|f\|_{L^2(E^c)} + \|\hat{f}\|_{L^2(F^c)} \end{aligned}$$

so that $C = \max(1, C')$ yields the desired result. □

Proof of Theorem 4.2. Fix E,F as above. We consider the operator T given by

$$Tf = \chi_E(\widecheck{\chi_F \hat{f}}).$$

If the L^2 operator norm of T is less than 1, then the hypothesis of Lemma 4.4 holds with $C' = \frac{1}{1-\|T\|}$. Thus, it suffices to show that $\|T\| < 1$.

Observe that T is a Hilbert-Schmidt integral operator with kernel

$$K(x,y) = \chi_E(x)\hat{\chi}_F(y)$$

and Hilbert-Schmidt norm σ given by

$$\sigma^2 = \|K\|^2_{L^2(\mathbb{R}^{2d})} = \int_{\mathbb{R}^{2d}} \chi_E^2(x)\chi_F^2(y)dxdy = |E||F|.$$

It follows that T is a compact operator. Note that, by compactness, the L^2 operator norm of T is 1 if and only if there exists a function $f \in L^2(\mathbb{R}^d)$ such that f is supported on E and $\hat{f}$ is supported on F. Therefore, the quantitative bound (4.3) is in fact equivalent to the claim that a nonzero function and its Fourier transform cannot both be supported on sets of finite measure.

We now show that the norm of T is less than 1. Since T is the product of projections, we know it has L^2 operator norm less than or equal to 1. Suppose by way of contradiction that $\|T\| = 1$, i.e. that there exists a $f \in L^2(\mathbb{R})$ with $\text{supp}(f) \subset E$ and $\text{supp}(\hat{f}) \subset F$. By repeatedly translating f by sufficiently small and vanishing amounts (say, 2^{-k}), we obtain an infinite collection of linearly independent functions compactly supported in some set E' of finite measure and whose Fourier transforms are all supported in F. It follows that these functions are eigenfunctions of the operator T' obtained by substituting E' for E in the definition of T with eigenvalue 1. But since T' is a compact operator, like T, its eigenspaces with nonzero eigenvalue are all finite-dimensional, so we have obtained a contradiction. □

Notice that, by abandoning complex-variable methods, we were able to easily prove Theorem 4.2 for multiple dimensions, whereas it is not immediately clear how the weaker Proposition 4.1 might be generalized to $\mathbb{R}^d$. In this case, the trick is simply to generalize Proposition 3.6 to $\mathbb{C}^d$, which can be done without too much difficulty, but we will encounter this principle again later in more desperate circumstances.

5. Logvinenko-Sereda Theorem and Applications to PDE

We now consider a version of the uncertainty principle that is more involved in its formulation (and is something of a slog to prove), but which proves to be useful in applications. We will consider one such application in the proof of the Malgrange-Ehrenpreis theorem, a remarkably general result in the theory of PDE. This result concerns the failure of a function with compactly supported Fourier transform to be localized on an appropriately "thin" set.

We first require a lemma. Below, $B_r(a)$ denotes the ball of radius r about a.

Lemma 5.1. *Let $f \in L^2(\mathbb{R}^d)$ satisfy $\text{supp}(\hat{f}) \subset B_r(0)$. Then*

$$\|\partial^\alpha f\|_2 \leqslant (2\pi r)^{|\alpha|}\|f\|_2$$

for all multi-indices α.

Proof. By Plancherel, we have

$$\begin{aligned}\|\partial^\alpha f\|_2 &= \|\widehat{\partial^\alpha f}\|_2\\ &= \|(-2\pi i)^{|\alpha|}\xi^\alpha \hat{f}\|_2\\ &= (2\pi)^{|\alpha|}\sqrt{\int_{B_r(0)} |\xi^\alpha|^2|\hat{f}(\xi)|^2 d\xi}\\ &\leqslant (2\pi)^{|\alpha|}\sqrt{\int_{B_r(0)} |\xi|^{2|\alpha|}|\hat{f}(\xi)|^2 d\xi}\\ &\leqslant (2\pi r)^{|\alpha|}\|f\|_2\end{aligned}$$

as desired. □

The statement of the Logvinenko-Sereda theorem requires two definitions.

