CALCULATION OF FUNDAMENTAL GROUPS OF SPACES

PETER ROBICHEAUX

Abstract. We develop theory, particularly that of covering spaces and the van Kampen Theorem, in order to calculate the fundamental groups of various spaces. We use topological and algebraic ideas, including notions from basic category theory. Some familiarity with group theory, category theory, and topology is assumed.

Contents

1. Introduction

One viewpoint of topology regards the study as simply a collection of tools to distinguish different topological spaces up to homeomorphism or homotopy equivalences. Many elementary topological notions, such as compactness, connectedness, path-connectedness, and any of the separation axioms, are simply conditions that a space can satisfy that are invariant under homeomorphism. For instance, it seems obvious that one cannot "continuously deform" an open line segment into a circle. No matter what you do with the ends of the segment, you cannot "meld them together" continuously. However, this statement is very informal, and one could argue the matter either way with no decisive mathematical outcome. However, using the machinery of elementary topology, one could use a connectedness argument to show that they are *not homeomorphic*. The argument is simple: deleting a point of the circle leaves a connected space behind; deleting a point of the line segment does not.

Algebraic topology expands on this idea. What exactly, then, is the difference between S^1 and D^2? Can we tell the difference between a circle and the wedge of two circles? In what ways is the 2-torus different from the 2-sphere? The fundamental group functor gives us the tools to answer many of these questions.

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2. A Few Simply Connected Spaces

Definition 2.1. A pointed space (X, x_0) is *simply connected* if its fundamental group $\pi_1(X, x_0)$ is trivial.

First, a quick result allows us to identify fundamental groups of a space based at different points.

Lemma 2.1. If a space X is path connected, then for any two points x_0 and x_1 in X, $\pi_1(X, x_0) \cong \pi_1(X, x_1)$.

pf. Consider a path α from x_0 to x_1. Note that we can define a map α^{-1} by setting $\alpha^{-1}(t) = \alpha(1-t)$. Define a map $\hat{\alpha} : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$ by

$$[f] \mapsto [\alpha^{-1}] * [f] * [\alpha]$$

This map is well defined by virtue of the fact that the path-product $*$ is well defined. α^{-1} starts at x_1 and ends at x_0, f is a loop at x_0, and α starts at x_0 and ends at x_1. Suppose $[\alpha^{-1}] * [f] * [\alpha] = [\alpha^{-1}] * [g] * [\alpha]$. Then $[f] = [\alpha] * [\alpha^{-1}] * [f] * [\alpha] * [\alpha^{-1}] = [\alpha] * [\alpha^{-1}] * [g] * [\alpha] * [\alpha^{-1}] = [g]$. This shows injectivity of $\hat{\alpha}$. Now consider a loop $[g'] \in \pi_1(X, x_1)$. Then $[\alpha] * [g'] * [\alpha^-1] \in \pi_1(X, x_0)$, and $\hat{\alpha}([\alpha] * [g'] * [\alpha^-1]) = [g']$, which shows surjectivity. Finally, $\hat{\alpha}$ is a homomorphism, as

$$\hat{\alpha}([f]) * \hat{\alpha}([g]) = [\alpha^{-1}] * [f] * [\alpha] * [\alpha^{-1}] * [g] * [\alpha] = [\alpha^{-1}] * [f] * [g] * [\alpha] = \hat{\alpha}([f] * [g])$$

This shows that $\hat{\alpha}$ is an isomorphism.

□

This allows us to mainly forget about the decision to use a pointed space (X, x_0), as the fundamental groups of a space at two different basepoints are isomorphic.

Definition 2.2. Let C be a category. $I \in \mathrm{Obj}(\mathsf{C})$ is *initial* if $\forall A \in \mathrm{Obj}(\mathsf{C})$, $\mathrm{Hom}(I, A)$ is a singleton. $F \in \mathrm{Obj}(\mathsf{C})$ is *final* if $\forall A \in \mathrm{Obj}(\mathsf{C})$, $\mathrm{Hom}(A, F)$ is a singleton.

