KOLMOGOROV-BARZDIN AND SPACIAL REALIZATIONS OF EXPANDER GRAPHS

MARIA SMITH

Abstract. One application of graph theory is to analyze connectivity of neurons and axons in the brain. We begin with basic definitions from graph theory including the Cheeger constant, a measure of connectivity of a graph. In Section 2, we will examine expander graphs, which are very sparse yet highly connected. Surprisingly, not only do expander graphs exist, but most random graphs have the expander property. Section 3 discusses the Kolmogorov-Barzdin realization of graphs in a sphere in $\mathbb{R}^3$. This can be used to model neurons and axons in the brain and yields the smallest possible radius for the sphere for any graph with the expander property.

Contents

1. Graphs

We begin with some basic definitions of graphs.

Definition 1.1. A *graph* G consists of sets $V(G)$ and $E(G)$, where $V(G)$ is a non-empty finite set of vertices and $E(G)$ is a finite set of ordered pairs called edges. A *weighted graph* assigns some value $x \in \mathbb{R}$ to each edge $(v, w) \in E(G)$.

Remark We do not require $e, f \in E(G)$ to be distinct and we allow $(v, v) \in E(G)$ if $v \in V(G)$. Thus, we can consider *multigraphs*, where several edges join two vertices and *loops*, where an edge joins a vertex to itself.

Definition 1.2. For a graph G, vertices $v, w \in V(G)$ are *adjacent* if $(v, w) \in E(G)$ and are *incident* to the edge (v, w). Two edges are incident if they share a vertex.

Definition 1.3. The degree of a vertex v, written $\deg(v)$, is the number of edges incident with v. The degree of a graph, $\deg(G)$, is defined as follows:

$$\deg(G) = \max_{v \in V(G)} \deg(v).$$

Definition 1.4. For $A \subset V(G)$ the *boundary* of A, denoted ∂A, is the set of $(v, w) \in E(G)$ such that $v \in A$ and $w \notin A$ or $v \notin A$ and $w \in A$.

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Definition 1.5. Let G be a graph. The *Cheeger constant* $C(G)$ is defined to be

$$C(G) = \min_{\substack{A \subset V \\ |A| \leq \frac{|V|}{2}}} \frac{|\partial A|}{|A|}.$$

Definition 1.6. Two graphs G_1 and G_2 are *disjoint* if $V(G_1) \cap V(G_2) = \emptyset$ and $E(G_1) \cap E(G_2) = \emptyset$.

Definition 1.7. A graph G is *connected* if it cannot be written as the union of disjoint graphs. If a graph is not connected it is *disconnected*.

Theorem 1.8. *A graph G is connected if and only if $C(G) > 0$.*

Proof. We will prove the equivalent statement that $C(G) = 0$ if and only if G is disconnected.

If $C(G) > 0$, then for all $A \subset V(G)$, $|\partial A| \geq 1$. This means there exists some $e \in E(G)$ originating at some $v \in A$ and terminating at some $w \notin A$, so G cannot be written as the union of disjoint graphs. Conversely, if G is connected then it cannot be written as the union of disjoint graphs. Then if $G = G_1 \cup G_2$, assume without loss of generality $|G_1| \leq \frac{|V(G)|}{2}$. Then there exists some $(v, w) \in E(G)$ such that $v \in V(G_1)$ and $w \in E(G_2)$. Thus $|\partial G_1| \geq 1$ so $C(G) > 0$. □

2. Expander Graphs

A graph G is said to have the expander property if it is sparse but also highly connected. This means for any $A \subset V(G)$, $|\partial A|$ is large, meaning that disjoint sets of vertices cannot be disconnected without removing many edges. The following theorem proves the existence of such graphs.

