FIXED POINT METHODS IN NONLINEAR ANALYSIS

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Abstract. In this paper we present a selection of fixed point theorems with applications in nonlinear analysis. We begin with the Banach fixed point theorem, which we use to prove the inverse and implicit mapping theorems and the Picard-Lindelöf theorem for Banach spaces. We then prove in succession the fixed point theorems of Brouwer, Schauder, and Schaeffer, after which we conclude with two example applications for semilinear and quasilinear PDE.

Contents

1. Introduction

We seek to provide a sampling of fixed point methods and their applications in analysis. By fixed point, we mean the following:

Definition 1.1. Let X be a topological space and let $T : X \to X$ be a map. A point $x \in X$ is a *fixed point* if $T(x) = x$.

Fixed point theorems guarantee the existence of a fixed point under appropriate conditions on the map T and the set X. Over the course of this paper we present several major fixed point theorems and prove some fundamental results in analysis by reducing nonlinear problems to fixed point problems. We assume knowledge of

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basic real analysis and linear algebra. Familiarity with select results from topology and functional analysis is also helpful, but not required; outlines of the necessary material can be found in the appendix.

Definition 1.2. A normed vector space X is a *Banach space* if the metric space (X, d) is complete, where

$$d(x, y) = \|x - y\| \text{ for all } x, y \in X.$$

The most common example of a Banach space is n-dimensional Euclidean space $\mathbb{R}^n$, where the norm $|\cdot|$ is given by the Euclidean distance. Another example is the space of continuous real-valued functions $C(X)$, where X is the domain and the norm $\|\cdot\|$ is given by

$$\|f\|_{C(X)} = \max_{x \in X} |f(x)| \text{ for } f \in C(X).$$

2. Differential Calculus on Banach Spaces

2.1. Banach Fixed Point Theory.

Definition 2.1. Let (X, d) be a metric space and $T : M \subseteq X \to X$ be a map. We say T is a *contraction* if, for all $x, y \in M$ with $x \neq y$, there exists $k \in (0, 1)$ such that

$$d\big(T(x), T(y)\big) \leqslant k\, d(x, y).$$

Theorem 2.2 (Banach's Fixed Point Theorem). *Let (X, d) be a complete metric space and $M \subseteq X$ be nonempty and closed. If a map $T : M \to M$ is a contraction, then T has a unique fixed point $x \in M$.*

Proof. Note that closed subsets of complete metric spaces are also complete metric spaces, so it is sufficient to consider the case $M = X$. Fix some point $x_0 \in X$ and define a sequence $\{x_n\}$ by $x_{n+1} = T(x_n)$. Then

$$d(x_2, x_1) = d\big((T(x_1), T(x_0)\big) \leqslant k\, d(x_1, x_0)$$

for some $k \in (0, 1)$. Continuing inductively gives

$$d(x_{n+1}, x_n) \leqslant k^n\, d(x_1, x_0).$$

Thus, for $n < m$, we have

$$\begin{aligned} d(x_n, x_m) &\leqslant d(x_n, x_{n+1}) + \cdots + d(x_{m-1}, d_m) \\ &\leqslant (k^n + k^{n+1} + \cdots + k^{m-1})\, d(x_1, x_0) \\ &\leqslant \frac{k^n}{1-k}\, d(x_1, x_0), \end{aligned}$$

where we have made use of the triangle inequality and the properties of sums. Since $|k| < 1$, $k^n/(1-k) \to 0$ as $n \to \infty$. Hence $\{x_n\}$ is Cauchy and has a limit $x \in X$ by completeness. Contraction maps are continuous, so it follows that

$$T(x) = \lim_{n \to \infty} T(x_n) = \lim_{n \to \infty} x_{n+1} = x,$$

as desired.

To see that the fixed point $x \in X$ is unique, suppose there is $x' \neq x$ in X such that x' is also a fixed point. Then $d\big(T(x), T(x')\big) = d(x, x')$ since both $T(x) = x$ and $T(x') = x'$. But $d\big(T(x), T(x')\big) < d(x, x')$ since T is a contraction, a contradiction. $\square$

Remark 2.3. Note that the requirement of $d\big(T(x), T(y)\big) \leqslant k\, d(x,y)$ for $k \in (0,1)$ as opposed to $d\big(T(x), T(y)\big) < d(x,y)$ is essential in the proof of theorem 2.2. When we bound $d(x_n, x_m)$ in the latter case we could set $k = \sup\{d\big(T(x), T(y)\big)/d(x,y) : x, y \in X\}$, but this might give $k = 1$, which breaks the limit.

When applicable, the Banach fixed point theorem is especially useful as it both guarantees the existence *and* uniqueness of a fixed point. Unfortunately, the requirement that our map be a contraction limits its utility. In later sections we will prove fixed point theorems that have relaxed conditions on the mapping but stronger conditions on the domain and codomain. These theorems only assert the existence of a fixed point, however, not the uniqueness.

2.2. **Inverse and Implicit Mapping Theorems.** We now turn our attention to two major theorems in analysis: the inverse and implicit mapping theorems. We first need to generalize our notion of differentiability to a Banach space.

Definition 2.4. Let X and Y be Banach spaces, and $U \subseteq X$ be an open set. A map $f : X \to Y$ is *Fréchet differentiable* at $x \in U$ if there exists a bounded linear operator $Df(x) : X \to Y$ such that

$$\lim_{\|h\|_X \to 0} \frac{\|f(x+h) - f(x) - (Df(x))h\|_Y}{\|h\|_X} = 0.$$

Such a map $Df(x)$ is unique and is known as the *Fréchet derivative* at x. It's worth noting that *bounded operator* does not mean bounded in the general sense, but rather that there exists $M > 0$ such that $\|Df(x)\|_Y \leqslant M\|x\|_X$ for all $x \in X$. In $B(X,Y)$, the space of bounded linear operators from X to Y, we define the norm $\|Df\|$ to be the infimum of all such M. This norm makes $B(X,Y)$ a Banach space. Furthermore, the elements of $B(X,Y)$ are all continuous mappings.

Lemma 2.5. *Let X and Y be Banach spaces and let U be an open convex subset of X. If $f : U \to Y$ is a continuously Fréchet differentiable mapping and $\|Df(x)\|_Y \leqslant M$ for all $x \in U$, then*

$$\|f(x_2) - f(x_1)\|_Y \leqslant M\|x_2 - x_1\|_X$$

for all $x_1, x_2 \in U$.