Definition 2.3. Let $J \in \mathbb{N}$, and $J \geq 1$. A space $X \subset \mathbb{R}^J$ is *convex* if the line segment joining any two points in X is contained in X.

Proposition 2.2. If $X \subset \mathbb{R}^J$ is convex, and f and g are any two paths in X with $f(0) = g(0) = x_0$ and $f(1) = g(1) = x_1$, then $f \simeq g \ \mathrm{rel}\{x_0, x_1\}$.

pf. Let $H : I \times I \to X$ be the straight line homotopy

$$(x, t) \mapsto (1-t)f(x) + tg(x)$$

H is well defined, as X is convex and thus contains every point in the image of the homotopy. H is continuous, as it is constructed out of continuous functions. Furthermore, $\forall x \in I$, $H(x, 0) = f(x)$, and $H(x, 1) = g(x)$, so H is a homotopy. Finally, $\forall t \in I$, $H(0, t) = (1-t)f(0) + tg(0) = (1-t)x_0 + tx_0 = x_0$, and similarly for $H(1, t)$ and x_1.

□

Corollary 2.3 If $X \subset \mathbb{R}^J$ is convex, then $\pi_1(X)$ is trivial.

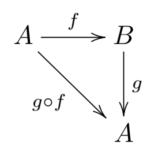
pf. Every loop is in the same homotopy class, so the fundamental group has only one element.

□

Definition 2.4. Two spaces A and B are said to be *homotopy equivalent* if there exists a continuous map $f : A \longrightarrow B$ and a continuous map $g : B \longrightarrow A$ such that $g \circ f \simeq 1_A$ and $f \circ g \simeq 1_B$.

Proposition 2.4. If spaces A and B are homotopy equivalent, then $\pi_1(A) \cong \pi_1(B)$.

pf. Consider the following commutative diagram, where $g \circ f \simeq 1_A$.



Choose a base point x_0 in A such that $x_0 \in g(B)$. Suppose $g \circ f \simeq 1_A$ by H. Define $\gamma : I \to A$ by $t \to H(x_0, 1-t)$. Using homotopy squares, it becomes clear that $\gamma_* = (g \circ f)_*$, which means $g_* \circ f_*$ is an isomorphism, which means that f_* is injective. We can construct the diagram the other way to show g_* injective.

□

Definition 2.5 A space X is called *contractible* if $1_X \simeq c_0$, where c_0 is the constant map at c_0 in X.

It is clear that a contractible space is homotopy equivalent to a one point space. Thus, it follows immediately that any contractible space has a trivial fundamental group. However the converse is not necessarily true.

Proposition 2.5 $\pi_1(S^2) = \{1\}$.

pf. Here we accept the smooth approximation theorem. The smooth approximation theorem allows us to state that for any continuous map $f : I \longrightarrow S^2$, $f \simeq f'$ where f' is smooth. We consider an arbitrary loop f. Let f be homotopic to f' with f' smooth. We aim to show f' nullhomotopic. Using Sard's theorem, f' cannot be surjective, or $\exists x \in S^2$ such that $x \notin \text{im}(f')$. We take this point to be a north pole, and project a map to the plane, and then use the straight line homotopy in the plane and the fact that this projection is a homeomorphism $p : S^2 - \{x\} \longrightarrow \mathbb{R}^2$ to show that f' is nullhomotopic. This completes the proof.

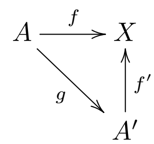
□

Now that we have seen a few simply connected spaces, it is time to move on to spaces that do not have trivial fundamental groups.

3. Covering Spaces

Definition 3.1 Let C be a category. Let X be an object of C. We construct the *slice category* of C over X, denoted by C/X, where the objects of C/X are ordered pairs (A, f) where A is an object and f is a morphism in C from A to X.

