EXPANDER GRAPHS

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Abstract. This paper will introduce expander graphs. Kolmogorov and Barzdin's proof on the three dimensional realization of networks will be discussed as one of the first examples of expander graphs. The last section will discuss error correcting code as an application of expander graphs to computer science.

Contents

1. Expander Graphs: The discovery of Kolmogorov and Barzdin

In this section, we will introduce the concept of expander graphs and attempt to provide an intuitive understanding of these graphs. As such, the definition will be informal and we will proceed to define the aspects of expanders to formalize the understanding. It is difficult to precisely define an expander graph because they are defined within different branches of mathematics distinctly, so a common definition is difficult to formulate.

Definition 1.1. An Expander graph is a sparsely populated graph that is well connected.

Definition 1.2. A sparse graph is a graph in which the total number of edges is few compared to the maximal number of edges.

Example 1.3. Consider a simple graph G with n vertices and 2 edges originating from each vertex. There are $2n$ edges in this graph. If this graph was a complete graph, every vertex connected to every other vertex, we would need n! edges. It is clear that this graph is sparse since $n! >> 2n$. A similar graph is used in the next section as the network that we are working with.

The next aspect of an expander graph is that it is well connected.

Definition 1.4. A graph G is connected if there exists a path between vertices α and β, for all $\alpha, \beta \in$G

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When we say that a graph has a high level of connectivity or that it is well connected, we mean that to make this graph disconnected, we would need to remove a sizable percentage of edges found in the graph.

Example 1.5. A tree T is a poorly connected graph. Removing any one edge from T will render it disconnected. See the figure below.

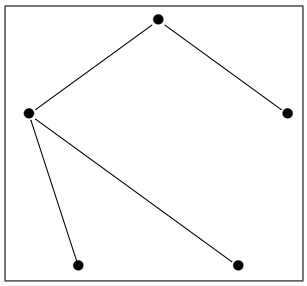

FIGURE 1. A simple tree.

Example 1.6. The figure below contains a bottleneck. If we remove the connection between the two pentagons, the network is disconnected. We can remove other points without it disconnecting the graph immediately, but that does not negate the easily disconnected nature of the graph. Clearly, this type of graph is not well connected either.

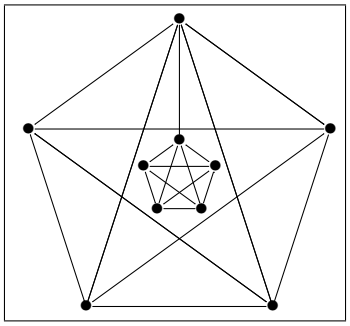

FIGURE 2. A basic bottleneck network.

Example 1.7. Figure 3 (found on the next page) is just one section of the previous example. But this graph is well connected. There are multiple paths connecting each point to another, so removing a few connections will not render the graph disconnected.

The previous examples attempt to convey an intuitive sense of a well connected graph, but we can make this concept more formal with the Cheeger Constant. This constant is used to determine the level of connectivity of a network.

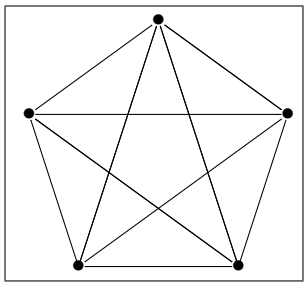

FIGURE 3. A complete graph.

Definition 1.8. The Cheeger constant of a finite graph, G

$$c(G) = \min_{A \subset V} \min_{|A| \leq |V|/2} \frac{|\delta A|}{|A|} \tag{1.9}$$

where the boundary of A is δA=$\{$(v,w)$\in$E : v$\in$A and w$\notin$A or v$\notin$A and w$\in$A$\}$

The larger the Cheeger constant, the better connected the graph is. Additionally, c(G) $>$ 0 if and only if G is a connected graph.

Due to the nature of expander graphs, formalizing the definition gives rise to three distinct perspectives of expanders. They are edge-expanders, vertex-expanders, and spectral-expanders.

Definition 1.10 (Edge Expansion)**.** Let G be a graph of n vertices. The edge expansion of G is c(G), i.e. the smallest ratio of the boundary of A and A, where A is any subset of G containing $\leq n/2$ vertices.

The larger that number is, the better expansion based on the edges G will have.

Definition 1.11 (Vertex Expansion)**.** Let G be a graph of n vertices. The vertex isoperimetric number h_{out}(G) is defined as

$$h_{out}(G) = \min_{0<|S|\leq n/2} \frac{|\delta_{out}(S)|}{|S|} \tag{1.12}$$

where δ_{out} is outer boundary of S, i.e. the set of vertices which have at least one neighbor in S.