Proof. Fix $x_1, x_2 \in X$ and define $\gamma : [0,1] \to X$ by $\gamma(t) := (1-t)x_1 + tx_2$. We have

$$(f \circ \gamma)'(t) = Df(\gamma(t))\gamma'(t).$$

Taking norms and applying the fundamental theorem of calculus then gives

$$\begin{aligned}\|f(x_2) - f(x_1)\|_Y &= \left\| \int_0^1 Df(\gamma(t))\gamma'(t)\, dt \right\|_Y = \left\| \int_0^1 Df(\gamma(t))(x_2 - x_1)\, dt \right\|_Y \\ &\leqslant \|x_2 - x_1\|_X \int_0^1 \|Df(\gamma(t))\|_Y\, dt \leqslant M\|x_2 - x_1\|_X.\end{aligned}$$

□

Theorem 2.6 (Inverse Mapping Theorem). *Let X be a Banach space and $U \subseteq X$ be an open set containing a. If $f : U \to X$ is continuously Fréchet differentiable and $Df(a)$ is invertible, then there are open sets V containing a and W containing*

$f(a)$ such that the restriction $f : V \to W$ has a differentiable inverse $f^{-1} : W \to V$. For $y \in W$, the Fréchet derivative of f^{-1} is given by

$$Df^{-1}(y) = \big[Df\big(f^{-1}(y)\big)\big]^{-1}.$$

Proof. We first show that f is injective on an open set V in its domain. This will guarantee that f^{-1} exists on $f(V)$. Set $A = Df(a)$. Since A is invertible, we may associate with each $y \in X$ a map $T : U \to X$ defined by

$$T(x) := x + A^{-1}(y - f(x)).$$

Observe that $f(x) = y$ if and only if x is a fixed point of T. Now

$$DT(x) = I - A^{-1}\, Df(x) = A^{-1}\big(Df(a) - Df(x)\big),$$

which is clearly continuous on U as Df is continuous on U. Since $DT(a) = 0$ and DT is continuous, we can choose $\delta > 0$ such that $V := B(a;\delta) \subseteq U$ and $\|DT(x)\| \leqslant 1/2$ for all $x \in V$. By lemma 2.5, it follows that

$$\|T(x_2) - T(x_1)\| \leqslant \|x_2 - x_1\|/2 \text{ for all } x_1, x_2 \in V. \tag{2.7}$$

Hence T can have at most one fixed point in V. Using our previous observation, for each $y \in X$ there is at most one $x \in V$ such that $f(x) = y$, implying that f is injective on V.

Next we show that $W := f(V)$ is open. Fix a point $y_0 \in W$. Then there exists $x_0 \in V$ such that $f(x_0) = y_0$. Choose $r > 0$ such that $\overline{B(x_0;r)} \subseteq V$, and consider the open ball $B(y_0;\lambda r)$ where $\lambda := (2\|A^{-1}\|)^{-1}$. We see that

$$\|T(x_0) - x_0\| = \|A^{-1}(y - y_0)\| \leqslant \|A^{-1}\| \cdot \|y - y_0\| < r/2.$$

Thus for all $x \in \overline{B(x_0;r)}$, (2.7) implies

$$\begin{aligned}\|T(x) - x_0\| &\leqslant \|T(x) - T(x_0)\| + \|T(x_0) - x_0\| \\ &< \|x - x_0\|/2 + r/2 \leqslant r.\end{aligned}$$

Hence $T(x) \in \overline{B(x_0;r)}$, so (2.7) shows that $T : \overline{B(x_0;r)} \to \overline{B(x_0;r)}$ is a contraction and the Banach fixed point theorem guarantees that there is an $x \in \overline{B(x_0;r)}$ such that $T(x) = x$. For this x, $f(x) = y$, which implies $y \in f(\overline{B(x_0;r)}) \subseteq W$. Our choice of y_0 was arbitrary, so each point in W is contained in some open ball that is a subset W, as desired.

To show that $f^{-1} : W \to V$ is differentiable, pick $y, (y+k) \in W$. Then there exists $x, (x+h) \in V$ such that $y = f(x)$ and $y + k = f(x+h)$. We want to show that

$$\lim_{\|k\| \to 0} \frac{\|f^{-1}(y+k) - f^{-1}(y) - Bk\|}{\|k\|} = 0,$$

where $B := [Df(x)]^{-1}$. Note that $Df(a)$ is invertible for some point a in the open set V, so we may assume $Df(x)$ is invertible for $x \in V$. To prove the limit above, we will first find a relation between $\|k\|$ and $\|h\|$. Using the same map T from earlier,

$$T(x+h) - T(x) = h + A^{-1}\big(f(x) - f(x+h)\big) = h - A^{-1}k.$$

Taking the norm and applying equation (2.7) gives $\|h - A^{-1}k\| \leqslant \|h\|/2$. Hence $\|A^{-1}k\| \geqslant \|h\|/2$ and

$$\|h\| \leqslant 2\|A^{-1}\| \cdot \|k\| = \|k\|/\lambda. \tag{2.8}$$

Thus $\|h\| \to 0$ as $\|k\| \to 0$. Moreover, since

$$f^{-1}(y+k) - f^{-1}(y) - Bk = h - Bk = -B\big(f(x+h) - f(x) - Df(x)h\big),$$

(2.8) implies

$$\frac{\big\|f^{-1}(y+k) - f^{-1}(y) - Bk\big\|}{\|k\|} \leqslant \frac{\|B\|}{\lambda} \cdot \frac{\big\|f(x+h) - f(x) - Df(x)h\big\|}{\|h\|}.$$

The right-hand side converges to 0 since f is differentiable, so the left-hand side also converges to 0. Therefore $f^{-1} : W \to V$ is differentiable and

$$Df^{-1}(y) = B = \big[Df\big(f^{-1}(y)\big)\big]^{-1}.$$

□

Corollary 2.9 (Implicit Mapping Theorem). *Let X and Y be Banach spaces and $f : X \times Y \to Y$ be a continuously Fréchet differentiable mapping. If there is a point $(a, b) \in X \times Y$ such that $f(a, b) = 0$ and the map $D_y f(a, b)$ defined by $y \mapsto Df(a, b)(0, y)$ is an invertible mapping from Y to Y, then there are open sets $A \subseteq X$ containing a and $B \subseteq Y$ containing b and a Fréchet differentiable function $g : A \to B$ with the following property: for each $x \in A$ there is a unique $g(x) \in B$ such that $f(x, g(x)) = 0$.*