Morphisms from an object (A, f) to an object (A', f') are morphisms $g : A \longrightarrow A'$ such that the diagram



commutes.

Definition 3.2. Let X and $\tilde{X}$ be spaces and let $p : \tilde{X} \longrightarrow X$ be continuous. An open set U is *evenly covered* by p if $p^{-1}(U)$ is a disjoint union of open sets S_i in $\tilde{X}$, where $p|S_i : S_i \longrightarrow U$ is a homeomorphism for every i. S_i is called a *sheet*.

Definition 3.3. An ordered pair $(\tilde{X}, p)$ is a *covering space* of X if:

i.) $\tilde{X}$ is path connected;
ii.) $p : \tilde{X} \longrightarrow X$ is continuous;
iii.) each $x \in X$ has an open neighborhood U_x that is evenly covered by p.

Lemma 3.1. $\mathbb{R}$ is a covering space of S^1 where $p : \mathbb{R} \to S^1$ is defined by $x \mapsto e^{2\pi i x}$.

pf. The first two conditions are satisfied trivially. If $x \in S^1 \neq 1$, then $S^1 - \{1\}$ is an open neighborhood of x, and p is a homeomorphism from $(n, n+1)$ to $S^1 - \{1\}$. with obvious inverse $\frac{1}{2\pi i}\ln(x)$ using the principal branch. If $x = 1$, $p : (\frac{1}{2} + n, \frac{1}{2} + n + 1) \longrightarrow S^1 - \{-1\}$ is a homeomorphism.

□

In the case of the circle, the covering map can be thought of as wrapping the real line around the circle.

Lemma 3.2. Let (X, x_0) be a pointed space, and let $(\tilde{X}, p)$ be a covering space of X. Pick a point $\tilde{x_0}$ in the fiber over x_0. Let (Y, y_0) be a connected space, and let $f : (Y, y_0) \longrightarrow (X, x_0)$ be continuous. Then there is at most one continuous $\tilde{f} : (Y, y_0) \longrightarrow (\tilde{X}, \tilde{x_0})$ with $p \circ \tilde{f} = f$.

pf. Suppose there were two such functions, f' and $\tilde{f}$. Consider the subsets of Y $A = \{y \in Y : f'(y) = \tilde{f}(y)\}$ and $B = \{y \in Y : f'(y) \neq \tilde{f}(y)\}$. $y_0 \in A$ by definition of continuous functions between pointed spaces, so $A \neq \emptyset$. Therefore, since Y is connected, A and B both being open implies that $B = \emptyset$, or that $f' = \tilde{f}$. Take an arbitrary point $a \in A$. Pick $U_{f(a)}$ as an admissible neighborhood of $f(a)$, or a neighborhood such that $U_{f(a)}$ is evenly covered by p. Let $S_{U_{f(a)}}$ be the sheet over $U_{f(a)}$ that contains $f'(a)$. $W = f'^{-1}(S_{U_{f(a)}}) \cap \tilde{f}^{-1}(S_{U_{f(a)}})$ is an open neighborhood of a in Y. $W \subset A$, because for $w \in W$, $f'(w)$ and $\tilde{f}(w)$ both lie in the same sheet, $S_{U_{f(a)}}$. Because of the defining nature of the two functions, $p \circ f'(w) = f(w) = p \circ \tilde{f}$. As $p|_{S_{U_{f(a)}}}$ is a homeomorphism, cancellation applies, yielding, $f'(w) = \tilde{f}(w)$. Thus $w \in A$, which means $W \subset A$. Since A is exactly the union of all these W, A is open.