This definition is similar to the previous one, using the boundary as a measure of the expansion of the graph. Instead of looking at the edges, we look at the vertices.

Definition 1.13 (Spectral Expansion)**.** If G is a d-regular graph, we can create a notion of expansion based on the spectral gap, where the spectral gap of G, s(G) is defined as

$$s(G) = \lambda_1 - \lambda_2 \tag{1.14}$$

where λ_1 is the largest eigenvalue and λ_2 is the second largest eigenvalue of the adjacency matrix of G.

We can further relate the spectral gap of a graph and the Cheeger constant with Cheeger's and Buser's inequalities.

$$\frac{s(G)}{2} \leq c(G) \leq \sqrt{2\lambda_1 s(G)} \tag{1.15}$$

We can utilize eigenvalues in this definition because the adjacency matrix of G is symmetric. With the spectral theorem, we choose to order the eigenvalue such that each subsequent eigenvalue is less than or equal to the previous one and we know that the eigenvalues are bounded above by the degree of G and below by the negative of the degree.

We can refine our definition of the spectral gap further because it is known that λ_1=d, the degree of G. Thus

$$s(G) = d - \lambda_2 \tag{1.16}$$

Similar to the other notions described above, the larger this value, the better the expander G is.

An example of a spectral expander is the Ramanujan graph, which will be discussed in section 3.

2. Kolmogorov and Barzdin's proof on the Realization of Networks in Three-Dimensional Space

As the story goes, Kolmogorov became interested in three dimensional realizations after learning about the structure and size of neuron networks. Neuron networks have a volume on the order of (number of neurons)$^{3/2}$ and the question became if the evolved network was the smallest sized network possible.

This proof contains the first notion of expander graphs as it showed that most random graphs are expanders. The term expander graphs was originally defined by Pinsker in 1973 [5], 6 years after this proof was published. Contained in this section, we see Kolmogorov and Barzdin construct a random graph with properties equivalent to an expander and use some of those properties in their proof.

In this section, we will follow the proof of Kolmogorov and Barzdin [6] with commentary and adjustments made for the sake of clarity. This proof is not original to this paper.

Definition 2.1. A (d, n)-network is an oriented graph with n vertices, labeled $\alpha_1.\alpha_2, ..., \alpha_n$ with dn total edges and d edges incident to each vertex and one of them is marked by the weight x_1, another by the weight x_2, etc., and the final one by x_d.

Definition 2.2. A (2,n)-network will be referred to as a network

Without essential loss of generality (for we are only dealing with values up to order), we shall consider only networks.

Definition 2.3. The network $\mathfrak{A}$ is realized in 3-Dimensional space if:

1. For all α in $\mathfrak{A}$, there exists a $\phi(\alpha) \in R^3$, where the point $\phi(\alpha)$ is surrounded by a ball of radius 1/2 and the ball shall be referred to as a ϕ-point. ϕ-points cannot intersect with each other.
2. to each edge $p = (\alpha, \beta)$ originating at α and ending at β, we create a continuous cylinder of radius 1/2, K_p in R^3, joining the points $\phi(\alpha)$ and $\phi(\beta)$. The curve will be called a conductor. Conductors cannot intersect with each other unless they share a vertex and are within 1 of that vertex.

Suppose R is a realization of the network $\mathfrak{A}$ in three-dimensional space.

Definition 2.4. The solid W contains R if all the ϕ-points and conductors in R are contained within W and are located at a distance no less than 1 from its boundary.

W enables us to analyze the volume (up to order) that a realization takes up.

Definition 2.5. By the volume of realization R, we shall mean the minimal volume of a solid containing the realization R.

Definition 2.6. By the volume of the network $\mathfrak{A}$, V($\mathfrak{A}$) we shall mean the minimal volume of its realization in three-dimensional space. Given all possible realizations of $\mathfrak{A}$, R_i, and their volumes of realization, U_i, we can determine which R_i will takes up the smallest volume.

Theorem 2.7. ***(Estimate from above)*** *For all networks $\mathfrak{A}$ with n vertices we have*

$$V(\mathfrak{A}) \leq C_1 n^{3/2} \tag{2.8}$$

C_1 is a certain constant not depending on n.

Definition 2.9. A network which has no more than two edges originating from each vertex shall be called a network with bounded branching.

Theorem 2.10. *Any network with n vertices may be realized in a sphere of radius $C\sqrt{n}$, where C is a constant not dependent on n.*

Theorem 2.10 is a stronger statement than Theorem 2.7 so a proof of Theorem 2.10 means Theorem 2.7 is true.