Proof. Define the map $F : X \times Y \to X \times Y$ by $F(x, y) = (x, f(x, y))$. Then $DF(a, b)$ is given by

$$DF(a,b) = \begin{bmatrix} I & 0 \\ D_x f(a,b) & D_y f(a,b) \end{bmatrix},$$

which is an invertible map from $X \times Y$ to $X \times Y$. By the inverse mapping theorem, there is an open set $W \subseteq X \times Y$ containing $F(a, b) = (a, 0)$ and an open set in $X \times Y$ containing (a, b), which we may take to be of the form $A \times B$, such that $F : A \times B \to W$ has a differentiable inverse $F^{-1} : W \to A \times B$. Clearly F^{-1} is of the form $F^{-1}(x, y) = (x, k(x, y))$ for some differentiable function k. Let $\pi : X \times Y \to Y$ be defined by $\pi(x, y) := y$. Then $\pi \circ F = f$ and

$$\begin{aligned} f(x, k(x, y)) &= f \circ F^{-1}(x, y) = (\pi \circ F) \circ F^{-1}(x, y) \\ &= \pi \circ (F \circ F^{-1})(x, y) = \pi(x, y) = y. \end{aligned}$$

Thus $f(x, k(x, 0)) = 0$, so we define the map $g : A \to B$ by $g(x) := k(x, 0)$. □

Remark 2.10. Although we used the inverse mapping theorem in our proof of the implicit mapping theorem, it is also possible to separately prove the implicit mapping theorem with fixed points, and then derive the inverse mapping theorem as a corollary.

2.3. **The Picard-Lindelöf Existence Theorem.**

Definition 2.11. Let (X, d_X) and (Y, d_Y) be two metric spaces. A function $f : X \to Y$ is said to be *Lipschitz continuous* on $U \subseteq X$ if there exists a real constant $L \geqslant 0$ such that, for all $x_1, x_2 \in U$,

$$d_Y\big(f(x_1), f(x_2)\big) \leqslant L\, d_X(x_1, x_2)$$

For the next theorem, we adopt the notation $C(I, X)$ to denote the set of continuous X-valued functions with domain I.

Theorem 2.12 (Picard-Lindelöf). *Let X be a Banach space with norm $\|\cdot\|$ and $x_0 \in \mathbb{R}$ and $y_0 \in X$ be given. Consider the initial value problem*

$$y' = f(x, y) \quad y(x_0) = y_0.$$

Suppose $f : \mathbb{R} \times X \to X$ is continuous and bounded on some region

$$Q = \{(x, y) : |x - x_0| \leqslant a, \|y - y_0\| \leqslant b\} \quad (a, b > 0)$$

and that f is Lipschitz continuous with respect to y on Q. Then there exists $\delta > 0$ and a continuous function $\phi : [x_0 - \delta, x_0 + \delta] \to X$ such that $y = \phi(x)$ is a unique solution to the initial value problem.

Proof. Since f is Lipschitz continuous with respect to y on Q, there exists $L > 0$ such that

$$\|f(x, y_1) - f(x, y_2)\|_X \leqslant L\|y_1 - y_2\|_X \text{ for all } (x, y_1), (x, y_2) \in Q;$$

since f is bounded on Q there exists $K > 0$ such that

$$\sup_{(x,y)\in Q} \|f(x, y)\| \leqslant K.$$

We then define $\delta := \min(a, b/K)$ and $I := [x_0 - \delta, x_0 + \delta]$. Now let $Z := C(I, X)$ be the Banach space with norm

$$\|y\|_Z := \max_{x\in I} \|y(x)\|_X.$$

Also consider the norm

$$\|y\|_{Z'} := \max_{x\in I} e^{-L|x-x_0|}\|y(x)\|_X.$$

Observe that, for all $y \in Z$,

$$e^{-L\delta}\|y\|_Z \leqslant \|y\|_{Z'} \leqslant \|y\|_Z$$

so the two norms $\|\cdot\|_Z$ and $\|\cdot\|_{Z'}$ are equivalent—that is to say, they are within a constant of each other. It follows that $(Z, \|\cdot\|_{Z'})$ is also a Banach space. Define $M := \{y \in Z : \|y - y_0\|_Z \leqslant b\}$ and a map $T : M \subseteq (Z, \|\cdot\|_{Z'}) \to (Z, \|\cdot\|_{Z'})$ by

$$T\big(y(x)\big) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt.$$

Our goal is to show that T and M satisfy the conditions of Banach's fixed point theorem.

We claim that M is closed. To see this, let $\{y_n\} \subseteq M$ such that $y_n \to y$ in $(Z, \|\cdot\|_{Z'})$. By the equivalence of norms, $y_n \to y$ in $(Z, \|\cdot\|_Z)$. Since $\{y_n\} \subseteq M$ we have $\|y_n - y_0\|_Z \leqslant b$ for all $n \in \mathbb{N}$ and passing the limit as $n \to \infty$, we find that $\|y - y_0\|_Z \leqslant b$. Thus $y \in M$ and M is closed.

We now show that $T : M \to M$. For $y \in M$ we have

$$\begin{aligned}
\big\|T(y) - y_0\big\|_Z &= \max_{x\in I} \left\| \int_{x_0}^{x} f(t, y(t))\, dt \right\|_X \\
&\leqslant \max_{x\in I} \int_{x_0}^{x} \|f(t, y(t))\|_X\, dt \\
&\leqslant K\delta \leqslant K\left(\frac{b}{K}\right) = b.
\end{aligned}$$

We conclude $T(y) \in M$, as desired.

Finally, we show that T is a contraction. Using the Lipschitz continuity of f in y yields

$$\begin{aligned}\|T(y_1) - T(y_2)\|_{Z'} &= \max_{x\in I} e^{-L|x-x_0|} \left\| \int_{x_0}^{x} \big(f(t, y_1(t)) - f(t, y_2(t))\big)\, dt \right\|_X \\ &\leqslant \max_{x\in I} e^{-L|x-x_0|} \int_{x_0}^{x} L\|y_1(t) - y_2(t)\|_X \, dt\end{aligned}$$

We multiply the integrand by $e^{-L|t-x_0|}e^{L|t-x_0|}$. Since

$$\|y_1(t) - y_2(t)\|_X e^{-L|t-x_0|} \leqslant \|y_1 - y_2\|_{Z'},$$

the inequality above becomes

$$\begin{aligned}\|T(y_1) - T(y_2)\|_{Z'} &\leqslant L\|y_1 - y_2\|_{Z'} \max_{x\in I} e^{-L|x-x_0|} \int_{x_0}^{x} e^{L|t-x_0|}\, dt \\ &= L\|y_1 - y_2\|_{Z'} \max_{x\in I} e^{-L|x-x_0|} \frac{1}{L}\left(e^{L|x-x_0|} - 1\right) \\ &\leqslant \left(1 - e^{-L\delta}\right) \|y_1 - y_2\|_{Z'}.\end{aligned}$$