Now take an arbitrary point $b \in B$. $f'(b)$ and $\tilde{f}(b)$ cannot both be in the same sheet over a neighborhood U of $f(b)$, or else $f'(b) = \tilde{f}(b)$ by the argument above. Denote their two sheets S' and $\tilde{S}$, respectively Let $W = f'^{-1}(S') \cap \tilde{f}^{-1}(\tilde{S})$. W is clearly open, and it is non-empty, as $b \in W$. Let $w \in W$. Since $w \in f'^{-1}(S') \cap \tilde{f}^{-1}(\tilde{S})$, $f'(w) \in S'$ and $\tilde{f}(w) \in \tilde{S}$. As the sheets are disjoint, $f'(w) \neq \tilde{f}(w)$, so $w \in B$, which implies that $W \subset B$. Thus, B is open by the same argument as earlier. This shows that $B = \emptyset$, otherwise A and B would constitute a separation of Y.

□

Definition 3.4. A *lebesgue number* $\delta \in \mathbb{R}$ of a metric space X with open cover $\mathcal{O}$ is a number such that if $U \subset X$ with $\text{diam}(U) < \delta$, then there exists $O \in \mathcal{O}$ so that $U \subset O$.

Proposition 3.3. Every compact metric space (X, d) has a lebesgue number for any open cover.

pf. Let $\mathcal{O}$ be an open cover of the metric space. Choose a finite subcover $\mathcal{O}' = \{O_1, \ldots O_n\}$. For each $i \in 1 \ldots n$, define $C_i := X - O_i$. Define a function $f : X \to \mathbb{R}$ by $f(x) := \frac{1}{n} \sum_{i=1}^{n} d(x, C_i)$. f is continuous and its domain is compact, so it attains a minimum, δ. $\delta > 0$ as $d(x, C_i) > 0$ for some i. If $U \subset X$ has diameter less than δ, then U is contained in an open ball $B(x_0, \delta)$. But $d(x_0, C_j) \geq \delta$ for some j, so $U \subset B(x_0, \delta) \subset O_j$.

□

Proposition 3.4. (Lifting Lemma) Let (X, x_0) be a pointed space, and let $(\tilde{X}, p)$ be a covering space of X. Pick a point $\tilde{x_0}$ in the fiber over x_0. Then if Cat is the subcategory of $\mathsf{Top}_*/(X, x_0)$ where objects take the form $((\tilde{X}, \tilde{x_0}), p)$ or $((I, 0), f)$, $((\tilde{X}, \tilde{x_0}), p)$ is final in Cat.

pf. By Lemma 3.2, there is at most one morphism from objects of the form $((I, 0) f)$ to $((\tilde{X}, \tilde{x_0}), p)$. And as far as morphisms $f : ((\tilde{X}, \tilde{x_0}), p) \longrightarrow ((\tilde{X}, \tilde{x_0}), p)$ go it is clear that the only morphism possible is the identity. Thus, we must only show that a morphism $\tilde{f} : ((I, 0) f) \longrightarrow ((\tilde{X}, \tilde{x_0}), p)$ exists.
Let $\mathcal{O}$ be a covering of X by evenly covered open sets. Define $f^{-1}(\mathcal{O}) := \{f^{-1}(O) : O \in \mathcal{O}\}$. I is a compact metric space, so by proposition 3.2, it has a lebesgue number δ for $f^{-1}(\mathcal{O})$. Pick $n \in \mathbb{N}$ such that $\frac{1}{n} < \delta$. Partition I into sets $\{I_1, I_2, \ldots I_n\} := \{[0, \frac{1}{n}], [\frac{1}{n}, \frac{2}{n}], \ldots, [\frac{n-1}{n}, 1]\}$. Each I_i has diameter $\frac{1}{n} < \delta$, so $f(I_i)$ lies in some O evenly covered by p. Then $p|_{I_i}$ is a homeomorphism, so $p^{-1} \circ f : I_i \longrightarrow \tilde{X}$ is defined. Let $\tilde{f}$ be the pasting of these functions, where $\tilde{f}(0) = \tilde{x}_0$, which we can do as $p(\tilde{x}) = f(0) = x_0$. This completes the proof.