Proof. First, we shall prove this theorem by using a network with bounded branching. The network with bounded branching is comparable to a network with the potential for missing edges.

Thus, suppose that $\mathfrak{A}$ is a network with n vertices and bounded branching. Let us show it can be realized in the parallelepiped ABCDA′B′C′D′, with side lengths of order $\sqrt{n}$. Without loss of generality, we shall assume that $\sqrt{n}$ is an integer. This means we can create a sphere to encompass the parallelepiped with radius $C_3\sqrt{n}$, where C_3 is some constant.

Suppose U is a set of points on the lower base ABCD of the form (i+2,2j,0), i= $0, 1, ..., \sqrt{n} - 1$; $j = 0, 1, ..., \sqrt{n} - 1$. Now, $\forall \alpha \in \mathfrak{A}$ $\exists \phi(\alpha) \in$ U and $\forall$ arc p = $(\alpha, \beta) \in \mathfrak{A}$ $\exists$ the polygonal line $K_p(\zeta)$, with the path $\phi(\alpha), a, b, c, d, e, f, g, h, \phi(\beta)$, depending on the parameter ζ in the following way:

Suppose the coordinates of the points $\phi(\alpha)$ and $\phi(\beta)$ are (i+2,j,0) and (i'+2,j',0) respectively, then the coordinates of the point h are (i'+2+t,j',0) where $t = 1$ if the arc has the weight x_1 and t=-1 if the arc has the weight x_2. The coordinates of the point a and g are (i+2,j,2-1) and (i'+2+t,j',2). The coordinates of the point b and f are respectively (i+2,j+1,2-1) and (i'+2+t,j'+1,2). The coordinates of the point c are (i,j+1,2-1). The coordinates of the point d and e are respectively (i,j'+1,2-1) and (i',j'+1,2).See figure 4 for a visualization.

Each point is at least one unit apart and we construct the first layer of our realization of $\mathfrak{A}$.

Definition 2.11. Two arcs are related if they share identical abscissas, identical ordinates, or both.

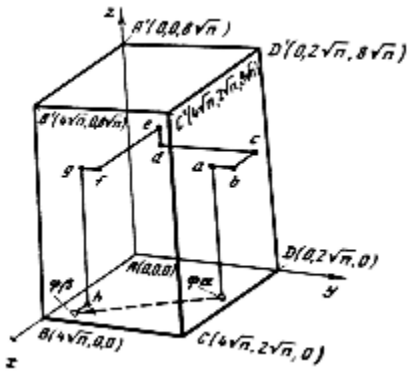

FIGURE 4. This figure is found in Kolmogorov and Barzdin's proof [6].

The number of related arcs in $\mathfrak{A}$ to the arc p is no greater than $4\sqrt{n}$ due to the nature of a bounded branching network.

The polygonal line $K_p(\zeta)$ has the following properties:

(1) If the arcs p=(α, β) and p′=(α', β') are not related, then the corresponding polygonal lines $K_p(\zeta)$ and $K'_p(\zeta')$ $\forall$ values of ζ and ζ' are located at a distance ≥ 1 from each other (excluding neighborhoods of radius 1 of the point $\phi(\alpha)(\phi(\beta))$ if $\alpha = \beta'$ or $\beta = \alpha'$).
(2) If the arcs p=(α, β) and p′=(α', β') are related, then the corresponding polygonal lines $K_p(\zeta)$ and $K'_p(\zeta')$ have the property that whenever $\zeta \neq \zeta'$ these lines are located at a distance ≥ 1 from each other (excluding neighborhoods of radius 1 of the point $\phi(\alpha)(\phi(\beta))$ if $\alpha = \beta'$ or $\beta = \alpha'$).

Thus our polygonal lines can be considered conductors in the realization of $\mathfrak{A}$.

To prove that $\mathfrak{A}$ can be realized in the parallelepiped, it suffices to show that the parameters ζ for the arcs in $\mathfrak{A}$ are natural numbers so that:

(1) These values for distinct related arcs are different.
(2) The corresponding polygonal lines are all inside the parallelepiped, i.e. $i \leq \zeta \leq 4\sqrt{n}$

Such a choice for the parameter ζ is always possible because the number of arcs related to any arc p is no greater than $4\sqrt{n}$ which is the number of admissible distinct values of ζ.