Definition 5.2. Let $r > 0$. A set $E \subset \mathbb{R}^d$ is B_r*-thick* if there exists a $\gamma > 0$ such that

$$|E \cap B_r(x)| > \gamma \text{ for all } x \in \mathbb{R}^d.$$

Definition 5.3. A set $F \subset \mathbb{R}^d$ is $B_1(0)$*-negligible* if there exists a $\epsilon > 0$ such that, for all $f \in L^2(\mathbb{R}^d)$, one has either

$$\|\hat{f}\|_{L^2(\mathbb{R}^d\setminus B_1(0))} \geqslant \epsilon\|f\|_2 \quad \text{or} \quad \|f\|_{L^2(\mathbb{R}^d\setminus F)} \geqslant \epsilon\|f\|_2.$$

Theorem 5.4 (Logvinenko-Sereda)**.** *A set $F \subset \mathbb{R}^d$ is $B_1(0)$-negligible if and only if its complement $E = F^c$ is B_r-thick for some $r > 0$.*

Proof. We begin with the "only if" direction. Let $F \subset \mathbb{R}^d$ be $B_1(0)$-negligible. Suppose its complement E is not B_r-thick for any $r > 0$. Then there exists a sequence $r_k > 0, x_k \in \mathbb{R}^d$ such that $r_k \to \infty$ and

$$|E \cap B_{r_k}(x_k)| \to 0 \text{ as } k \to \infty.$$

Let $B_k = B_{r_k}(x_k)$. Take any $f \in \mathcal{S}(\mathbb{R}^d)$ with $\operatorname{supp}(\hat{f}) \subset B_1(0)$. Let $f_k(x) = f(x - x_k)$. Note that $\hat{f}_k$ is still supported in $B_1(0)$ for all k. Thus, there is some $\epsilon > 0$ for which

$$\begin{aligned}\epsilon\|f\|_2 \leqslant \|f_k\|_{L^2(E)} &\leqslant \|f_k\|_{L^2(B_k\cap E)} + \|f_k\|_{L^2(\mathbb{R}^d\setminus B_k)}\\ &\leqslant \|f\|_\infty\sqrt{|E \cap B_{r_k}(x_k)|} + \sqrt{\int_{|x-x_k|>r_k} |f(x)|^2 dx}\end{aligned}$$

since f is Schwarz, the RHS above vanishes as $k \to \infty$, a contradiction.

The "if" direction is more difficult. We will prove the weaker claim that, if E is B_r-thick for some $r > 0$, then there exists a $C > 0$ such that

$$\operatorname{supp}(\hat{f}) \subset B_1(0) \implies \|f\|_2 \leqslant C\|f\|_{L^2(E)}. \tag{5.5}$$

If this claim is true, and if $\|\hat{f}\|_{L^2(\mathbb{R}^d\setminus B_1(0))} < \epsilon\|f\|_2$, then we can take

$$f_1 = \widecheck{(\chi_{B_1(0)}\hat{f})}$$

and, writing $f = f_1 + f_2$, obtain

$$\begin{aligned}\|f\|_2 &\leqslant \|f_1\|_2 + \|f_2\|_2\\ &\leqslant C\|f\|_{L^2(E)} + \epsilon\|f\|_2\end{aligned}$$

which, for sufficiently small $\epsilon < \frac{1}{C}$, gives $\|f\|_{L^2(E)} \geqslant \epsilon \|f\|_2$, as desired.

It remains to show (5.5). We begin by partitioning $\mathbb{R}^d$ into cubes Q of side length $2r$. Fix A large and let $f \in L^2(\mathbb{R}^d)$. We say Q is "good" with respect to f if

$$\|\partial^\alpha f\|_{L^2(Q)} \leqslant A^{|\alpha|} \|f\|_{L^2(Q)} \text{ for all } \alpha.$$

We make three claims about this partition:

Claim 1. Let $B = \bigcup_{\text{bad}} Q$. Then there is some constant $M > 0$ such that

$$\|f_k\|_{L^2(B)} \leqslant MA^{-1}\|f\|_2.$$

Proof of Claim 1. Note that

$$\begin{aligned}
\|f\|^2_{L^2(B)} &\leqslant \sum_{\alpha \neq 0} \sum_{Q \text{ bad}} A^{-2|\alpha|} \|\partial^\alpha f\|^2_{L^2(Q)} \\
&\leqslant \sum_{\alpha \neq 0} \sum_{Q \text{ bad}} A^{-2|\alpha|} (2\pi)^{2|\alpha|} \|f\|^2_{L^2(Q)} \\
&\leqslant \left(\sum_{\alpha \neq 0} \frac{(2\pi)^{2|\alpha|}}{A^{2(|\alpha|-1)}} \right) A^{-2} \|f\|_2^2 \\
&\leqslant MA^{-2}\|f\|_2^2
\end{aligned}$$

for A sufficiently large, where we use Lemma 5.1 to pass to the second line. □

Claim 2. For any good Q, there exists $x_0 \in Q$ such that

$$|\partial^\alpha f(x_0)| \leqslant A^{2|\alpha|+1} \|f\|_{L^2(Q)} \text{ for all } \alpha.$$

