PROOFS OF THE FUNDAMENTAL THEOREM OF ALGEBRA

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ABSTRACT. The fundamental theorem of algebra states that a polynomial of degree $n \geq 1$ with complex coefficients has n complex roots, with possible multiplicity. Throughout this paper, we use f to refer to the polynomial $f : \mathbb{C} \longrightarrow \mathbb{C}$ defined by $f(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$, with $n \geq 1$. We provide several proofs of the fundamental theorem of algebra using topology and complex analysis. We also suppose that $a_0 \neq 0$. Otherwise, 0 itself is a root. The first proof is a topological proof. The next three use complex analysis.

CONTENTS

1. A TOPOLOGICAL PROOF

Let f be the previously defined polynomial. We first show that there exists at least one root of f in the complex numbers. With one root we can use an inductive argument to show that there are n roots, possibly with multiplicity greater than one. Let I be the interval $[0, 1]$. For positive real numbers r and R, let $\gamma : I \longrightarrow \mathbb{C}$ and $\Gamma : I \longrightarrow \mathbb{C}$ be paths in the complex plane defined by $\gamma(x) = re^{2\pi ix}$ and $\Gamma(x) = Re^{2\pi ix}$. These paths are circles of radius r and R. They are homotopic in $\mathbb{C}$, so their images $f \circ \gamma$ and $f \circ \Gamma$ are homotopic because polynomials are continuous. To show that there is at least one root, we show that there are values of r and R such that $f \circ \gamma$ and $f \circ \Gamma$ are not homotopic in $\mathbb{C} \setminus \{0\}$. This will be a contradiction to what we have just shown, showing that 0 must lie in the image of f.

Let $g_1 : I \longrightarrow \mathbb{C}$ and $g_2 : I \longrightarrow \mathbb{C}$ be defined as $g_1(x) = 1$ and $g_2(x) = e^{2\pi inx}$. We show that there exist values of r and R such that both $f \circ \gamma \simeq a_0 g_1$ and $f \circ \Gamma \simeq a_0 g_2$, and, moreover, that there exist homotopies between them that are nowhere zero. We then show that $a_0 g_1$ and $a_0 g_2$ are not homotopic in $\mathbb{C} \setminus \{0\}$, which will give our desired contradiction.

The existence of the first homotopy follows from the continuity of f. Note that $f(0) = a_0$. Then, there exists $\delta > 0$ such that if $|z| < \delta$, then $|f(z) - a_0| < |\frac{a_0}{2}|$. Suppose that $r < \delta$. If z is in the path γ, then $|z| = r$. Then $|f(z) - a_0| < |\frac{a_0}{2}|$

Date: January 10, 2015.

for $z \in \gamma$. By the reverse triangle inequality, $||f(z)| - |a_0|| \leq |f(z) - a_0| < |\frac{a_0}{2}|$. Then, $-|\frac{a_0}{2}| < |f(z)| - |a_0| < |\frac{a_0}{2}|$, and, adding $|a_0|$ to both sides, $|\frac{a_0}{2}| < |f(z)| < |\frac{3a_0}{2}|$. Consider the homotopy between $f \circ \gamma$ and $a_0 g_1$, $h : I \times I \longrightarrow \mathbb{C}$, defined as $h(x,t) = (1-t)(f \circ \gamma)(x) + a_0 t$. If $r < \delta$, then $|h(x,t)| > 0$, as shown below:

$$\begin{aligned} |h(x,t)| &= |(1-t)(f \circ \gamma)(x) + a_0 t| \\ &= |(f \circ \gamma)(x) - t((f \circ \gamma)(x) - a_0)| \\ &\geq |(f \circ \gamma)(x)| - t|(f \circ \gamma)(x) - a_0| \\ &> \left|\frac{a_0}{2}\right| - t\left|\frac{a_0}{2}\right| \\ &\geq 0 \end{aligned}$$

Thus, with r chosen small enough, $f \circ \gamma \simeq a_0 g_1$ with a nowhere zero homotopy.

To show that $f \circ \Gamma \simeq a_0 g_2$ with a nowhere zero homotopy, we first show that $f \circ \Gamma \simeq R^n g_2$ with a nowhere zero homotopy for large enough $|R|$. Then, we show that $R^n g_2 \simeq a_0 g_2$ with a nowhere zero homotopy.

For the first part, we make use of the fact that polynomials eventually behave like their leading term. We can pick some real M such that if $|z| > M$, then $|z|^{n-j} > 2n|a_j|$ where a_j is a coefficient in f and $0 \leq j < n$. Then, rearranging, $\frac{|z^n|}{2n} > |a_j z^j|$. By picking M large enough, we make this inequality true for all $0 \leq j < n$. Then, adding the ineqalities, $\frac{|z^n|}{2} > |a_{n-1}z^{n-1}| + \cdots + |a_1 z| + |a_0| \geq |a_{n-1}z^{n-1} + \cdots + a_1 z + a_0|$. Then, with $z > M$,

$$\begin{aligned} |f(z)| &= \left|z^n + a_{n-1}z^{n-1} + \cdots + a_0\right| \\ &\geq |z^n| - \left|a_{n-1}z^{n-1} + \cdots + a_0\right| \\ &= \left|\frac{z^n}{2}\right| + \left(\left|\frac{z^n}{2}\right| - \left|a_{n-1}z^{n-1} + \cdots + a_0\right|\right) \\ &> \left|\frac{z^n}{2}\right| \end{aligned}$$