And this shows that the statement is correct for networks with bounded branching. We can extend this to arbitrary networks because any network $\mathfrak{A}$ with n vertices may be reconstructed into a network $\mathfrak{A}'$ with bounded branching by adding no more than 2n new vertices. As a result, we obtain a network $\mathfrak{A}'$ with bounded branching and $n' \leq 3n$ (We added up to $2n$ vertices, but now we limit the vertices to have only 2 incident edges. The end result is the same, but the paths from one vertex to another now passes through intermediary vertices) By the proved theorem, this network may be realized in a sphere of radius $C\sqrt{(}n')$. In order to obtain the realization of the network $\mathfrak{A}'$ in the sphere from the realization of the network $\mathfrak{A}$, we must interpret the new vertices as branching points of polygonal lines joining

the vertices found in $\mathfrak{A}$ to other vertices in $\mathfrak{A}$. Thus we have created a realization of a network which has volume $\approx C_1 n^{3/2}$. □

Theorem 2.12. ***(Estimate from below)*** *For almost all networks $\mathfrak{A}$ with n vertices we have*

$$V(\mathfrak{A}) \geq C_2 n^{3/2} \tag{2.13}$$

C_2 is a certain positive constant not depending on n

Suppose w is a certain partition of the vertices of the network $\mathfrak{A}$ into three ordered parts which are denoted respectively by w_1, w_2, w_3.

Definition 2.14. An (ϵ, δ)-partition is a w partition of $\mathfrak{A}$ if w_1 contains [ϵn] vertices, w_2 contains [δn] vertices and w_3 contains 1-[ϵn]-[δn] vertices.

The remainder of the proof will focus on the vertices in w_1 and w_3, ignoring those in w_2. Since all of these sets are of variable size, we can adjust the sizes to account for all possible permutations on the edges between vertices.

Definition 2.15. By the degree of connectivity S($\mathfrak{A}$,w) of the network $\mathfrak{A}$ with respect to the partition w, we shall mean the maximal number of non-incident arcs (originating in w_1 and terminating in w_3) joining the vertices of w_3 to the vertices of w_1.

The larger this value, the more edges and thus the more connected these two sets are.

Lemma 2.16. *There exists $0 < \epsilon_0 < 1$, $a_0 > 0$ and $b_0 > 0$ such that for all $\epsilon \leq \epsilon_0$ and $a \leq a_0$ the degree of connectivity of almost all networks A with n vertices with respect to any $(\epsilon, a\epsilon)$-partition is no less than $b_0 \epsilon n$.*

Remark 2.17. Referring back to our previous discussion of expander graphs, we can see how this lemma ties in. It claims that nearly all networks are well connected, a necessary part of our definition of expander graphs.

Instead of an (ϵ, δ)-partition, we have an $(\epsilon, a\epsilon)$-partition, where aϵ is equivalent for our purposes to δ. The key portion of this lemma is that there will be a minimum (for most networks) degree of connectivity between all possible partitions of the network.

Definition 2.18. By the degree of incidence, Z($\mathfrak{A}$,w) of the network $\mathfrak{A}$ with respect to the partition w, we shall mean the number of vertices belonging to w_3 from which there are arcs going to vertices belonging to w_1.

Since no more than two arcs can enter any one vertex, it follows that S($\mathfrak{A}$,w)$\leq$ 1/2Z($\mathfrak{A}$,w). Thus if we prove the degree of incidence is bounded, then S(A,w) must have a similar lower bound and our lemma will be proven.

Proof of Lemma 2.16. We shall begin by constructing networks with n vertices $\alpha_1, \alpha_2, ..., \alpha_n$. Next, there are $2n$ arcs such that precisely two arcs will be connected to each vertex while the other end of each arc remains unconnected. The free extremity of each arc is randomly, with equal probability $1/n$, joined to one vertex in the set $(\alpha_1, ...\alpha_n)$. As a result, we obtain a network $\mathfrak{A}$ with vertices $\alpha_1, ..., \alpha_n$.

Suppose a certain (ϵ, δ)-partition w of the set $(\alpha_1, ...\alpha_n)$ is fixed. Denote $P_w(\epsilon, \delta, c, n)$ as the probability of obtaining, as the result of the procedure described above, a network which has a degree of incident with respect to w satisfying Z($\mathfrak{A}, w$)<cn. Denote $P(\epsilon, \delta, c, n)$ as the probability of obtaining a network which has at least one (ϵ, δ)-partition satisfying Z($\mathfrak{A}, w$) <cn.

A successful outcome means there is not minimum level of connectivity for networks. Our end goal is for almost all partitions to fail. If $P(\epsilon, \delta, c, n)$ goes to 0 as n increases, then for most cases, the degree of connectivity will have a minimum value.