Proof of Claim 2. Suppose not; then there is some good Q such that, for every $x \in Q$, there exists a multi-index α with

$$|\partial^\alpha f(x_0)| > A^{2|\alpha|+1} \|f\|_{L^2(Q)}.$$

Therefore, we may cover Q by the sets

$$A_\alpha = \{x \in Q \mid |\partial^\alpha f(x)| > A^{2|\alpha|+1} \|f\|_{L^2(Q)}\}$$

and so we obtain

$$\begin{aligned}
\|f\|^2_{L^2(Q)} &\leqslant \frac{1}{|Q|} \sum_\alpha \int_{A_\alpha} \|f\|^2_{L^2(Q)} dx \\
&\leqslant \frac{1}{|Q|} \sum_\alpha \int_{A_\alpha} A^{-4|\alpha|-2} |\partial^\alpha f(x)|^2 dx \\
&\leqslant \frac{\|f\|_2^2}{|Q|} \sum_\alpha A^{-4|\alpha|-2} (2\pi)^{|\alpha|}
\end{aligned}$$

whence we can obtain a contradiction by choosing large enough A. □

Claim 3. There exists $\eta > 0$ such that, for all $f \in L^2(\mathbb{R}^d)$ and any Q good with respect to f,

$$\|f\|_{L^2(E \cap Q)} \geqslant \eta \|f||_{L^2(Q)}.$$

Proof of Claim 3. Suppose not; then we can find a sequence $\{f_k\} \subset L^2(\mathbb{R}^d)$ with $\|f_k\|_{L^2(Q_k)} = 1$ and $\text{supp}(\hat{f}_k) \subset B_1(0)$ and a sequence $\{Q_k\}$ of cubes in the partition such that Q_k is good with respect to f_k and

$$\|f\|_{L^2(E \cap Q_k)} \to 0 \text{ as } k \to \infty.$$

Since translations of f_k do not affect the support of $\hat{f}_k$, we may take each Q_k to be the cube $Q = [-r, r]^d$ centered at 0. Now, by Theorem 3.6, each f_k admits an entire extension F_k to all of $\mathbb{C}^d$. We take the Taylor expansion of each F_k about an $x_k \in Q_k$ as in Claim 2 and obtain

$$\begin{aligned} |F_k(z)| &\leqslant \sum_\alpha \frac{A^{2|\alpha|+1}}{\alpha!} |(z - x_k)^\alpha| \\ &\leqslant \left(\sum_{j=0}^{\infty} \frac{A^{2j+1}}{j!} (2 + |z|)^j \right)^d \leqslant C. \end{aligned}$$

So F_k is a normal family, and so it has a locally uniform limit F by Theorem 3.2. Since $\|f_k\|_{L^2(Q)} = 1$ for all k, we have $\|F\|_{L^2(Q)} = 1$. On the other hand, since $\|f_k\|_{L^2(E \cap Q)} \to 0$, we have $F = 0$ on $E \cap Q$. But Q contains the ball $B_r(0)$, so $E \cap Q \geqslant \gamma$, whence F vanishes on a set whose intersection with $\mathbb{R}^d$ has positive measure. By Theorem 3.1, we conclude that $F = 0$, a contradiction. □

We are finally ready to prove (5.5). Let $f \in L^2(\mathbb{R}^d)$ satisfy $\operatorname{supp}(\hat{f}) \subset B_1(0)$. Then, by Claims 1 and 3, we have

$$\begin{aligned} \|f\|_2 &= \sum_{\text{good } Q} \|f\|^2_{L^2(Q)} + \sum_{\text{bad } Q} \|f\|^2_{L^2(Q)} \\ &\leqslant \eta^{-2} \|f\|^2_{L^2(E)} + M^2 A^{-2} \|f\|_2^2 \end{aligned}$$

which completes the proof for $C \geqslant \frac{1}{\eta\sqrt{1 - M^2 A^{-2}}}$.