Let $|R| > M$, and consider the homotopy $h : I \times I \longrightarrow \mathbb{C}$ defined by $h(x,t) = (1-t)(f \circ \Gamma)(x) + t(R^n e^{2\pi i n x})$. We show that h is nowhere zero:

$$\begin{aligned} |h(x,t)| &= |(f \circ \Gamma)(x) - t((f \circ \Gamma)(x) - R^n e^{2\pi i n x})| \\ &= |(R^n e^{2\pi i n x} + \cdots + a_0) \\ &\qquad - t(a_{n-1}R^{n-1}e^{2\pi i (n-1)x} + \cdots + a_1 R e^{2\pi i x} + a_0)| \\ &> \left|\frac{R^n}{2}\right| - \left|a_{n-1}R^{n-1}e^{2\pi i(n-1)x} + \cdots + a_1 R e^{2\pi i x} + a_0\right| \\ &> 0 \end{aligned}$$

We now show that $R^n g_2 \simeq a_0 g_2$ with a nowhere zero homotopy. Let $R^n = R^* e^{i\theta_1}$, and let $a_0 = a^* e^{i\theta_2}$, where R^* and a^* are positive real numbers. Consider the homotopy $h : I \times I \longrightarrow \mathbb{C}$ defined by $h(x,t) = ((1-t)R^* + ta^*)e^{i((1-t)\theta_1 + t\theta_2)}e^{2\pi i n x}$. The function $(1-t)R^* + ta^*$ has a zero at $t = \frac{R^*}{R^* - a^*}$. If $R^* = a^*$, then the function is constant with no zero. If $R^* < a^*$, then the zero is negative. If $R^* > a^*$, then the zero is greater than 1. In every case, $t \notin I$ when the function is zero. Thus, the homotopy h is never 0, and, by running each homotopy at twice speed, we have that $f \circ \Gamma \simeq a_0 g_2$ with a nowhere zero homotopy.

We now consider the fundamental group $\pi_1(\mathbb{C} \setminus \{0\}, a_0)$, and we show that a_0g_1 and a_0g_2 belong to different elements of the group. We use the fact that the complex plane is a covering space of $\mathbb{C} \setminus \{0\}$ and that the exponential function is a covering map. Also, the paths a_0g_1 and a_0g_2 have unique liftings to paths in $\mathbb{C}$ beginning at a_0. We show that the endpoints of these paths are different. This is sufficient to show that they belong to different equivalence classes in the fundamental group [1].

Consider the covering map $p : \mathbb{C} \longrightarrow \mathbb{C} \setminus \{0\}$ defined as $p(z) = e^{iz}$. Let e_0 be the point in $\mathbb{C}$ such that $a_0 = e^{ie_0}$. We consider liftings of a_0g_1 and a_0g_2 that begin at a_0 in $\mathbb{C}$. They are $\widetilde{a_0g_1} : I \longrightarrow \mathbb{C}$, defined as $\widetilde{a_0g_1}(x) = e_0$, and $\widetilde{a_0g_2} : I \longrightarrow \mathbb{C}$, defined as $\widetilde{a_0g_2}(x) = 2\pi nx + e_0$. As a check, $(p \circ \widetilde{a_0g_1})(x) = e^{ie_0} = a_0 = a_0g_1(x)$, and $(p \circ \widetilde{a_0g_2})(x) = e^{i(2\pi nx+e_0)} = a_0e^{2\pi inx} = a_0g_2(x)$, as required. Also, note that $\widetilde{a_0g_1}(1) = e_0$ and that $\widetilde{a_0g_2}(1) = 2\pi n + e_0$. Because $n \geq 1$, the liftings have different ending points, so the paths a_0g_1 and a_0g_2 are not homotopic in $\mathbb{C} \setminus \{0\}$. It follows from the transitivity of homotopy relations that $f \circ \gamma$ and $f \circ \Gamma$ are also not homotopic in $\mathbb{C} \setminus \{0\}$. This is a contradiction of the fact that they are homotopic. Thus, the image of f must contain 0, so f has a root.

As mentioned previously, the existence of n roots follows from an inductive argument once it is known that at least one root exists. Let $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ be a polynomial of degree n. If z_0 is a root of P, then $P(z) = (z - z_0)Q(z)$, where Q is a polynomial of degree $n-1$. Because Q also has a root, we can continue expanding in this manner until P is written as the product of n linear factors. Thus, P has a total of n roots. This completes our first proof of the fundamental theorem of algebra.

□

2. A Similar Proof Using the Language of Complex Analysis

We now present a proof of the fundamental theorem of algebra that is similar to the above but written in the language of complex analysis. We show that for a large enough circle centered at the origin, the image of the circle will wrap around the origin n times, but, under the assumption that the image of the polynomial lies in $\mathbb{C} \setminus \{0\}$, it does not wrap around the origin at all.