□

In the case of the circle, the lifted map can be thought of as a "flattened" version of the loop. First we flatten a loop via the lift, and then we use it to wrap around the circle a specified amount by using the projection map. When we look at the fundamental group of the circle, we will use the fact that if two "flattened loops" begin at the same point and end at different points, then they must be in different classes. In fact, we will use the fact that lifts are unique to define an isomorphism between the endpoints of the lift and the fundamental group of the circle.

Proposition 3.5 Homotopies can likewise be lifted.

pf. The proof is similar, however we bound the size of subsets of $I \times I$ instead of I. Pick the same covering, yielding the same lebesgue number δ. Let n be large enough that $\frac{\sqrt{2}}{n} < \delta$. Subdivide the unit square into n^2 even squares. Then each square has diameter $\frac{\sqrt{2}}{n}$. The rest of the argument proceeds the same from here. Thus, we must only make sure that the lifted map remains a homotopy. Since a homotopy can be thought of as a family of maps f_t with $f_t(0) = x_0$ and $f_t(1) = x_1$, the lifted homotopy can be thought of as a family of maps $\tilde{f}_t$, and by definition $\tilde{f}_t(0) = 0$. It thus suffices to show that $\tilde{f}_t(1)$ is constant for any t. Now $\tilde{f}_t(1) \in p^{-1}(x_1)$ (where p is the covering map) and $\{\tilde{f}_t(1) | t \in [0,1]\}$ is connected, which proves the result.

□

Corollary 3.6. Suppose f and f' are two paths in X. If $f \simeq f' \,\mathrm{rel}\{0,1\}$, then $\tilde{f} \simeq \tilde{f}' \,\mathrm{rel}\{0,1\}$

As the lifted map from the unit interval to the real line can be thought of as a path, it makes sense to think of path homotopies between lifted maps. However, any path is homotopic to any other in $\mathbb{R}$, provided they have the same endpoints. Thus, what the preceding proposition tells us with respect to the covering space $\mathbb{R}$ in particular is that the two lifted maps of homotopic loops have the same endpoint, which should match our intuition: if they "wrap around the circle" the same amount, they should "flatten" to the same length.

Lemma 3.7. Let H be any group. If G is a cyclic group with generator g_0, and $\phi : H \longrightarrow G$ is a homomorphism such that $\phi(h) = g_0$ for some $h \in H$, then ϕ is surjective.

pf. Let $g \in G$. $g = g_0^n$ for $n \in \mathbb{Z}$, as G is cyclic. Then $g = g_0^n = \phi(h)^n = \phi(h^n)$, which shows that ϕ is surjective.

□

Proposition 3.8. $\pi_1(S^1, 1) \simeq \mathbb{Z}$.

pf. We use the same notation for functions as in Lemma 3.1. Consider a path $f : I \longrightarrow S^1$. Let $\phi : \pi_1(S^1, 1) \longrightarrow \mathbb{Z}$ be defined by $[f] \mapsto \tilde{f}(1)$, where $\tilde{f}$ is the unique path lifting to $(\mathbb{R}, 0)$. This map is well defined, because $p \circ \tilde{f}(1) = f(1) = 1$, and $p(x) = 1 \iff x \in \mathbb{Z}$. Furthermore, if f and f' are two maps in the same class, then there must be a homotopy between them rel ∂I, by corollary 3.6. Thus, they have the same endpoint at all times, showing that the map is well defined.
Now suppose that$[f]$ and $[g]$ are two classes of maps. Define q by

$$q(x) := \begin{cases} \tilde{f}(2x) & \text{if } x \in [0, \frac{1}{2}] \\ \tilde{g}(2x-1) + \tilde{f}(1) & \text{if } x \in [\frac{1}{2}, 1] \end{cases}$$