□

Recall that by "domain" we mean a connected, open set. If q is a polynomial with

$$q(x) = \sum_\alpha a_\alpha x^\alpha,$$

then below we write

$$q(D) = \sum_\alpha a_\alpha \partial^\alpha.$$

Theorem 5.6 (Malgrange-Ehrenpreis). *Let Ω be a bounded domain in $\mathbb{R}^d$ and let p be a nonzero polynomial. Then, for all $g \in L^2(\Omega)$, the partial differential equation $p(D)f = g$ has a solution $f \in L^2(\Omega)$ in the sense of distributions, i.e.*

$$\langle g, \varphi \rangle = \langle f, \overline{p}(D)\varphi \rangle \text{ for all } \varphi \in C_0^\infty(\Omega).$$

Proof. Note that, if

$$p(x) = \sum_\alpha a_\alpha x^\alpha,$$

then the adjoint of the operator $p(D)$ is the operator $\bar{p}(D)$ where

$$\overline{p}(x) = \sum_\alpha \overline{a}_\alpha x^\alpha.$$

Define the linear subspace $X \subset L^2(\Omega)$ by

$$X = \{\overline{p}(D)\varphi \mid \varphi \in C^\infty(\Omega)\}.$$

Define a linear functional ℓ on X by

$$\ell(\overline{p}(D)\varphi) = \langle g, \varphi \rangle.$$

If we had a bound of the form

$$\|\varphi\|_2 \leqslant C\|\overline{p}(D)\varphi\|_2 \tag{5.7}$$

then we would obtain the bound

$$|\ell(\overline{p}(D)\varphi)| \leqslant \|g\|_2\|\varphi\|_2 \leqslant C\|g\|_2\|\overline{p}(D)\varphi\|_2$$

i.e. $\|\ell\| \leqslant C$, so that we could apply the Hahn-Banach theorem to extend ℓ to a bounded linear functional ℓ' on $L^2(\Omega)$. By the Riesz representation theorem, we could find $f \in L^2(\Omega)$ satisfying, for all $\varphi \in C_0^\infty(\Omega)$,

$$\langle g, \varphi\rangle = \ell'(\overline{p}(D)\varphi) = \langle f, \overline{p}(D)\varphi\rangle$$

which is the desired result.

It remains then to show that (5.7) holds for all $\varphi \in C_0^\infty(\Omega)$. This is where the uncertainty principle will be of use. If we can show that there is some $\epsilon_0 > 0$ such that the set

$$E = \{x \in \mathbb{R}^d \mid |p(x)| > \epsilon_0\}$$

is B_1-thick, then Theorem 5.4 will imply that

$$\begin{aligned}\|\overline{p}(D)\varphi\|_2 &= \|\overline{p}(\xi)\hat{\varphi}(\xi)\|_2 \\ &\geqslant \|\overline{p}\hat{\varphi}\|_{L^2(E)} \\ &\geqslant \epsilon_0\|\hat{\varphi}\|_{L^2(E)} \\ &\geqslant \epsilon_1\|\varphi\|_2\end{aligned}$$

for some $\epsilon_1 > 0$. So we need to show that the set E is B_1-thick. We isolate this result (merely a fact about polynomials) in the following lemma. □

Lemma 5.8. *Let p be any nonzero polynomial in $\mathbb{R}^d$. Then there exists $\epsilon_0 > 0$, $\gamma > 0$ such that*

$$|\{x \in B \mid |p(x)| > \epsilon_0\}| \geqslant \gamma \tag{5.9}$$

for all unit balls $B \subset \mathbb{R}^d$.

Proof. Here we rely crucially on the fact that any two norms on a finite-dimensional vector space are equivalent; that is, for any two norms $\|\cdot\|_1$ and $\|\cdot\|_2$, there exists a constant C such that $C^{-1}\|v\|_1 \leqslant \|v\|_2 \leqslant C\|v\|_1$ for all vectors v. Let $n = \deg(p)$. The particular finite-dimensional vector space we are concerned with is the space of polynomials in $\mathbb{R}^d$ of degree at most n. For any fixed unit ball B, one norm on this space is $\|q\| = \max_{x\in B}(|q(x)| + |\nabla q(x)|)$, another is $\|q\| = \max_{x\in B}|p(x)|$, and yet another is $\|\sum b_\alpha x^\alpha\| = \sum |b_\alpha|$. Therefore, there are some constants $C_1, C_2 > 0$ for which we have