Theorem 2.1. *Let G_n denote a graph with n vertices. There exists a sequence of graphs $G_1, G_2, G_3, \dots$ such that $\deg(G_n) \leq B$ for some bound B and $C(G_n) > 0$ for all n.*

Proof. Construct a random graph G_n by assigning each vertex a label $1, \dots, n$ and randomly selecting permutations $\pi_1, \pi_2, \dots, \pi_N \in \Sigma_n$ from a uniform distribution, where $N = \frac{1}{2} \deg G_n$. Vertices $i, j \leq n$ are adjacent if there exists some permutation π_k that permutes i to j or j to i. There are $(n!)^N$ ways to choose $\pi_1, \pi_2, \dots, \pi_N$ permutations.

We want to show that for all $A \subset V(G_n)$ where $|A| \leq \frac{|V|}{2}$,

$$\left| \bigcup_{i=1}^{\infty} \pi_i(A) \right| \geq \frac{3}{2} |A|.$$

Then it would follow that $|\partial A| \geq \frac{1}{2} |A|$ and $C(G_n) \geq \frac{1}{2}$. Thus, we want to rule out the existence of $A, B \subset V(G_n)$ such that $|A| \leq \frac{|V(G_n)|}{2}$, $|B| \leq \frac{3}{2} |A|$, and $\pi_i(A) \subset B$ for all $i \leq N$.

Fix $A, B \subset V(G_n)$ such that $|A| \leq \frac{|V(G_n)|}{2}$, $|B| \leq \frac{3}{2} |A|$, and $\pi_i(A) \subset B$ for all $i \leq N$ and let $a = |A|$ and $b = |B|$. Under these conditions there are

$$[b \cdot (b-1) \cdots (b-a+1) \cdot (n-a)!]^N = \left(\frac{b!(n-a)!}{(b-a)!} \right)^N$$

ways to choose $\pi_1, \pi_2, \ldots, \pi_N$.

Set

$$R(a) = \binom{n}{a}\binom{n}{b}\left(\frac{b!(n-a)!}{(b-a)!}\right)^N.$$

Note that the number of tuples $(\pi_1, \ldots, \pi_N)$ that satisfy the supposed conditions is bounded above by $\sum_a R(a)$. We want to show that $R(a)$ is monotonically decreasing.

Case 1 If a is even, then $\lfloor \frac{3}{2}(a+1) \rfloor = b+1$ so

$$\begin{aligned}\frac{R(a)}{R(a+1)} &= \frac{\binom{n}{a}}{\binom{n}{a+1}} \cdot \frac{\binom{n}{b}}{\binom{n}{b+1}} \cdot \left(\frac{b!(n-a)!}{(b-a)!} \cdot \frac{(b-a)!}{(b+1)!(n-a-1)!}\right)^N \\ &= \frac{a+1}{n-a} \cdot \frac{b+1}{n-b} \cdot \left(\frac{n-a}{b+1}\right)^N \\ &> 1,\end{aligned}$$

provided $N \geq 4$ and $a < \frac{n}{3}$.

Case 2 If a is odd, then $\lfloor \frac{3}{2}(a+1) \rfloor = b+2$. We have

$$\begin{aligned}\frac{R(a)}{R(a+1)} &= \frac{\binom{n}{a}}{\binom{n}{a+1}} \cdot \frac{\binom{n}{b}}{\binom{n}{b+2}} \cdot \left(\frac{b!(n-a)!}{(b-a)!} \cdot \frac{(b-a+1)!}{(b+2)!(n-a-1)!}\right)^N \\ &= \frac{a+1}{n-a} \cdot \frac{(b+2)(b+1)}{(n-b)(n-b-1)} \cdot \left(\frac{(n-a)(b-a+1)}{(b+2)(b+1)}\right)^N \\ &> 1,\end{aligned}$$

if $N \geq 5$ and $a < \frac{n}{3}$.

In both cases, when $N \geq 5$ and $a < \frac{n}{3}$, $R(a)$ is monotonically decreasing. Therefore, if $\deg(G_n) = 10$, then $R(a) \leq R(1)$.