Therefore T is a contraction on M in $(Z, \|\cdot\|_{Z'})$ with contractive factor $1 - e^{-L\delta}$. Banach's fixed point theorem then implies the existence of a unique fixed point $\phi \in C(I, X)$, which is the unique solution to the initial value problem. □

3. Topological Fixed Point Theory

3.1. Brouwer Fixed Point Theory.

Notation 3.1. Denote the unit ball in $\mathbb{R}^n$ by $B^n := \overline{B(0;1)} = \{x \in \mathbb{R}^n : |x| \leqslant 1\}$ and the unit sphere (the boundary of the unit ball) by $S^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\} = \partial B^n$.

Definition 3.2. Let A be a subset of a topological space X. A *retraction* is a map $r : X \to A$ such that $r(x) = x$ for all $x \in A$. If there exists a retraction from X to A, we say A is a *retract* of X.

Lemma 3.3 (No-Retraction Theorem). *There is no continuous retraction $r : B^n \to S^{n-1}$.*

Intuitively, it is not difficult to see why this lemma holds. If we fix every point on the surface of the sphere, there is no function that continuously "makes room" for every mapped point from the interior of the sphere. Proving the no-retraction theorem for n-dimensional space, however, is not as trivial as it might seem. The most common methods make use of tools far out of the scope of this paper, so we will simply assume lemma 3.3. Proofs using algebraic topology can be found in both [3] and [4].

Theorem 3.4 (Brouwer's Fixed Point Theorem). *Every continuous map $T : B^n \to B^n$ has a fixed point.*

Proof. Suppose there exists a map $T : B^n \to B^n$ with no fixed points. Construct the map $r : B^n \to S^{n-1}$ by extending a ray along the path from x to $T(x)$ and defining $r(x)$ to be the intersection of the ray with the sphere S^{n-1} (see Figure 1). The map r is well-defined since $x \neq T(x)$ for any $x \in B^n$, and continuous since T

is continuous. Moreover, $r(x) = x$ for all $x \in S^{n-1}$, so r is a retraction from B^n to S^{n-1}. But this contradicts lemma 3.3, which says that no such retraction exists. Hence T must have a fixed point. $\square$

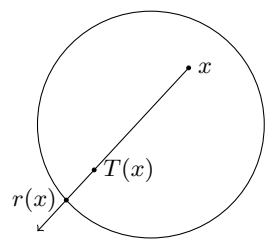

FIGURE 1. The map r for $n = 2$.

Corollary 3.5. *Let K be a nonempty, compact and convex subset of $\mathbb{R}^n$. Every continuous map $T : K \to K$ has a fixed point.*

The general idea behind corollary 3.5 is that K is homeomorphic to the closed unit ball B^n. For a detailed proof, see [1], [5], or [8].

3.2. **Ascoli-Arzelà Theory.**

Definition 3.6. Let $\epsilon > 0$. A subset S of a metric space X is an *ϵ-net* of X if

$$X \subseteq \bigcup_{x \in S} B(x; \epsilon).$$

A metric space X is said to be *totally bounded* if there is a finite ϵ-net for all $\epsilon > 0$.

Definition 3.7. A subset S of a metric space X is *precompact* if $\overline{S}$ is compact.

Definition 3.8. Let X and Y be Banach spaces. A map $f : X \to Y$ is *compact* if $f(S)$ is precompact in Y whenever $S \subseteq X$ is bounded.

We state, but do not prove, the well-known Heine-Borel theorem, which we will use later in our proof of the Ascoli-Arzelà theorem.

Theorem 3.9 (Heine-Borel). *A subset S of a metric space X is compact if and only if it is complete and totally bounded.*

Corollary 3.10. *A totally bounded subset S of a complete metric space X is precompact.*

Proof. If S is totally bounded, then so is its closure $\overline{S}$. A closed subset of a complete metric space is itself a complete metric space, so $\overline{S}$ is compact by theorem 3.9. $\square$

Definition 3.11. A subset $A \subseteq C(X)$ is *equicontinuous* at $x \in X$ if, given $\epsilon > 0$, there exists $\delta_x > 0$ such that

$$d(x, y) < \delta_x \text{ implies } |f(x) - f(y)| < \epsilon \text{ for all } f \in A.$$

If A is equicontinuous at each point $x \in X$, then we simply say that A is equicontinuous.

Theorem 3.12 (Ascoli-Arzelà). *Let X be a compact metric space. If A is an equicontinuous, bounded subset of $C(X)$, then A is precompact.*

Proof. Let $\epsilon > 0$. Since A is equicontinuous, for every $x \in X$ there exists $\delta_x > 0$ such that $|f(x) - f(y)| < \epsilon/4$ for any $f \in A$ and y such that $d(x, y) < \delta_x$. The collection $\{B(x; \delta_x)\}_{x \in X}$ is an open cover of the compact metric space X, so we can extract a finite subcover, which we denote $\{B(x_j; \delta_j)\}$ for $j = 1, \ldots, n$. Thus the equicontinuity of A gives

$$|f(x) - f(x_j)| < \epsilon/4 \text{ for all } x \in B(x_j; \delta_j) \text{ and } f \in A.$$

The set A is bounded, so the set $F := \{f(x_j) : 1 \leqslant j \leqslant n, f \in A\}$ is bounded as well. A bounded subset of $\mathbb{R}$ is totally bounded, so there are points $y_1, \ldots, y_m \in \mathbb{R}$ such that

$$F \subseteq \bigcup_{i=1}^{m} B(y_i; \epsilon/4).$$

For any map $k : \{1, \ldots, n\} \to \{1, \ldots, m\}$, define

$$A_k := \{f \in A : f(x_j) \in B(y_{k(j)}; \epsilon/4), j = 1, \ldots, n\}.$$

Note that there are only finitely many sets A_k since there are a finite number of maps k. Also, each $f \in A$ belongs to one of the sets A_k. If we take $f, g \in A_k$ and $x \in X$, then $x \in B(x_j; \delta_j)$ for some j and

$$\begin{aligned} |f(x) - g(x)| \leqslant |f(x) - f(x_j)| + |f(x_j) - y_{k(j)}| \\ + |y_{k(j)} - g(x_j)| + |g(x_j) - g(x)| < \epsilon. \end{aligned}$$