$$\max_{x\in B}|\nabla p(x)| \leqslant \max_{x\in B}(|p(x)| + |\nabla p(x)|) \leqslant C_1 \max_{x\in B}|p(x)| \tag{5.10}$$

and

$$\sum |a_\alpha| \leqslant C_2 \max_{x\in B}|p(x)| \tag{5.11}$$

where $p(x) = \sum a_\alpha x^\alpha$. Now, let p achieve its maximum at $x_0 \in B$. (5.10) together with the mean value theorem gives

$$|p(x) - p(x_0)| \leqslant |x - x_0| \max_{x\in B}|\nabla p(x)| \leqslant C_1|x - x_0||p(x_0)|$$

so that, for $C_1^{-1} > \gamma > 0$,

$$|x - x_0| < \gamma \implies |p(x)| < \frac{|p(x_0)|}{1 - C_1\gamma}.$$

At the same time, (5.11) gives that

$$|p(x_0)| \geqslant C_2^{-1} \sum |a_\alpha| \geqslant C_2^{-1} \sum_{|\alpha|=n} |a_\alpha|.$$

where the quantity on the RHS is invariant under translations of p. Therefore, by translating p, we obtain (5.9) for *all* unit balls B, with $\epsilon_0 = \frac{\sum_{|\alpha|=n} |a_\alpha|}{C_2(1-C_1\gamma)} > 0$. □

6. Hardy's Uncertainty Principle

We know that the set of Gaussian functions is invariant under Fourier transform; to be precise, if $f(x) = e^{-ax^2}$ for some $a > 0$, then the Fourier transform satisfies

$$\begin{aligned} \frac{\partial}{\partial \xi_k} \hat{f}(\xi) &= \int_{\mathbb{R}^d} -2\pi i x_k e^{-2\pi i x\cdot\xi} e^{-ax^2} dx \\ &= \frac{-2\pi^2 \xi_k}{a} \hat{f}(\xi) \end{aligned}$$

for all $k = 1, \dots, d$. From this we can conclude, after solving for the appropriate constant coefficient in front, that

$$\hat{f}(\xi) = \left(\frac{\pi}{a}\right)^{\frac{d}{2}} e^{-\pi^2\xi^2/a}. \tag{6.1}$$

Note that, in this case, both f and $\hat{f}$ decay at comparable rates—and very fast ones at that. In the spirit of our previous uncertainty principle results, one might expect to say that if a nonzero function decays at a super-exponential rate, then its Fourier transform cannot. The Gaussian functions provide a counterexample to this strong claim, but Hardy's uncertainty principle states that they are in some sense the "only" counterexample possible.

We begin with the one-dimensional case.

Theorem 6.2 (Hardy's uncertainty principle)**.** *Let $f : \mathbb{R} \to \mathbb{C}$ satisfy*

$$|f(x)| \leqslant C_1 e^{-ax^2} \tag{6.3}$$

and

$$|\hat{f}(\xi)| \leqslant C_2 e^{-b\xi^2} \tag{6.4}$$

for some $C_1, C_2, a, b > 0$. If $ab = \pi^2$, then there is some scalar A such that

$$f(x) = Ae^{-ax^2}. \tag{6.5}$$

If $ab > \pi^2$, then $f = 0$.

Proof. It suffices to prove the theorem for the case $a = b = \pi$, for, if f satisfies (6.3) and (6.4), then $g = f(\sqrt{\pi}\cdot/\sqrt{a})$ satisfies

$$|g(x)| \leqslant C_1 e^{-\pi x^2}$$

and

$$|\hat{g}(\xi)| = \left| \frac{\sqrt{\pi}}{\sqrt{a}} \hat{f}(\sqrt{a}\xi/\sqrt{\pi}) \right| \leqslant C_2' e^{-ab\xi^2/\pi}. \tag{6.6}$$

This reduces the case of general $ab = \pi^2$ to the case $a = b = \pi$. On the other hand, if $ab > \pi^2$, then (6.6) implies (6.4) for $\beta = \pi$, and so applying the theorem to g gives $g(x) = Ae^{-\pi x^2}$. But then (6.1) gives

$$|\hat{g}(\xi)| = \left|Ae^{-\pi\xi^2}\right| > O(e^{-ab\xi^2/\pi})$$

if $A \neq 0$, a contradiction.