Let us estimate $P_w(\epsilon, \delta, c, n)$. Consider the following probabilistic model. There are n boxes of which ϵn are white, δn are black and (1-$\epsilon - \delta$) are red. Consider the sequence of trials $W = s_1, ..., s_{2\epsilon n}$ consisting of random throws of 2ϵn balls into the boxes. One trial is one throw of one ball. By a successful result of a trial, we mean the event in which a white ball falls into an empty red box. Denote by $P'(\epsilon, \delta, c, n)$ the probability that the number of successful outcomes in a series of trials W is $< cn$.

The 2ϵn balls are analogous to the 2 edges originating from each vertex in the w_1 partition. We then see how many of those balls (or edges) enter the red boxes (or w_3 vertices). The number of balls entering the red box is compared to the fixed value, cn. If the value is less than cn, the trial was successful, i.e. Z($\mathfrak{A}$,w)$< cn$. Clearly

$$P_w(\epsilon, \delta, c, n) = P'(\epsilon, \delta, c, n) \tag{2.19}$$

Our next step will be to estimate $P'(\epsilon, \delta, c, n)$. Now, the smallest probability of a successful outcome is 1-3$\epsilon - \delta$, which is obtained when all other trials were successful. If we assume that all the outcomes have that smallest probability, we can construct $P''(\epsilon, \delta$,c,n) with the probability of each outcome as 1-3$\epsilon - \delta$. It is easy to see that

$$P'(\epsilon, \delta, c, n) < P''(\epsilon, \delta, c, n) \tag{2.20}$$

Let us now estimate $P''_w(\epsilon, \delta, c, n)$. Because this probability is a $2\epsilon n$ Bernoulli trial

$$P''(\epsilon, \delta, c, n) = \sum_{i=0}^{cn} \binom{2\epsilon n}{i} (1 - 3\epsilon - \delta)^i (3\epsilon + \delta)^{2\epsilon n - i} \tag{2.21}$$

Although it is not intuitively obvious based off of the equation alone, we do end up discovering that $P''(\epsilon, \delta, c, n)$ tends towards 0 as n approaches ∞. This portion of the proof relies mainly on techniques which are beyond the scope of this paper, but can be found in [6, 204]

Recalling the order of the probabilities:

$$P''(\epsilon, \delta, c, n) > P'(\epsilon, \delta, c, n) = P_w(\epsilon, \delta, c, n) \tag{2.22}$$

Which implies that as

$$P_w(\epsilon, \delta, c, n) \to 0 \quad as \quad n \to \infty \tag{2.23}$$

Our lemma is proven. □

Remark 2.24. The random networks used in the previous proof are equivalent to expander graphs and are indirectly proved to be as such in the conclusion of the proof.

Now we know that for most networks, (not all because we are dealing with limits so there is a non-zero chance of a network not meeting the criteria) that there is a minimum degree of connectivity for the network. Independent of which combination of vertices we look at in a network $\mathfrak{A}$, there will be some minimum number of edges connecting the selected subsets of vertices. Let us take into consideration the following two circumstances:

(1) According to the definition of realization, the distance between any two conductors corresponding to non-incident arcs is ≥ 1.
(2) According to Lemma 2.16, almost all networks with n vertices are such that any $\epsilon_0 n$ vertices are joined to the remaining vertices by at least $b_0\epsilon_0 n$ non-incident arcs.

This implies the validity of the following lemma.

Lemma 2.25. *Almost all networks $\mathfrak{A}$ with n vertices posses the following property: for any realization of the $\mathfrak{A}$ in R_3, there are less than ϵn points in any parallelepiped of surface area $L \leq 1/2 b_0\epsilon_0 n$.*

The direct consequence of this lemma is that most network′s realization are sparse. It also implies that there is a certain minimum space needed to contain all n vertices.

Remark 2.26. With that, we have shown that almost all networks (in this proof) are sparse, well connected graphs, i.e. they are expanders. For almost any random graphs of any size n, we can construct them so that they have a fixed amount of sparsity and connectivity.

Proof of Estimate from Below. Suppose u and s are certain numbers (their values will be chosen later). Suppose $\mathfrak{A}$ is a certain network with n vertices for which we have the assertions of the previous Lemmas (almost all networks are such). Suppose further that the network $\mathfrak{A}$ is realized in three-dimensional space and let T be a certain parallelepiped whose regular edges are of length $\geq su\sqrt{n}$ and which contains all the ϕ points and conductors arising for the given realization of the network $\mathfrak{A}$.