Hence $\operatorname{diam}(A_k) \leqslant \epsilon$ and A can be covered by finitely many sets of diameter less than ϵ. The set A is totally bounded, so we conclude A is precompact by corollary 3.10. □

3.3. **Schauder Fixed Point Theory.**

Definition 3.13. Let X be a normed vector space and $F = \{x_1, x_2, \ldots, x_n\}$ a finite subset of X. Then $\operatorname{conv}(F)$, the *convex hull* of F, is defined by

$$\operatorname{conv}(F) = \left\{\textstyle\sum_{j=1}^{n} t_j x_j : \sum_{j=1}^{n} t_j = 1, t_j \geqslant 0\right\}.$$

For future applications, we will need a more general definition to handle the case in which F is infinite:

Definition 3.14. Let X be a normed vector space and F a subset of X. The convex hull $\operatorname{conv}(F)$ is the intersection of all convex sets $S \subseteq X$ such that $F \subseteq S$.

Proposition 3.15. *Definitions 3.13 and 3.14 are equivalent for finite sets.*

Lemma 3.16 (Schauder Projection Lemma). *Let K be a compact subset of a normed vector space X, with metric d induced by the norm $\|\cdot\|$. Given $\epsilon > 0$, there exists a finite subset $F \subseteq X$ and a map $P : K \to \operatorname{conv}(F)$ such that $d(P(x), x) < \epsilon$ for all $x \in K$. This map is called the Schauder projection.*

Proof. Take a finite ϵ-net for the compact set K to obtain a set $F = \{x_1, \ldots, x_n\}$. For $i = 1, \ldots, n$, define functions $\phi_i : K \to \mathbb{R}$ by

$$\phi_i(x) := \begin{cases} \epsilon - d(x, x_i) & \text{if } x \in B(x_i; \epsilon) \\ 0 & \text{otherwise.} \end{cases}$$

We see that ϕ_i is strictly positive on $B(x_i;\epsilon)$ and vanishes elsewhere. Therefore $\sum_{i=1}^n \phi_i(x) > 0$ for all $x \in K$. We define the Schauder projection $P : K \to \mathrm{conv}(F)$ by

$$P(x) = \sum_{i=1}^n \frac{\phi_i(x)}{\phi(x)} x_i \quad \text{where} \quad \phi(x) = \sum_{i=1}^n \phi_i(x).$$

The map P is continuous since all the ϕ_i are. Moreover,

$$\begin{aligned} d(P(x), x) &= \left\| \sum_{i=1}^n \frac{\phi_i(x)}{\phi(x)} x_i - \sum_{i=1}^n \frac{\phi_i(x)}{\phi(x)} x \right\| = \left\| \sum_{i=1}^n \frac{\phi_i(x)}{\phi(x)} (x_i - x) \right\| \\ &\leqslant \sum_{i=1}^n \frac{\phi_i(x)}{\phi(x)} \|x_i - x\| < \sum_{i=1}^n \frac{\phi_i(x)}{\phi(x)} \epsilon = \epsilon \end{aligned}$$

because $\phi_i(x) = 0$ if $\|x_i - x\| \geqslant \epsilon$. □

Theorem 3.17 (Schauder's Fixed Point Theorem). *Let X be a Banach space and let $M \subseteq X$ be nonempty, convex, and closed. If $T : M \to M$ is compact, then T has a fixed point.*

Proof. Let K denote the closure of $T(M)$ which, by hypothesis, is compact. For each natural number n, let F_n be a finite $\frac{1}{n}$-net for K and let $P_n : K \to \mathrm{conv}(F_n)$ be the corresponding Schauder projection. The convexity of M implies that $\mathrm{conv}(F_n) \subseteq K$; define $T_n : \mathrm{conv}(F_n) \to \mathrm{conv}(F_n)$ by $T_n := (P_n \circ T)|_{\mathrm{conv}(F_n)}$. Corollary 3.5 guarantees that T_n has fixed points. For each $n \in \mathbb{N}$, we choose one such fixed point of T_n and call it x_n. Since K is compact $\{x_n\}$ has a convergent subsequence, which we denote $\{x_{n'}\}$. This sequence converges to some $x \in K$ as $n' \to \infty$, which we claim is the desired fixed point. From lemma 3.16 we obtain

$$d(T(x), x_{n'}) \leqslant d(T(x), T(x_{n'})) + d(T(x_{n'}), T_{n'}(x_{n'})) \to 0 \quad \text{as} \quad n' \to \infty$$

since T is continuous and $d(T(x_{n'}), T_{n'}(x_{n'})) = d(T(x_{n'}), x_{n'}) < 1/n'$. Thus $\{x_{n'}\}$ converges to both x and $T(x)$. Limits are unique, so $T(x) = x$, as desired. □

In practice, it is often awkward to apply Schauder's fixed point theorem as one needs to find an appropriate set M—and such a set is rarely obvious. This gives rise to an alternative formulation, known as Schaeffer's fixed point theorem, in which we do not have to identify an explicit convex, compact set.

Theorem 3.18 (Schaeffer's Fixed Point Theorem). *Let X be a Banach space and $T : X \to X$ be a continuous and compact mapping. If the set*

$$\{x \in X : x = \lambda T(x) \text{ for some } \lambda \in [0,1]\}$$

is bounded, then T has a fixed point.

Proof. By hypothesis, we can choose a constant M so large that

$$\|x\| < M \text{ if } x = \lambda T(x) \text{ for some } \lambda \in [0,1].$$

Define a retraction $r\colon X \to B(0;M)$ by

$$r(x) = \begin{cases} x & \text{if } \|x\| \leqslant M \\ (M/\|x\|)\, x & \text{if } \|x\| > M \end{cases}$$

and observe that the composition $(r \circ T) : B(0;M) \to B(0;M)$ is compact since T is compact. Let K denote the closed convex hull of $(r \circ T)(B(0;M))$. The set

K is convex by definition, and the compactness of $r \circ T$ implies K is compact. By Schauder's fixed point theorem, there exists a fixed point $x \in K$ of the restriction $(r \circ T)|_K : K \to K$. We claim that x is also a fixed point of T. To show this, it is sufficient to prove that $T(x) \in K$. Suppose not. Then $\|T(x)\| > M$ and

$$x = r(T(x)) = \frac{M}{\|T(x)\|} T(x), \tag{3.19}$$

which implies

$$\|x\| = \left\| \frac{M}{\|T(x)\|} T(x) \right\| = M.$$

On the other hand, $M/\|T(x)\| \in (0,1)$, so our choice of M and (3.19) also imply $\|x\| < M$, a contradiction. □

3.4. **The Cauchy-Peano Existence Theorem.** We now revisit the ODE example from section 2.3. Armed with the Schauder fixed point theorem, we can relax the assumption of Lipschitz continuity to regular continuity.