We now prove the theorem for the case $a = b = \pi$. We proceed by tackling the even and odd components of f separately. Let f satisfy (6.3) and (6.4) for $a = b = \pi$. Then Proposition 3.5 implies that $\hat{f}$ is analytic. It is also easy to see that (6.4) implies

$$|\hat{f}(z)| \leqslant C_2 e^{\pi|z|^2} \tag{6.7}$$

for complex z.

If f is even, then $\hat{f}$ is even and analytic and so we can write

$$\hat{f}(\xi) = \sum a_\alpha \xi^{2\alpha} = h(\xi^2)$$

where h is analytic. (6.7) gives

$$|h(z)| \leqslant C_2 e^{\pi|z|} \tag{6.8}$$

while, for positive ξ, (6.4) gives

$$|h(\xi)| \leqslant C_2 e^{-\pi\xi}. \tag{6.9}$$

We want to use (6.8) to apply Theorem 3.3 to $h(z)$; to do this, we must consider a sector of angle smaller than π. So let $0 < \delta < \pi$. Writing $z = Re^{i\theta}$, we have

$$\left|e^{\frac{i\pi z e^{-i\delta/2}}{\sin(\delta/2)}} h(z)\right| = e^{\frac{-\pi R\sin(\theta-\delta/2)}{\sin(\delta/2)}}|h(z)| \leqslant C_2$$

when $\theta = 0, \delta$ by (6.9) and (6.8), respectively. Applying Theorem 3.3, we obtain

$$|h(z)| \leqslant C_2 e^{\frac{\pi R\sin(\theta-\delta/2)}{\sin(\delta/2)}}$$

for $0 < \theta < \delta$. Letting $\delta \to \pi$ gives

$$|h(z)| \leqslant C_2 e^{\pi|z|\cos(\arg z)} \leqslant C_2 e^{\pi|z|}$$

for $0 \leqslant \arg z \leqslant \pi$. The case of z in the lower half plane is handled similarly. So we see that $e^{\pi z}h(z)$ is a bounded, entire function, and, by Proposition 3.1, we conclude that

$$\hat{f}(\xi) = Ae^{-\pi x^2}$$

for some scalar A, and with (6.1) we obtain (6.5), as desired.

On the other hand, if f is odd, then we proceed in the same way, but with

$$\frac{\hat{f}(\xi)}{\xi} = \sum a_\alpha \xi^{2\alpha} = h(\xi^2).$$

In this case we obtain

$$|h(z^2)| \leqslant \left|\frac{\hat{f}(z)}{z}\right| \leqslant \frac{C_2}{|z|}e^{\pi|z|^2}.$$

Since h is bounded in the disk $\{|z| < 1\}$, we can write

$$|h(z)| \leqslant C' e^{\pi|z|^2}$$

for $C' = \max(\sup_{|z|<1}|h(z)|, C_2)$. From here the argument proceeds as above in order to obtain (6.5), which, since f is odd, implies $A = 0$, i.e. $f = 0$.

Finally, in the general case, we split f into its even and odd components and apply the triangle inequality. Then the even and odd components of f are each seen to satisfy (6.3) and (6.4) with $a = b = \pi$, and so the work we have just done implies (6.5). □

Observe that the above proof relies crucially on complex-variable techniques; in particular, Theorem 3.3. In this case, it is not clear how these techniques might be generalized into higher dimensions. The Phragmén-Lindelöf theorem seems to be a unique artifact of $\mathbb{C}$. However, if we abandon complex-variable methods for real-variable ones, we may be able to make progress towards a multidimensional Hardy's uncertainty principle. The approach discussed below is due to Escauriaza, Kenig, Ponce, and Vega, and the version of Hardy's uncertainty principle we prove is from [1].

We consider the Schrödinger equation in $\mathbb{R}^d \times [0, \infty)$:

$$i\partial_t u + \Delta u = 0, \qquad u(x, 0) = f(x). \tag{6.10}$$

Taking a Fourier transform in the spatial variable yields

$$i\partial_t \hat{u}(\xi, t) - 4\pi^2|\xi|^2\hat{u}(\xi, t) = 0, \qquad \hat{u}(\xi, 0) = \hat{f}(\xi).$$

which is an ordinary differential equation in t with solution

$$\hat{u}(\xi, t) = e^{-4\pi^2 it|\xi|^2}\hat{f}(\xi).$$

Taking an inverse Fourier transform gives us an explicit form for the solution u to (6.10):