First note that this map is well defined. Now, $q(0) = \tilde{f}(0) = 0$, and furthermore

$$p \circ q = \begin{cases} e^{2\pi i x} \circ \tilde{f}(2x) = f(2x) & \text{if } x \in [0, \frac{1}{2}] \\ e^{2\pi i x} \circ (\tilde{f}'(2x-1) + \tilde{f}(1)) = g(2x-1) \cdot e^{\tilde{f}(1)2\pi i} = g(2x-1) & \text{if } x \in [\frac{1}{2}, 1] \end{cases}$$

Thus, $p \circ q = f * g$, which shows that $q = \widetilde{f * g}$. Note that $q(1) = \tilde{g}(1) + \tilde{f}(1)$. This shows that ϕ is a group homomorphism. Now let f_0 be the path given simply by $f_0(x) := e^{2\pi i x}$. Then $\tilde{f_0}$ is simply the inclusion into $\mathbb{R}$. Thus, $\tilde{f_0}(1) = 1$. Since $\mathbb{Z}$ is the cyclic group generated by 1, by Lemma 3.7, ϕ is surjective. Finally, let f_1 be any map such that $\tilde{f_1}(1) = 0$. Then $\tilde{f_1} \simeq c_0$, where c_0 is the constant map at 0. Then $f = p \circ \tilde{f_1} \simeq p \circ c_0 = c_1$. Thus, f is homotopic to the constant map at 1, which is the identity element of $\pi_1(S^1)$. Thus, $\ker(\phi) = [c_1]$, so ϕ is injective. This completes the proof.

□

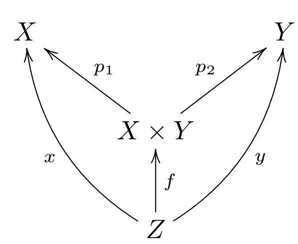
Lemma 3.9 If two objects A and B are initial in a category C, then $A \simeq B$.

pf. Because A and B are initial, their automorphism sets are singletons, so the only automorphisms of A and B are 1_A and 1_B. Furthermore there is only one morphism $f : A \longrightarrow B$ and one morphism $g : B \longrightarrow A$. Composition must be defined, so the only choice for $f \circ g$ is 1_B and the only choice for $g \circ f$ is 1_A. Thus, $g = f^{-1}$ and f is an isomorphism.

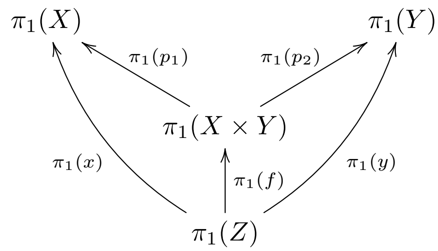
□

Proposition 3.10 Let X and Y be two path connected spaces. $\pi_1(X \times Y) \simeq \pi_1(X) \times \pi_1(Y)$.

pf. The cartesian product of two spaces is just the product in the category Top defined by the fact that for any Z, f exists and is uniquely determined by x and y. This is the same as requiring that $(X \times Y, p_1, p_2)$ be final in the slice category to the two spaces (X, Y) (this is called the *universal property* of the product).



We can apply the functor π_1 to see that $\pi_1(X \times Y)$ is also final in the approriate slice category over Group (as the limit operation commutes with the hom functor $\mathrm{Hom}(I, -)$:



Thus, $\pi_1(X \times Y)$ is the product of $\pi_1(X)$ and $\pi_1(Y)$, and in Group the product is simply the direct product $\pi_1(X) \times \pi_1(Y)$. Thus, since $\pi_1(X \times Y)$ and $\pi_1(X) \times \pi_1(Y)$ are both final in the appropriate slice category, by Lemma 3.9, $\pi_1(X \times Y) \simeq \pi_1(X) \times \pi_1(Y)$.