Theorem 3.20 (Cauchy-Peano). *Let $(x_0, y_0) \in \mathbb{R} \times X$ be given. Consider the initial value problem*

$$y' = f(x, y) \quad y(x_0) = y_0.$$

Suppose $f : Q \subseteq \mathbb{R} \times X \to X$ is continuous and bounded on some region

$$Q = \{(x, y) : |x - x_0| \leqslant a, \|y - y_0\| \leqslant b\} \quad (a, b > 0).$$

Then there exists $\delta > 0$ and a continuous function $\phi : [x_0 - \delta, x_0 + \delta] \to X$ such that $y = \phi(x)$ is a (not necessarily unique) solution to the initial value problem.

Proof. Let $K := \max_{(x,y)\in Q} \|f(x, y)\|_X$ and define $\delta := \min(a, b/K)$. We also define sets

$$I := [x_0 - \delta, x_0 + \delta] \text{ and } M := \{y \in Z : \|y - y_0\|_Z \leqslant b\},$$

where Z is the Banach space $C(I, X)$ with norm $\|y\|_Z = \max_{x\in I} \|y(x)\|_X$. The set M is nonempty, convex, closed, and bounded; if we define the map $T : M \to Z$ by

$$T(y(x)) := y_0 + \int_{x_0}^{x} f(t, y(t))\, dt$$

we have

$$\|T(y) - y_0\|_Z \leqslant \max_{x\in I} \left\| \int_{x_0}^{x} f(t, y(t))\, dt \right\|_X \leqslant \delta K \leqslant b.$$

Thus $T(M) \subseteq M$.

Next we show that T is continuous. Let $\{y_n\} \subseteq M$ be such that $y_n \to y$ in M. Then

$$\begin{aligned} \|T(y_n) - T(y)\|_Z &= \max_{x\in I} \|T(y_n(x)) - T(y(x))\|_X \\ &= \max_{x\in I} \left\| \int_{x_0}^{x} \big[f(t, y_n(t)) - f(t, y(t))\big]\, dt \right\|_X \\ &\leqslant \int_{x_0-\delta}^{x_0+\delta} \|f(t, y_n(t)) - f(t, y(t))\|_X\, dt. \end{aligned}$$

Observe that f is uniformly continuous since f is continuous on a compact interval, so we may pass to the limit as $n \to \infty$ to obtain

$$\lim_{n\to\infty} \|T(y_n) - T(y)\|_Z \leqslant \int_{x_0-\delta}^{x_0+\delta} \lim_{n\to\infty} \|f(t, y_n(t)) - f(t, y(t))\|_X \, dt = 0.$$

Hence $T(y_n) \to T(y)$ in $T(M)$, so T is indeed continuous. $T(S)$ is equicontinuous for every bounded set $S \subseteq M$ because

$$\sup_{y\in S} \|T(y(x_1)) - T(y(x_2))\|_X \leqslant K|x_1 - x_2| \to 0 \text{ as } |x_1 - x_2| \to 0.$$

Moreover, $T(S)$ is bounded since

$$\sup_{y\in S} \|T(y(x))\|_X = \sup_{y\in S} \left\| y_0 + \int_{x_0}^{x} f(t, y(t)), dt \right\|_X \leqslant \|y_0\|_X + b.$$

Thus $T(S)$ is precompact by the Ascoli-Arzelà theorem for each bounded $S \subseteq M$, so T is a compact map. Schauder's fixed point theorem then implies T has a fixed point $\phi \in M$. By our choice of T, the map $\phi : I \to X$ is a continuous solution to our initial value problem. □

4. Example Applications in Nonlinear PDE

4.1. **Some Results in Functional Analysis.** Before beginning our study of partial differential equations, we first need some prerequisite facts about L^p and Sobolev spaces. Readers unfamiliar with L^p and Sobolev spaces should consult the appendix for an outline of their definitions. It should be noted that we restrict our treatment to smooth, bounded subsets $\Omega \subseteq \mathbb{R}^n$, and that all the integrals appearing in this section and the appendix are Lebesgue integrals.

Theorem 4.1 (Dominated Convergence Theorem). *Let $\Omega \subseteq \mathbb{R}^n$ be open, $p \in [1, \infty)$, and $\{f_n\}$ a sequence of measurable functions with domain Ω. Suppose $\{f_n\}$ converges pointwise almost everywhere to f and is dominated by some function $g \in L^p(\Omega)$, i.e., $|f_n(x)| \leqslant g(x)$ for each $n \in \mathbb{N}$ and $x \in \Omega \setminus N$ where N is a set of measure 0. Then each f_n as well as f is in $L^p(\Omega)$ and $f_n \to f$ in L^p.*

Corollary 4.2. *Given $f \in C(\mathbb{R})$ such that $|f(t)| \leqslant a(1 + |t|)$ where $a > 0$, the map $u \mapsto f(u)$ is continuous from $L^2(\Omega)$ to $L^2(\Omega)$.*

Definition 4.3. Define the *gradient* ∇ of a function $f : \mathbb{R}^n \to \mathbb{R}$ to be

$$\nabla f := \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right),$$

provided all the partial derivatives exist.

Definition 4.4. The *Laplacian* Δ of a function $f : \mathbb{R}^n \to \mathbb{R}$ is given by

$$\Delta f := \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2},$$

provided all the second-order unmixed partial derivatives exist.

Proposition 4.5. *Let $g \in H^{-1}(\Omega)$ and $\mu \in [0, \infty)$. Then there exists a unique $v \in H_0^1(\Omega)$ such that $-\Delta v + \mu v = g$ in $\mathcal{D}'(\Omega)$ and this v is the unique solution to the variational problem*

$$\int_\Omega \nabla v \cdot \nabla w \, dx + \mu \int_\Omega vw \, dx = \langle g, w \rangle \text{ for all } w \in H_0^1(\Omega).$$

Furthermore, the map $g \mapsto v$ is continuous from $H^{-1}(\Omega)$ to $H_0^1(\Omega)$.

This is a special case of the Lax-Milgram theorem; consult [2] for a proof.