When $\frac{n}{3} \leq a \leq \frac{n}{2}$, $R(a)$ is not necessarily monotonically decreasing. Since $\binom{n}{a}$ and $\binom{n}{b}$ are the number of subsets of $V(G_n)$ with size a and size b respectively and 2^n is the number of all possible subsets, we have

$$\binom{n}{a} \cdot \binom{n}{b} \leq 2^n \cdot 2^n.$$

Furthermore, on the interval $\frac{n}{3} \leq a \leq \frac{n}{2}$, we know $\frac{n}{2} \leq b \leq \frac{3n}{4}$ so

$$\frac{b!(n-a)!}{(b-a)!} \leq \frac{\lfloor \frac{3}{4}n \rfloor! \lfloor \frac{2}{3}n \rfloor!}{\lfloor \frac{1}{6}n \rfloor!}.$$

We are ready to calculate the probability of picking permutations $\pi_1, \cdots, \pi_N$ such that $\pi_i(A) \subset B$ for all $i \leq N$:

$$\begin{aligned}\frac{1}{(n!)^N}\sum_{a=1}^{\frac{n}{2}} R(a) &\leq \frac{1}{(n!)^N}\left[\frac{n}{3} \cdot R(1) + \frac{n}{6} \cdot 2^{2n} \cdot \left(\frac{\lfloor \frac{3}{4}n \rfloor! \lfloor \frac{2}{3}n \rfloor!}{\lfloor \frac{1}{6}n \rfloor!}\right)^N\right] \\ &= \frac{1}{(n!)^N}\left[\frac{n^3}{3}[(n-1)!]^N + \frac{n}{6} \cdot 2^{2n} \cdot \left(\frac{\lfloor \frac{3}{4}n \rfloor! \lfloor \frac{2}{3}n \rfloor!}{\lfloor \frac{1}{6}n \rfloor!}\right)^N\right].\end{aligned}$$

Thus, if $deg(G_n) = 10$, then

$$\lim_{n\to\infty} \frac{1}{(n!)^N} \sum_{a=1}^{\frac{n}{2}} R(a) = 0.$$

Since we can construct all G_n so that the degree is bounded as such, as n approaches infinity, the probability that for all $A \subset V(G_n)$, $C(G_n) \geq \frac{1}{2}$ goes to 1. $\square$

Not only do expander graphs exist, but almost all random graphs have the expander property as the number of vertices goes to infinity.

3. Realizations of a Graph in $\mathbb{R}^3$

In 1967 Kolmogorov and Barzdin published a paper *On the Realization of Networks in Three-Dimensional Space.* The motivation for this paper was the biological observation that neurons were arranged on the outer layer of the brain and the axons run through the inside of the brain connecting neurons. By interpreting the neurons as vertices and the axons as edges, we will see this is the optimal and most efficient arrangement of neurons and axons in the brain through Kolmogorov and Barzdin's methodology of realizing a "thick" graph in $\mathbb{R}^3$.

Definition 3.1. A weighted graph G is realized in $\mathbb{R}^3$ if the following criteria are satisfied:

(1) There exists an injective function $\varphi : V(G) \to \mathbb{R}^3$ such that every $v \in V(G)$ is sent to a point $\varphi v \in \mathbb{R}^3$.
(2) For every edge $e = (v, w) \in E(G)$ there exists some continuous curve K_e connecting the points φv and φw.
(3) The distance between any two distinct φv and φw is greater than or equal to 1.
(4) If $e, e' \in E(G)$ are non-incident edges, then the distance between K_e and $K_{e'}$ is greater than or equal to 1.
(5) If $e, e' \in E(G)$ are both incident to $v \in V(G)$, then the distance between K_e and $K_{e'}$ outside a neighborhood of radius 1 around φv is greater than or equal to 1.

The requirement that vertices and edges must be converted into points and curves that are sufficiently far apart means we can think of the points in $\mathbb{R}^3$ as small balls with radius $\frac{1}{2}$ and the curves in $\mathbb{R}^3$ as flexible tubes where each tube has radius $\frac{1}{2}$ and cannot intersect itself or other curves. We need one more definition before we are ready to prove the next theorem.