Then:

(1) By means of the planes $z = iu\sqrt{n}$ (i=0,1,2) let us divide the parallelepiped T into parallel slices of thickness $\mathrm{u}\sqrt{(n)}$. For our purposes, let us assume that the dimensions of T are $lus\sqrt{n}$, where l is an integer such that we have exactly ls slices. Let us number the slices in the direction of the z axis, using the numbers 1,2,...(thus the lowest slice will be denoted with the number 1, and the next one will be 2, and so on). We then divide the slices into groups $A_1, A_2, ..., A_s$; in A_i we choose the slices such that the slices number (1,2,...,ls) mod i =0. Each A_i contains l slices. Then, from all of these parts let us choose the one which contains the smallest number of ϕ-points. Suppose it is the partition A_z. It is easy to see that A_z contains no more than n/s ϕ-points (the best case for the smallest number of points is if all points were split between the partitions, i.e. n/s points). Slices belonging to A_z will be called distinguished.
(2) By means of the planes $y = iu\sqrt{n}$ (i=0,1,2) let us partition the parallelepiped T into parallel slices, then construct a partition $B_1, B_2, , B_s$ similar to the one in (1) and let us call the slices belonging to the part B_y containing the least number of ϕ-points distinguished.

(3) By means of the planes $x = iu\sqrt{n}$ (i=0,1,2) let us partition the parallelepiped T into parallel slices, then construct a partition $C_1, C_2, , C_s$ similar to the one in (1) and let us call the slices belonging to the part C_x containing the least number of ϕ-points distinguished.

It follows from the above that the distinguished slices of the parallelepiped T contain no more than $3n/s$ ϕ-points since at most, each axial directions set of distinguished slices contain $\leq n/s$ points, the union of the three directions must be $\leq$ to $3n/s$.

Now, let us suppose that the numbers u and s are such that the cubes bounded by distinguished slices have a surface area no greater than $1/2b_0\epsilon_0$n (here we mean each cube separately). Refer to the figure 5 for a 2 dimensional example.

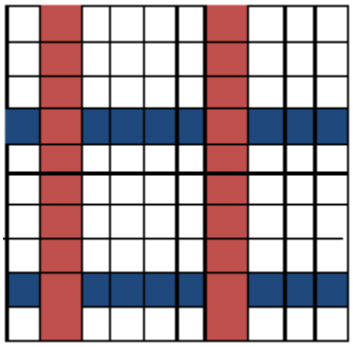

FIGURE 5. The red and blue represent the distinguished slices that are acting as borders for the rest of the slices, the white squares.

According to Lemma 2.25, each of these cubes contain less than $\epsilon_0 n$ points. Now, **if** the non-distinguished slices contain greater than $\epsilon_0 n$ ϕ-points, we see that

$$\text{number of undistinguished points} \geq n - 3n/s \geq \epsilon_0 n \tag{2.27}$$

Next, we create a subset, G, of the set of all cubes. Using equation 2.27, the fact that the total number of undistinguished points is greater than $\epsilon_0 n$ and each cube contains less than ϵ_0 points means we can find a set G such that

$$1/2\epsilon_0 n \leq \text{number of points in G} \leq \epsilon_0 n \tag{2.28}$$

Suppose further that the number s is such that the distinguished slices contain no more than $a_0\epsilon_0 n/2$ ϕ-points, i.e. let s satisfy the inequality

$$3n/s \leq a_0\epsilon_0 n/2 \tag{2.29}$$

Note: a_0, b_0, and ϵ_0 all come from the previous lemmas and are constants dependent on each specific network

Consider the following (ϵ, δ)-partition of the network $\mathfrak{A}$:

(1) w_1 is the set of vertices of the network $\mathfrak{A}$ such that their corresponding ϕ-points exist in G; the number of elements ϵn of the set w_1 is equal to the number of ϕ-points contained in G, i.e. $\epsilon_0/2 \leq \epsilon \leq \epsilon_0$

(2) w_2 is the set of all vertices of the network $\mathfrak{A}$ such that their corresponding ϕ-points exist in the distinguished slices of the realization; the number of elements δn of the set w_2 is equal to the number of ϕ-points contained in the distinguished slices, i.e. $\delta \leq a_0\epsilon_0/2$