Corollary 4.6. *The map $g \mapsto (-\Delta + \mu I_d)^{-1} g = v$ is continuous from $L^2(\Omega)$ to $H_0^1(\Omega)$, i.e.,*

$$\|v\|_{H_0^1(\Omega)} \leqslant C\|g\|_{L^2(\Omega)}$$

where C is a constant dependent on Ω.

Proof. Simply note that $L^2(\Omega)$ continuously embeds into $H^{-1}(\Omega)$ and apply proposition 4.5. □

Theorem 4.7 (Poincaré's Inequality). *For $p \in [1, \infty)$ and $u \in W_0^{1,p}$, there exists a constant C dependent on Ω and p such that*

$$\|u\|_{L^p(\Omega)} \leqslant C\|\nabla u\|_{L^p(\Omega)}.$$

Corollary 4.8. *A norm can be defined on $H_0^1(\Omega)$ by*

$$\|u\|_{H_0^1(\Omega)} = \|\nabla u\|_{L^2(\Omega)}$$

for $u \in H_0^1(\Omega)$. This norm is equivalent to the standard norm on $H^1(\Omega)$ as described in proposition A.9.

We finally have all the tools we need to solve some PDEs.

4.2. **Semilinear Elliptic Equations.** In this section we study semilinear partial differential equations of the form

$$\begin{aligned} -\Delta u &= f(u) \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{4.9}$$

where $\Omega \subseteq \mathbb{R}^n$ is open, bounded and smooth and $f : \mathbb{R} \to \mathbb{R}$ is a given function.

Theorem 4.10. *Let $\Omega \subseteq \mathbb{R}^n$ be an open, bounded and smooth domain and $f \in C(\mathbb{R})$ be a given bounded function. Then the boundary value problem (4.9) has a weak solution $u \in H_0^1(\Omega)$, i.e., the following formulation holds:*

$$\int_\Omega \nabla u \cdot \nabla \phi \, dx = \int_\Omega f(u)\phi \, dx.$$

Sketch of proof. Define a map $T : L^2(\Omega) \to L^2(\Omega)$ by $u \mapsto (-\Delta)^{-1}(f(u))$. Our strategy is to show that T satisfies the hypotheses of Schauder's fixed point theorem, which will then yield the desired weak solution.

Step 1: T is continuous. Corollary 4.2 shows that $u \mapsto f(u)$ is continuous from $L^2(\Omega)$ to itself; corollary 4.6 shows that $(-\Delta)^{-1}$ is continuous from $L^2(\Omega)$ into $H_0^1(\Omega)$, which is continuously embedded in $L^2(\Omega)$ by proposition A.8.

Step 2: Find a closed non-empty bounded convex set M such that $T : M \to M$. Given $u \in L^2(\Omega)$, $T(u)$ satisfies

$$\int_\Omega \nabla T(u) \cdot \nabla T(u) \, dx = \int_\Omega f(u)T(u) \, dx \leqslant a|\Omega| \|T(u)\|_{L^2(\Omega)} \tag{4.11}$$

by the Cauchy-Schwarz inequality. Using Poincaré's inequality then gives

$$\|T(u)\|^2_{L^2(\Omega)} \leqslant C\|\nabla T(u)\|^2_{L^2(\Omega)} \leqslant a|\Omega|C\|T(u)\|_{L^2(\Omega)}$$

for some constant C. Set $r = a|\Omega|C$ and choose $M := \{u \in L^2(\Omega) : \|u\|_{L^2(\Omega)} \leqslant r\}$. Hence $T : M \to M$.

Step 3: T is compact. Using Poincaré's inequality on the right-hand side of (4.11), we obtain

$$\|\nabla T(u)\|^2_{L^2(\Omega)} \leqslant K\|\nabla T(u)\|_{L^2(\Omega)}$$

for some constant K. Thus $\|\nabla Tu\|_{L^2(\Omega)} \leqslant K$, which implies Tu is bounded in $H^1(\Omega)$ by corollary 4.8, and since the embedding of $H^1(\Omega)$ into $L^2(\Omega)$ is compact, T is compact.

Step 4: Apply Schauder's fixed point theorem to conclude T has a fixed point $u \in M$. By our choice of solution operator, this u lies in the Sobolev space $H^1_0(\Omega)$. □

4.3. **Quasilinear Elliptic Equations.** Lastly, we consider a quasilinear partial differential equation of the form

$$\begin{aligned} -\Delta u + g(\nabla u) + \mu u &= 0 \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{4.12}$$

where $\Omega \subseteq \mathbb{R}^n$ is an open, bounded and smooth domain and $g : \mathbb{R}^n \to \mathbb{R}$ is smooth and Lipschitz continuous.

Theorem 4.13. *Let $\Omega \subseteq \mathbb{R}^n$ be open, bounded and smooth and $g : \mathbb{R}^n \to \mathbb{R}$ be smooth and Lipschitz continuous. If $\mu > 0$ is sufficiently large, then there exists a function $u \in H^2(\Omega) \cap H^1_0(\Omega)$ such that u is a weak solution of the boundary value problem (4.12).*

Sketch of proof. Given $u \in H^1_0(\Omega)$, set $f(u) := -g(\nabla u)$. Since g is Lipschitz continuous, it can be shown to satisfy the growth condition

$$|g(t)| \leqslant C(1 + |t|)$$

for some constant C and all $t \in \mathbb{R}^n$. Therefore $f(u) \in L^2(\Omega)$. By proposition 4.5, there exists a $w \in H^1_0(\Omega)$ that is a weak solution of the linear problem

$$\begin{aligned} -\Delta w + \mu w &= f(u) \quad \text{in } \Omega \\ w &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{4.14}$$

By the elliptic regularity theorem (see [2]), we have $w \in H^2(\Omega)$ and the estimate

$$\|w\|_{H^2(\Omega)} \leqslant K\|f\|_{L^2(\Omega)}$$

for some constant K. We now write $T(u) = w$ whenever w is derived from u via our definition of f and (4.14). Our strategy is to show that T satisfies the hypotheses of Schaeffer's fixed point theorem, which will yield our desired weak solution.