Consider a circle $\Gamma = Re^{2\pi ix}$, with $x \in I$ and R chosen large enough that $|R^n e^{2\pi ixn}| > |a_{n-1}R^{(n-1)}e^{2\pi ix(n-1)} + \cdots + a_0|$. Then, using Rouche's Theorem,

$$\begin{aligned}
\int_{f(\Gamma)} \frac{dz}{z} &= \int_\Gamma \frac{f'(z)}{f(z)} dz \\
&= \int_\Gamma \frac{(z^n)'}{z^n} dz \\
&= \int_\Gamma \frac{n}{z} dz \\
&= \int_0^1 \frac{n}{Re^{2\pi ix}} 2\pi i Re^{2\pi ix} dx \\
&= \int_0^1 2\pi i n dx \\
&= 2\pi i n
\end{aligned}$$

We now show that under the assumption that f is never 0, the integral must be 0. Polynomials are holomorphic functions, and the inverses of holomorphic functions are holomorphic wherever the function is nonzero. If the image of f is contained in $\mathbb{C} \setminus \{0\}$, then $\frac{1}{f}$ is holomorphic everywhere. Also, f' is a polynomial of degree $n-1$, so it is holomorphic everywhere. Then, $\frac{f'}{f}$ is holomorphic everywhere. By Cauchy's integral theorem, the integral of $\frac{f'}{f}$ over a closed path is 0. Then,

$$\begin{aligned}\int_\Gamma \frac{f'(z)}{f(z)}dz &= \int_{f(\Gamma)} \frac{dz}{z} \\ &= 0\end{aligned}$$

This is a contradiction, so the image of f must contain 0. Then, there is at least one zero of f. The existence of n zeros, with possible multiplicity, follows by induction as in the previous proof. □

3. A Proof Using the Maximum Modulus Principle

We now provide a proof of the fundamental theorem of algebra that makes use of the maximum modulus principle, i.e., the modulus of a holomorphic function on a connected, open set has no local maximum unless the function itself is constant. Consider the disk of radius R centered at the origin such that $|f(Re^{i\theta})| > \frac{1}{2}|Re^{i\theta}|^n$ and $\frac{1}{2}|Re^{i\theta}|^n > |a_0|$. There exists an R that satisfies these conditions, as shown in the first proof. Then, for $z \geq |R|$, $|f(z)| > |a_0|$. The function $|f|$ is continuous and the disk is compact, so, by the extreme value theorem, $|f|$ has a minimum on the disk. Call it a, and let α be a point such that $f(\alpha) = a$. For any z on the boundary of the disk, $|f(z)| > |a_0| \geq a$. Thus, the minimum is not achieved on the boundary. It is achieved in the open disk of radius R centered at the origin.

Suppose that $a \neq 0$. Then, f is never 0, so the function $\frac{1}{f}$ is holomorphic. This allows us to apply the maximum modulus principle to it. The open disk of radius R is an open connected subset of $\mathbb{C}$, and $\frac{1}{a}$ is a local maximum of the function $|\frac{1}{f}|$. The maximum modulus principle then implies that $|\frac{1}{f}|$ is constant. However, $|f|$ is then constant, but this is a contradiction because polynomials are not constant. Thus, $a = 0$, and α is a root of f. The existence of n roots follows as it does in the first proof.

4. A Proof Using Liouville's Theorem

Liouville's Theorem, i.e. that a bounded, entire function is constant, provides a proof of the fundamental theorem of algebra that is very similar to the proof using the maximum modulus principle. Consider the same disk of radius R used in the previous proof. As above, there exists some α on the disk such that $|f(\alpha)|$ is a minimum on the disk. We suppose again that $f(\alpha) \neq 0$. For any z such that $|z| \geq |R|$, $|f(z)| > |f(\alpha)|$, so $|\frac{1}{f(\alpha)}| > |\frac{1}{f(z)}|$. Then, $|\frac{1}{f(\alpha)}|$ is a maximum of $|\frac{1}{f}|$ over the whole complex plane. As above, $|\frac{1}{f}|$ is holomorphic on all of $\mathbb{C}$. Then, by Liousville's Theorem, $|\frac{1}{f}|$ is constant, so $|f|$ is constant. This is again a contradiction, so $f(\alpha) = 0$. There are again n roots, following from an induction argument. [2]

□

Acknowledgments. It is a pleasure to thank my mentor, Yiwen Zhou, for suggesting as a project proofs of the fundamental theorem of algebra and for helping me learn some of the basics of topology and complex analysis. Our meetings were always a lot of fun, and he showed me how interesting topology is. Before the REU I did not know anything about it, but it is now one of my favorite subfields of mathematics. I am glad to have had Yiwen as my mentor. It was a great experience.

References

[1] James Munkres, *Topology*, Prentice-Hall, 2000.
[2] Lars Ahlfors, *Complex Analysis*, McGraw-Hill, 1979.