It follows from Lemma 2.16 that the network $\mathfrak{A}$ for such an (ϵ, δ)-partition has a degree of connectivity $\geq b_0\epsilon n > b_0\epsilon n/2$. This means that the ϕ-points contained in the set G and ϕ-points contained in the other cubes bounded by distinguishing slices are joined to each other by at least $b_0(\epsilon_0/2)$n non-incident conductors. The number of conductors here is a repeat of the minimum value for the surface area applied in Lemma 2.25. If we take a slice of all these conductors, the total surface area is on the same order as the surface area for Lemma 2.25 because we have defined them to have thickness of diameter 1. The set of these conductors will be denoted by E. According to the definition of the realization, the distance between conductors from E must be $\geq$1. Since the thickness of the distinguished slices is equal to $u\sqrt{n}$, it follows that each of the conductors from the set E joins points located at a distance $\geq$ u$\sqrt{n}$. Hence we obtain that any solid W which contains all the conductors from E and in such a way that they are $\geq$1 from its surface, has a volume greater than

$$b_0\epsilon_0 n/2\pi(1/2)^2 u\sqrt{n} = (\pi/8)b_0\epsilon_0 un\sqrt{n} \tag{2.30}$$

(minimum number of conductors*surface area of conductors* minimum distance to travel)

Now let us recall that these estimates were obtained using the estimates for u and s being $>$0 and independent of n. It is easy to verify that there exists s and $u >$0 not depending on n to satisfy the earlier inequalities: 2.27, 2.29. This means that the volume of the network satisfies

$$V(\mathfrak{A}) > \pi/8 b_0\epsilon_0 un^{3/2} = C_2 n^{3/2} \tag{2.31}$$

C_2 is a certain constant > 0 not depending on n.

Thus the theorem is proved. □

3. A Computer Science Application of Expander Graphs

In this section, we will discuss how expander graphs, specifically Ramanujan graphs, can be applied to Computer Science in order to create error correcting codes.

Sending a message from one object to another is a common occurrence in our digital age. Televisions, radios, cell-phones, and the internet all use transmissions. And, as anyone who has used these services can attest to, they are not perfect. Perhaps someone's words sound garbled on the phone or the picture on the television is fuzzy, the basic problem is that somehow there were errors in the transmission. One solution is for the transmission method to detect and correct any errors that occur. Error correcting codes is a field within coding theory that concerns itself with this very issue.

We shall begin by considering the problem mentioned above, a message between a sender and a receiver. Now, considering that this message can be of any form, a picture, a document, vocals, etc. we shall represent the original message as a series

of encoded binary bits. The message is sent then the receiver needs to decode the bits to return the original message.

In order to encode/decode the message, there needs to be an agreed upon cipher. Our choice of cipher is important for we use it to correct any mistakes that may be made in the transmission process. The solution here is simple. We shall create a code, a subset of all possible n-bit blocks that contains the only valid encoded data, i.e. codewords. Anything in the transmission that is not found in our code is a mistake.

Consider the case when n =2 as an illustrative example. Let us graph the $n = 2$ case. To do this, let the first bit represent the x-coordinate, the next represent the y-coordinate. An example bit is 01, with coordinates (0,1). There are 4 possible 2-bit blocks and we will select 2, say (0,0) and (0,1) and remove the others. The points left on the graph are our codewords, 00 and 01, and the set of those points is our code. When the code is received we place each 2-bit block on the corresponding

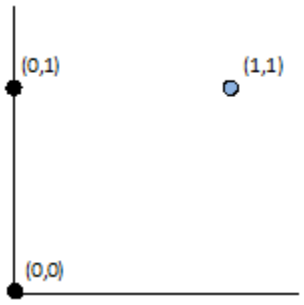

FIGURE 6. The black dots are the codewords and the blue dot is the received message.

spot on the graph. Let the original code bit be 01. If there is no error, the received bit will be place at (0,1). Suppose that there was an error and that the first bit was changed to 1. Now, the bit's coordinate is (1,1) instead of (0,1). We can correct this error by finding its closest neighbor that is a codeword and making it that codeword. In this case, (0,1) is the closest codeword so we change our received bit to 01. Obviously, this method is not reliable at n=2, but the process remains the same as n increases since at larger n, each individual error will change the distance from the correct codeword less.

The question now becomes how many codewords do we want? There are two extreme cases: a very small code which means the error correcting will almost always be correct but we severely limit our efficiency or a large code that has many potential messages but the correcting process is likely to make mistakes. We need to strike a balance between these two opposites to create a worthwhile error correcting code. It is easy to understand how expander graphs, graphs that are sparse, but well connected, become useful in this situation of balancing two opposing properties.

Definition 3.1. A code C is a set of n-bit binary strings, and its elements are called codewords.

The space we will be working with is $\mathbb{F}_2 = \{\overline{0}, \overline{1}\} \cong \mathbb{Z}/2\mathbb{Z}, \mathbb{F}_2^n$

Definition 3.2. The Hamming distance between x,y $\in \mathbb{F}_2^n$, denoted d_H, is the number of coordinates on which x and y differ,

$$d_H(x,y) = |i : x_i \neq y_i| \tag{3.3}$$

The Hamming distance will be the metric used in our code.