$$\begin{aligned} u(x,t) &= \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} e^{-4\pi^2 it|\xi|^2}\hat{f}(\xi)d\xi \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} e^{-4\pi^2 it|\xi|^2} e^{-2\pi i\xi\cdot y} f(y)dyd\xi \\ &= \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} e^{-2\pi i\xi\cdot(y-x)} e^{-4\pi^2 it|\xi|^2} d\xi\right) f(y)dy \\ &= \int_{\mathbb{R}^d} (4\pi it)^{-d/2} e^{i|x-y|^2/4t} f(y)dy \\ &= (4\pi it)^{-d/2} e^{i|x|^2/4t}\int_{\mathbb{R}^d} e^{-ix\cdot y/2t} e^{i|y|^2/4t} f(y)dy \\ &= (4\pi it)^{-d/2} e^{i|x|^2/4t}\widehat{(e^{i|\cdot|^2/4t}f)}\left(\frac{x}{4\pi t}\right). \end{aligned} \tag{6.11}$$

So for fixed $T > 0$ we have

$$u(x, T) = (4\pi iT)^{-d/2} e^{i|x|^2/4T}\widehat{(e^{i|\cdot|^2/4T}f)}\left(\frac{x}{4\pi T}\right). \tag{6.12}$$

Notice that $u(\cdot, T)$ is expressed in terms of the Fourier transform of $f = u(\cdot, 0)$. We may therefore rewrite our result about a function and its Fourier transform as a result about a solution to the Schrödinger equation and its evolution at time T. In particular, $u(\cdot, T)$ satisfies

$$|u(x, T)| \leqslant Ce^{-4T^2 b|x|^2}$$

if and only if f satisfies

$$|\hat{f}| \leqslant Ce^{-b|x|^2}$$

and so we have the following restatement of Theorem 6.2.

Theorem 6.13 (Hardy's uncertainty principle)**.** *Let $u : \mathbb{R} \times [0, \infty) \to \mathbb{C}$ be a solution to the Schrödinger equation*

$$i\partial_t u + \partial_x^2 u = 0$$

and satisfy

$$|u(x, 0)| \leqslant C_1 e^{-ax^2}, \qquad |u(x, T)| \leqslant C_2 e^{-4T^2 bx^2}$$

for some $C_1, C_2, a, b > 0$ and some fixed $T > 0$. If $ab = \pi^2$, then there is some scalar A such that

$$u(x, 0) = Ae^{-(a+i/4T)x^2}.$$

If $ab > \pi^2$, then $u = 0$.

This relationship between the Fourier transform of a function and its Schrödinger evolution over time is the key idea we will use to generalize Hardy's uncertainty principle to multiple dimensions. It will allow us to forego complex-variable methods, and instead bring results from the theory of PDE to bear on the problem. As such, we will black-box the following lemmas, technical results regarding particular partial differential equations. The proof of Lemma 6.14 can be found in [2], while the proof of Lemma 6.15 can be found in [1].

Lemma 6.14. *Let $u : \mathbb{R}^d \times [0, T] \to \mathbb{C}$, L^2 in the first variable and continuous in the second, be a solution to (6.10). Then, for any $a, b > 0$ with $ab \leqslant 4$, we have*

$$\|e^{\alpha(t)|\cdot|^2} u(\cdot, t)\|_2 \leqslant \|e^{|\cdot|^2/b^2} u(\cdot, 0)\|_2 + \|e^{|\cdot|^2/a^2} u(\cdot, T)\|_2$$

for

$$\alpha(t) = \frac{abRT}{2(at + b(T-t))^2 + 2R^2(at - b(T-t))^2}$$

and R the smallest root of the equation

$$2T(1 + R^2) = abR.$$

Lemma 6.15. *Let $v : \mathbb{R}^d \times [0, \infty) \to \mathbb{C}$ be a solution to the partial differential equation*

$$\partial_t v - i\Delta v + \frac{x}{t - i} \cdot \nabla v = 0. \tag{6.16}$$

If there is some $C > 0$ such that v satisfies

$$\|v(\cdot, t)\|_2 \leqslant C(1 + t^2)^{\frac{d}{4}} \tag{6.17}$$

for all $t \geqslant 0$, then $v = 0$.

We are now in a position to prove the following multidimensional L^2 variant of Hardy's uncertainty principle.