Definition 3.2. For a graph G, edges $e = (v, w) \in E(G)$ and $e' = (v', w') \in E(G)$ are *related* if the points φv and $\varphi v'$ have identical abscissas or the points φw and $\varphi w'$ have identical ordinates.

Theorem 3.3. *Any weighted graph G with n vertices can be realized in a sphere of radius R such that $R \leq C\sqrt{n}$, where C is a constant not dependent on n.*

Proof. We will first prove the case for a weighted graph G with n vertices and where the number of edges originating at each vertex is no more than two by showing G can be realized in a parallelepiped in $\mathbb{R}^3$ $ABCDA'B'C'D'$ given by the following coordinates:

$$\begin{aligned} A &= (0,0,0) & A' &= (0,0,8\sqrt{n}) \\ B &= (4\sqrt{n},0,0) & B' &= (4\sqrt{n},0,8\sqrt{n}) \\ C &= (4\sqrt{n},2\sqrt{n},0) & C' &= (4\sqrt{n},2\sqrt{n},8\sqrt{n}) \\ D &= (0,2\sqrt{n},0) & D' &= (0,2\sqrt{n},8\sqrt{n}). \end{aligned}$$

If we can realize some graph with n vertices in $\mathbb{R}^3$ and $m < n$, then it follows we can realize a graph with m vertices in $\mathbb{R}^3$, so without loss of generality, assume $\sqrt{n} \in \mathbb{Z}$. Let U be the set of points in $ABCD$ with the coordinates $(4i+2, 2j, 0)$ for $i = 0, 1, \dots, \sqrt{n}-1$ and $j = 0, 1, \dots, \sqrt{n}-1$.

Fix an edge $e = (v, w)$ and assign points φv and φw on U such that

$$\varphi v = (4i+2, 2j, 0)$$

and

$$\varphi w = (4i'+2, 2j', 0).$$

Pick an integer ζ such that $1 \le \zeta \le 4\sqrt{n}$ and let

$$\tau = \begin{cases} 1 & \text{if weight of } e = x_1 \\ -1 & \text{if weight of } e = x_2. \end{cases}$$

Construct a polygonal line $K_e(\zeta) = (\varphi v)abcdefgh(\varphi w)$ from the following coordinates, as shown in Figure 1:

$$\begin{aligned} a &= (4i+2, 2j, 2\zeta-1) \\ b &= (4i+2, 2j+1, 2\zeta-1) \\ c &= (4i, 2j+1, 2\zeta-1) \\ d &= (4i, 2j'+1, 2\zeta-1) \\ e &= (4i, 2j'+1, 2\zeta) \\ f &= (4i'+2+\tau, 2j'+1, 2\zeta) \\ g &= (4i'+2+\tau, 2j', 2\zeta) \\ h &= (4i'+2+\tau, 2j', 0). \end{aligned}$$

Note that the number of edges related to e is at most $2(\sqrt{n}+\sqrt{n}) = 4\sqrt{n}$. Now for every edge in $E(G)$, assign a polygonal line similarly, and for related edges, choose a distinct ζ.

To show this assignment of points and polygonal lines to vertices and edges satisfies a realization of G in $\mathbb{R}^3$, we will show it satisfies all five parts of Definition 3.1.

(1) Since $|U| = n$, for all $v, v' \in V(G)$ such that $v \neq v'$, we can pick $\varphi v \neq \varphi v'$.
(2) It is clear that $K_e(\zeta)$ is continuous. To show that $K_e(\zeta)$ does not intersect itself, we only need to examine the case where $j = j'$, as all other cases are trivial. If $j = j'$, then $c = d$ and the segment ce ensures that ef does not intersect bc.
(3) Distinct φv and φw have coordinates of integer values, so $d(\varphi v, \varphi w) \ge 1$. Furthermore, since φv and φw are in U, $d(\varphi v, \varphi w) \ge 2$.