$\square$

Corollary 3.11. The fundamental group of the torus, $S^1 \times S^1$, is $\mathbb{Z} \oplus \mathbb{Z}$ (where we write $\oplus$ instead of $\times$ since $\mathbb{Z}$ is additive).

$\square$

We have now achieved calculation of non-trivial fundamental groups for two distinct spaces!

4. The van Kampen Theorem

We state and prove the van Kampen Theorem categorically, as in *A Concise Course in Algebraic Topology*. First, we must develop our category theory a little further.

Definition 4.1 Let C and C' be categories and let $F : \mathsf{C} \longrightarrow \mathsf{C}'$ be a covariant functor. We call F a *diagram* in C' (specifically, a C shaped diagram in C').

Definition 4.2 Let C be a category. Let F be a C shaped diagram in C'. Let $A \in \text{Obj}(\mathsf{C}')$. A *co-cone* from F to A is a collection of morphisms $\{f_i\}$ indexed by $\text{Obj}(\mathsf{Cat})$ such that

i.) for every $B \in \text{Obj}(\mathsf{C})$, there is exactly one $f_B : F(B) \longrightarrow A$.
ii.) for any morphism $F(g) : F(B) \longrightarrow F(B')$ in F, $F(g) \circ f_B = f_{B'}$.

Definition 4.3 A *colimit* of a diagram F in a category C is simply an object that is initial in the category of co-cones from F.

Definition 4.4 The *fundamental groupoid* of a space X (denoted $\Pi(X)$) is simply a category having objects elements of X and morphisms between x and y homotopy equivalence classes of maps $I \to X$ such that $0 \mapsto x$ and $1 \mapsto y$.

Proposition 4.1. (First van Kampen Theorem) Let $\mathcal{O}$ be a cover of a connected space X by path connected open subsets. Let $\mathcal{O}$ be closed under finite intersection. We can look at $\mathcal{O}$ as a category with morphisms simply inclusions between subsets. Then $\Pi(X) \simeq \text{colim}_{O_i \in \mathcal{O}}(\Pi(O_i))$.

pf. Let $\mathcal{O} = \{O_i\}_{i \in I}$ be the desired open cover. We must show that $\Pi(X) \simeq \text{colim}(\Pi|_{\mathcal{O}} : \mathcal{O} \longrightarrow \mathsf{Groupoid})$. Let grpoid be a groupoid. Suppose there is

a co-cone from $\mathcal{O}$ to grpoid denoted $(\mathsf{grpoid}, \{\phi_i\}_{i\in I})$ where each ϕ_j is a morphism from O_j to grpoid. It is clear that $\Pi(X)$ with the morphisms induced by the necessary inclusion functors is a co-cone from $\mathcal{O}$. Thus, we must show there is a unique map $\Phi : \Pi(X) \longrightarrow \mathsf{grpoid}$ such that the whole diagram commutes. This is simply checking the universal property of the colimit.
Keep in mind that the objects of the groupoids are simply the points and the morphisms are the homotopy classes of paths. Let $x \in O_i$ simply define $\Phi(x) = \phi_i(x)$. Note that this map is well defined, as the cone to grpoid respects inclusions and the cover is closed under finite intersections, so if $x \in O_k \cap O_j = O_l$, $\phi_l(x) = \phi_j \circ i_{l\to j} = \phi_j \circ i_{l\to k}$, which means that these two maps must agree on values. By the the same general reasoning, $\Phi([f]) := \phi_j[f]$ is a perfectly good definition, even if a path in $\Pi(X)$ does not lie in one member of the open cover, as we can break it up into finitely many paths that are contained in one member (as a path is a map from a compact space), and these paths must agree on the intersection of the elements of the open cover. If we differ from these definitions, the maps do not compose, so this must be the unique map.
Furthermore, if two maps are homotopic to one another, we can break up the unit square to construct homotopies of paths in the elements of the open cover, so Φ is well defined and unique.