Theorem 6.18. *Let $f : \mathbb{R}^d \to \mathbb{C}$ satisfy*

$$\|e^{a|x|^2} f\|_2 + \|e^{b|\xi|^2} \hat{f}\|_2 \leqslant \infty$$

for some $a, b > 0$. If $ab \geqslant \pi^2$, then $f = 0$.

Proof. By scaling, we may assume $a = 1/4$ and $b = 4\pi^2$. Let u be the solution to the Schrödinger equation (6.10) with initial condition f as given. Then, by Lemma 6.14 and the relationship between $u(\cdot, T)$ and $\hat{f}$ discussed above, there is some constant $C > 0$ such that

$$\|e^{\frac{|\cdot|^2}{4(1+t^2)}} u(t)\|_2 \leqslant C.$$

To show this, it suffices to show that, for $0 < t < 1$,

$$\sup_{[0,1]} \|e^{\frac{|\cdot|^2}{4(1-2t(1-t))}} u(x/t, 1/t - 1)\| < \infty$$

We define

$$u'(x,t) = (it)^{\frac{-d}{2}} e^{\frac{-|x|^2}{4it}} \overline{u}\left(\frac{x}{t}, \frac{1}{t} - t\right). \tag{6.19}$$

Substituting (6.11), we get

$$\begin{aligned} u'(x,t) &= (it)^{\frac{-d}{2}} e^{\frac{-|x|^2}{4it}} \int_{\mathbb{R}^d} e^{\frac{-2\pi i x \cdot \xi}{t}} e^{4\pi^2 i |\xi|^2 (\frac{1}{t} - 1)} \bar{\hat{f}}(\xi) d\xi \\ &= (4\pi i t)^{\frac{-d}{2}} e^{\frac{i|x|^2}{4t}} \int_{\mathbb{R}^d} e^{-2\pi i \frac{x}{4\pi t} \cdot \xi} e^{\frac{i|\xi|^2}{4t}} e^{\frac{-i|\xi|^2}{4}} \bar{\hat{f}}\left(\frac{\xi}{4\pi}\right) d\xi \end{aligned}$$

so that (6.12) gives that u' solves the Schrödinger equation (6.10) with initial conditions

$$u'(x,0) = e^{\frac{-i|\xi|^2}{4}} \bar{\hat{f}}\left(\frac{\xi}{4\pi}\right).$$

This gives

$$\|e^{\frac{|\cdot|^2}{4}} u'(\cdot, 0)\|_2 < \infty.$$

Moreover, (6.19) gives

$$\|e^{\frac{|\cdot|^2}{4}} u'(\cdot, 1)\|_2 < \infty.$$

So, by Lemma 6.14, there is a constant $C > 0$ such that

$$\|e^{\frac{|\cdot|^2}{4(1-2t(1-t))}} u'(x,t)\|_2 < C$$

for all $0 \leqslant t \leqslant 1$, which, changing variables, gives

$$\|e^{\frac{|\cdot|^2}{4(1+t^2)}} u(t)\|_2 < C.$$

for all $0 \leqslant t < \infty$.

Now, let

$$g(x,t) = (t-i)^{\frac{-d}{2}} e^{\frac{-|x|^2}{4i(t-i)}}$$

and define $v = u/g$. By a tedious computation, which we omit, v solves the partial differential equation (6.16). Moreover, v satisfies the bound

$$\begin{aligned} \|v(\cdot, t)\|_2 &= \left\| \frac{u(\cdot, t)}{G(\cdot, t)} \right\|_2 \\ &= (1+t^2)^{\frac{d}{4}} \left\| e^{\frac{|\cdot|^2}{4(1+t^2)}} u(\cdot, t) \right\|_2 \\ &\leqslant C(1+t^2)^{\frac{d}{4}}. \end{aligned}$$

Thus, Lemma 6.15 gives that $v = 0$, and we are done. ∎

We conclude by remarking that a familiar-looking L^∞ form of Hardy's uncertainty principle follows from the L^2 one above. The proof of this may be found in [1].

Corollary 6.20 (Hardy's uncertainty principle). *Let $f : \mathbb{R}^d \to \mathbb{C}$ satisfy*

$$\|e^{a|x|^2} f\|_\infty + \|e^{b|\xi|^2} \hat{f}\|_\infty \leqslant \infty$$

for some $a, b > 0$. If $ab = \pi^2$, then there is some scalar A such that

$$f(x) = Ae^{-a|x|^2}.$$

If $ab > \pi^2$, then $f = 0$.

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