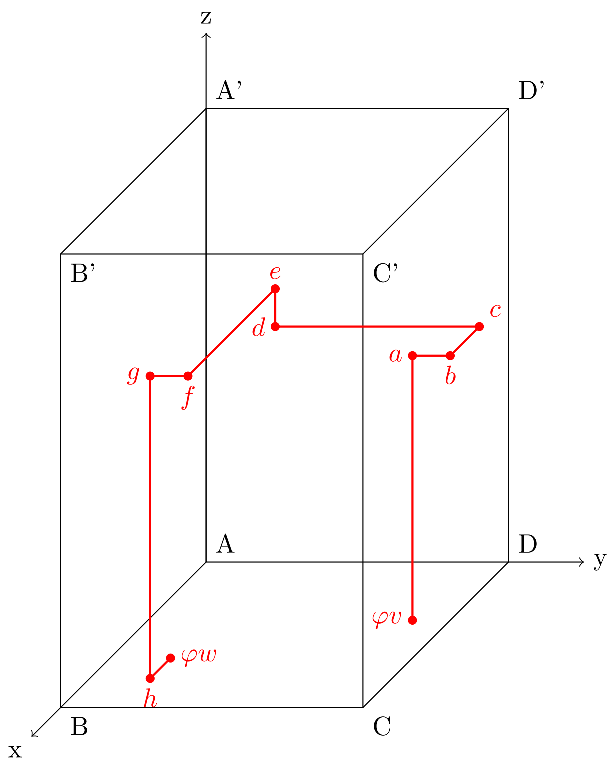
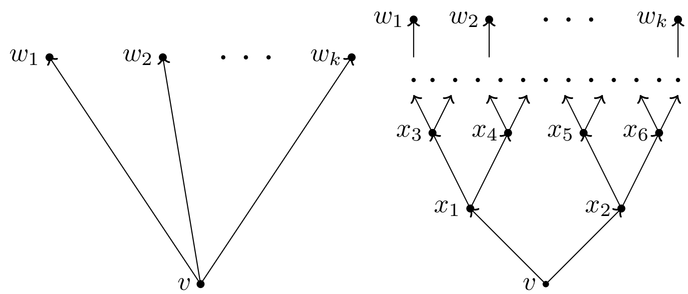


FIGURE 1. The polygonal line $K_e(\zeta)$

(4) If $e = (v, w)$ and $e' = (v', w')$ are non-incident edges, then v, w, v', w' distinct implies $\varphi v, \varphi w, \varphi v', \varphi w'$ are distinct. Since every point in K_e and $K_{e'}$ has integer coordinates and every line segment runs parallel or perpendicular to every other segment, to show that the distance between K_e and $K_{e'}$ is no less than 1, it is enough to show they do not intersect.
Case 1 If φv has the same abscissas as $\varphi v'$ it is clear that the line segments $(\varphi v)a$ and $(\varphi v')a'$ are disjoint. Since e and e' are related $\zeta \neq \zeta'$. Also, the segment bc ensures that ab doesn't intersect $c'd'$, so $abcdefg$ cannot not intersect with $a'b'c'd'e'f'g'$. Because φw and $\varphi w'$ are distinct, $d(\varphi w, \varphi w') \geq 2$ so $d(h, h') \geq 1$. Therefore $gh(\varphi w)$ does not intersect $g'h'(\varphi w')$.
Case 2 If φw has the same ordinate as $\varphi w'$, $d(\varphi w, \varphi w') \geq 4$. Therefore, $d(h, h') \geq 2$, so it is clear $gh(\varphi w)$ does not intersect $g'h'(\varphi w')$. Furthermore, since $\zeta \neq \zeta'$, $abcdefg$ and $a'b'c'd'e'f'g'$ cannot intersect. Also $\varphi v \neq \varphi v'$ so $(\varphi v)a$ and $(\varphi v')a'$ are disjoint.
Case 3 If e and e' are not related, then $d(\varphi v, \varphi v') \geq 4$ ($d(\varphi v, \varphi v') = 4$ when they have identical ordinates). Then $d(b, c') \geq 2$ so the line segments $(\varphi v)abc$ and $(\varphi v')a'b'c'$ do not intersect. Also φw and $\varphi w'$ have different ordinates, so $d(\varphi w, \varphi w') \geq 2$. Furthermore, $d(f, g') \geq 1$, and the segment ed ensures that $e'f'$ does not intersect cd. Thus, $K_e(\zeta)$ and $K_{e'}(\zeta')$ do not intersect.