Definition 3.4. The distance of a code C $\subset \mathbb{F}_2^n$ denoted dist(C), is the minimum Hamming distance between a pair of distinct codewords x,y $\in$ C.

$$dist(C) = \min_{x \neq y \in C} d_H(x,y) \tag{3.5}$$

The rate of a code C is defined as

$$rate(C) = \frac{log|C|}{n} \tag{3.6}$$

The relative distance is defined as

$$\delta(C) = dist(C)/n \tag{3.7}$$

Definition 3.8. A family of codes $C_n \subset \mathbb{F}_2^n$ is asymptotically good if there exist some fixed constants d $>$ 0 and r $>$0, such that for all n we have both $\delta(C)$ $>$d and rate(C)$>$r.

Definition 3.9. A code C $\subset \mathbb{F}_2^n$ is called a linear code of dimension k and length n if it is a k-dimensional vector subspace of $\mathbb{F}_2^n$.

Given a d-regular graph $G = (V, W)$, we will use an auxiliary bipartite graph G' based on G.

Let $G' = (V \cap W, W')$ where $ab \in W'$ iff $a \in V$, $b \in W$ and $\exists$ $c \in V$ such that $ac = b$.

We can see that edges only exist between V and W. Also, every vertex of V has exactly d edges connecting it to W. Label the vertices of W = 1,2,...,n. For any vertex $v \in V$ define $v(1)$,...,$v(d)$ to be the d vertices in W which are connected to v. Let C_0 be a linear code of length $n_0 = d$, with minimum distance d_0. We have constructed a bipartite graph.

Let $C \subset \mathbb{F}_2^n$ be the set of binary vectors $(x_1, ..., x_n)$ such that for every vertex $v \in V$ we have the smaller vector $(x_{v(1)}, ..., x_{v(d)}$ as the codeword in C_0.

The distance between codewords is at least $n\delta_0^2$(1-ϵ). The relative minimum distance of C_0 is $\delta_0 = \frac{d_0}{n_0}$. Additionally, ϵ is a constant that depends only on d_0, d, and $\lambda_2(G)$. Additionally, as $\lambda_2/d_0 \to 0$, then $\epsilon \to 0$.

Our next action will be to use a Ramanujan graph, one of the best possible spectral expanders. And while it is the best, these graphs are still difficult to construct explicitly; they were constructed by Sarnak utilizing deep number theory [4].

Definition 3.10. Let G be a connected k-regular graph with n vertices and let $\lambda_0 \leq \lambda_1 \leq ... \leq \lambda_{n-1}$ be the eigenvalues of the adjacency matrix for G. Because G is connected and d-regular, its eigenvalues are d=$\lambda_0 \leq \lambda_1 \leq ... \leq \lambda_{n-1} \leq$-$d$. Let

$$\lambda(G) = \$e_{aa}(2)a_{|\lambda_i|<d}|\lambda_i| \tag{3.11}$$

A d-regular graph G is a Ramanujan graph if $\lambda(G) \leq 2\sqrt{d-1}$

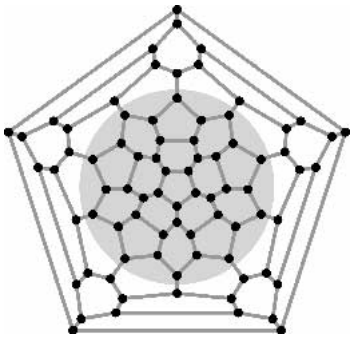

FIGURE 7. A Ramanujan graph. This graph contains 80 vertices. [4]

As was shown in the previous section and was done more explicitly by Pinsker [5], Expander graphs exist. Although they exist, it is difficult to create an explicit representation of an expander graph. A Ramanujan graph is an explicit construction of an expander, specifically spectral expanders. This makes them the ideal expander to use in our formulation of an error correcting code. Let us now assume that our graph G is a Ramanujan graph. With large enough d we find that $\lambda_2/d_0 \leq 2\sqrt{d-1}/(d * d_0) \to 0$ which means $\epsilon \to 0$. As such, we can use a Ramanujan graph to create asymptotically good codes.

We have thus shown a method to encode data using expander graphs. The use of the Ramanujan graph allowed us to maximize the minimum distance between codewords so that our code would not correct words falsely. And due to the expander properties, we do not need to harshly limit the size of our code to allow for this minimum distance. Thus our code will contain many codewords while still maintaining effectiveness in correcting errors.

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