Step 1: $T : H^1_0(\Omega) \to H^1_0(\Omega)$ is continuous and compact. Let $u_n \to u$ in $H^1_0(\Omega)$. By our growth condition we have

$$\|T(u)\|_{H^2(\Omega)} \leqslant L(\|u\|_{H^1_0(\Omega)} + 1),$$

for some constant L, so it follows that

$$\sup_{n\in\mathbb{N}} \|w_n\|_{H^2(\Omega)} < \infty,$$

where $w_n := T(u_n)$. Thus there is a subsequence $\{w_{n'}\}$ and a function $w \in H^1_0(\Omega)$ such that $w_{n'} \to w$ in $H^1_0(\Omega)$. Now

$$\int_\Omega \nabla w_{n'} \cdot \nabla v \, dx + \mu \int_\Omega w_{n'} v \, dx = -\int_\Omega g(\nabla u_{n'}) v \, dx$$

for each $v \in H_0^1(\Omega)$. Consequently using the continuity imposed by our growth condition yields

$$\int_\Omega \nabla w \cdot \nabla v \, dx + \mu \int_\Omega wv \, dx = -\int_\Omega g(\nabla u) v \, dx$$

for each $v \in H_0^1(\Omega)$. Thus $w = T(u)$.

We have shown $T(u_n) \to T(u)$ in $H_0^1(\Omega)$, so T is continuous. A similar argument shows that T is compact, since if $\{u_n\}$ is bounded in $H_0^1(\Omega)$, the growth condition asserts that $\{T(u_n)\}$ is bounded in $H^2(\Omega)$. The space $H^2(\Omega)$ embeds compactly into $H_0^1(\Omega)$, so $\{T(u_n)\}$ possesses a convergent subsequence in $H_0^1(\Omega)$.

Step 2: For sufficiently large $\mu > 0$, the set

$$M := \{u \in H_0^1(\Omega) : u = \lambda T(u) \text{ for some } \lambda \in [0,1]\}$$

is bounded in $H_0^1(\Omega)$. Assume $u \in H_0^1(\Omega)$ and $u = \lambda T(u)$ for some $\lambda \in [0,1]$. Then $u/\lambda = T(u)$; or, in other words, $u \in H^2(\Omega) \cap H_0^1(\Omega)$ and

$$-\Delta u + \mu u = -\lambda g(\nabla u) \text{ almost everywhere in } \Omega.$$

Multiply this identity by u and integrate over Ω to compute

$$\int_\Omega |\nabla u|^2 \, dx + \mu \int_\Omega |u|^2 \, dx = -\int_\Omega \lambda g(\nabla u) u \, dx \leqslant \int_\Omega C(|\nabla u| + 1)|u| \, dx.$$

Applying the inequality $ab \leqslant a^2/(2\epsilon) + \epsilon b^2/2$ for an appropriate choice of $\epsilon > 0$ to the right-hand side, we see that

$$\int_\Omega |\nabla u|^2 \, dx + \mu \int_\Omega |u|^2 \, dx \leqslant \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx + B \int_\Omega (|u|^2 + 1) \, dx$$

for some constant B. We then subtract terms to obtain

$$\frac{1}{2} \int_\Omega |\nabla u|^2 \, dx + (\mu - B) \int_\Omega |u|^2 \, dx \leqslant B|\Omega|$$

As noted in proposition A.9, the left-hand side is of the same form as the norm on $H_0^1(\Omega)$. Thus if $\mu > 0$ is large, we have $\|u\|_{H_0^1(\Omega)} \leqslant R$, for some constant R that does not depend on $\lambda \in [0,1]$.

Step 3: Applying Schaeffer's fixed point theorem in the space $X = H_0^1(\Omega)$, we conclude T has a fixed point $u \in H_0^1(\Omega) \cap H^2(\Omega)$, which in turn solves our boundary value problem (4.12). □

Appendix: L^p and Sobolev Spaces

Definition A.1. Let $p \in (0, \infty)$ and $\Omega \subseteq \mathbb{R}^n$. If $f : \Omega \to \mathbb{R}$ is a measurable function, then we define

$$\|f\|_{L^p(\Omega)} := \left(\int_\Omega |f(x)|^p \, dx \right)^{1/p}.$$

The space $L^p(\Omega)$ is defined to be

$$L^p(\Omega) := \{f : \Omega \to \mathbb{R} : \|f\|_{L^p(\Omega)} < \infty\}.$$

Definition A.2. The *support* of a real-valued function f with domain Ω is given by

$$\operatorname{supp}(f) := \overline{\{x \in \Omega : f(x) \neq 0\}}.$$

A function f is said to have *compact support* if its support is compact; for $\Omega \subseteq \mathbb{R}^n$, the support is compact if and only if it is bounded.

Notation A.3. Let $C^\infty(\Omega)$ by the set of infinitely differentiable real-valued functions with domain Ω. Denote the set of all C^∞ functions with compact support by $C_0^\infty(\Omega)$.

Definition A.4. Let $\Omega \subseteq \mathbb{R}^n$ be open and connected. Let $\alpha := (\alpha_1, \ldots, \alpha_n)$ be a multi-index[1]. For any $\phi \in C^\infty(\mathbb{R}^n)$, define the differential operator D^α by

$$D^\alpha \phi := \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} \phi.$$

Now let $f : \Omega \to \mathbb{R}$ be given. Then $g : \Omega \to \mathbb{R}$ is the *α-weak derivative* of f for some multi-index α, if for each $\phi \in C_0^\infty(\Omega)$, the following formula holds:

$$\int_\Omega f(x) D^\alpha \phi(x)\, dx = (-1)^{|\alpha|} \int_\Omega g(x)\phi(x)\, dx$$

where $|\alpha| = |\alpha_1| + \cdots + |\alpha_n|$.

Definition A.5. A *Sobolev space* $W^{k,p}(\Omega)$ consists of all locally integrable functions[2] $u : \Omega \to \mathbb{R}$ such that for each multi-index α with $|\alpha| \leqslant k$, $D^\alpha u$ exists in the weak sense and belongs to $L^p(\Omega)$.

Proposition A.6. *The Sobolev space $W^{k,p}(\Omega)$ is a Banach space.*

Notations A.7. Denote the closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$ by $W_0^{k,p}(\Omega)$. For $p = 2$ we usually write $W^{k,p}(\Omega) = H^k(\Omega)$. The space of bounded linear maps $f : H_0^1(\Omega) \to \mathbb{R}$ is denoted by $H^{-1}(\Omega)$.

Proposition A.8. *The Sobolev space $H_0^1(\Omega)$ embeds continuously into $L^2(\Omega)$.*

Proposition A.9. *$H^1(\Omega)$ is a Hilbert space with inner product*

$$\langle f, g \rangle = \int_\Omega \nabla f \cdot \nabla g\, dx + \int_\Omega fg\, dx$$

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[1] an n-tuple of non-negative integers
[2] integrable on any compact subset of its domain