(a) (b)

FIGURE 2

(5) If e and e' are incident edges, without loss of generality, assume $e = (v, w)$ and $e' = (w, x)$. The coordinate $4i' + 2 + \tau$ at the point h ensures that fgh does not intersect with $(\varphi w)a'b'$. Therefore, outside a neighborhood of radius 1, $K_e(\zeta)$ and $K_{e'}(\zeta')$ do not intersect.

We have shown G can be realized in a parallelepiped $ABCDA'B'C'D'$. Now we can inscribe $ABCDA'B'C'D'$ in a sphere with radius

$$\frac{\sqrt{8^2n + 4^2n + 2^2n}}{2} = \sqrt{\frac{84n}{4}} = \sqrt{21}\sqrt{n}.$$

In the general case, if G is an arbitrary weighted graph with n vertices, it can be reconstructed into a graph G' with no more than two edges originating at each vertex. As illustrated in Figure 2, to attain G' from G, the number of vertices added to each $v \in V(G)$ is

$$\sum_{i=1}^{\log_2 k} 2^i = 2(k-1) = 2k - 2$$

so if $n' = |V(G')|$, $n' \leq 3n$. Thus G may be realized in a sphere with radius at most $C'\sqrt{3n}$ by interpreting the points φx_i as branching points on polygonal lines joining φv to $\varphi w_1, \ldots, \varphi w_k$.

□

Theorem 3.4. *For any expander graph G with n vertices realized in a sphere of radius R, $C\sqrt{n} \leq R$, where C is a constant not dependent on n.*

Proof. Let G be an expander graph with n vertices realized in a sphere with radius R. We can find a cross-section of this sphere with radius r such that at least $\frac{n}{4}$ of the vertices are on both sides. Choose the side of this cross-section that has fewer vertices, and call these vertices $A \subset V(G)$. In Theorem 2.1 we have shown that for most graphs, $|\partial A| \geq \frac{1}{2}|A|$, and therefore we have the following inequalities:

$$\pi R^2 \geq \pi r^2 \geq |\partial A| \pi \left(\frac{1}{2}\right)^2 \geq \frac{1}{8}\pi |A| \geq \frac{1}{32}\pi n.$$

Thus, we have $C\sqrt{n} \leq R$ where $C = \frac{\sqrt{2}}{8}$. □

For graphs with the expander property, the Kolmogorov-Barzdin realization from Theorem 3.2 is the smallest possible realization of the graph in $\mathbb{R}^3$.

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References

[1] A.N Kolmogorov, Y. M. Barzdin. On The Realization of networks in Three-Dimensional Space. Kluwer Academic Publishers. 1987.
[2] Joel Friedman. Expanding Graphs. American Mathematical Society. 1993.
[3] Ronald Gould. Graph Theory. The Benjamin/Cummings Publishing Company, Inc. 1988.
[4] Robin J. Wilson. Introduction to Graph Theory. Pearson Education Limited. 2010.
[5] Terence Tao. Distinguished Lecture Series II: Avi Wigderson, Expander graphs constructions and applications. https://terrytao.wordpress.com/2008/01/11/distinguished-lecture-series-ii-avi-wigderson-expander-graphs-constructions-and-applications/.
[6] Misha Gromov and Larry Guth. Generalizations of the Kolmogorov-Barzdin Embedding Estimates. http://arxiv.org/pdf/1103.3423v1.pdf