$\square$

Lemma: (van Kampen Theorem finite version) Let $\mathcal{O}$ be a finite cover of a connected space X by path connected open subsets, with $x \in O_i$, $\forall O_i \in \mathcal{O}$. Let $\mathcal{O}$ be closed under finite intersection. We can look at $\mathcal{O}$ as a category with morphisms simply inclusions between subsets. Then $\pi_1(X, x) \simeq \mathrm{colim}_{O_i\in\mathcal{O}}(\pi_1(O_i, x))$.
pf. Let $\mathcal{F}$ be a finite open cover of a space X. Let G be a group and (G, ϕ_i) be a cone to G, with homomorphisms $\phi_i : \pi_1(F_i) \longrightarrow G$. We must show $\exists!$ homomorphism $\Phi : \pi_1(X, x) \longrightarrow G$ such that $\Phi : \pi_1(F_i) \longrightarrow G = \phi_i$ for all $i \in I$ (where $\imath_i : \pi_1(F_i) \longrightarrow \pi_1(X)$ is the standard inclusion map). Because $\pi_1(X, x)$ is a skeleton of the category $\Pi(X)$, the inclusion functor $J : \pi_1(X, x) \longrightarrow \Pi(X)$ is an equivalence of categories. We can define an inverse equivalence $H : \Pi(X) \longrightarrow \pi_1(X, x)$ by picking for each $y \in X$, a path class $[f]_y$ where $[f]_y(0) = x$ and $[f]_y(1) = y$. Then, we create a map $\alpha : \hom_{\Pi(X,x)}(x, y) \longrightarrow \hom_{\pi_1(X,x)}(x, x)$ defined by $\alpha([g]) = [g] * [f^{-1}]_y$ this is clearly full and faithful, so picking an alpha for each $y \in X$ is an equivalence of categories. For $y = x$, we pick $[f]_y = c_x$, to ensure that $H \circ J = 1_{\pi_1(X,x)}$.
Furthermore, since $\mathcal{F}$ is finite and closed under finite intersections, for $y \in X$, we take the intersection over all F_i that contain y. Since x is in all of these and each F is path connected, we can pick a path $[g]$ that is completely contained in all F containing y. Let $[f]_y := [g]$. This ensures that $H|_{F_i}$ is an inverse equivalence to the inclusion $J_{F_i} : \pi_1(F_i, x) \longrightarrow \Pi(F_i, x)$. Thus, the following family of maps yields a co-cone to G from $\mathcal{F}$.

$$\Pi(F_i, x) \xrightarrow{H|_{F_i}} \pi_1(F_i, x) \xrightarrow{\phi_i} G$$

Note because each of these is an inverse equivalence to J_{F_i}, these maps restrict appropriately under conditions of Proposition 4.1, and they yield a commutative diagram when the co-cone is constructed, so we can use Proposition 4.1 to obtain a map $\zeta : \Pi(X,x) \longrightarrow G$ that restricts to $\phi_i \circ H|_{F_i}$ on each F_i. We let $\Phi := \zeta \circ J$. First off, this makes sense, as $\zeta \circ J$ is a functor from the fundamental group to the group G, so it is a homomorphism. Secondly, we can see that $\Phi \circ \imath_i$ is given simply by:

$$\pi_1(F_i,x) \xrightarrow{\imath_i} \pi_1(X,x) \xrightarrow{J} \Pi(X,x) \xrightarrow{\zeta} G$$

And noting that $J \circ \imath_i = J_{F_i}$, we obtain an equivalent diagram::

$$\pi_1(F_i,x) \xrightarrow{J_{F_i}} \Pi(X,x) \xrightarrow{\zeta} G$$

And if we restrict completely to F_i, we see that this is equivalent to the following diagram (since ζ restricts to $\phi_i \circ H|_{F_i}$):

$$\pi_1(F_i,x) \xrightarrow{J_{F_i}} \Pi(F_i,x) \xrightarrow{H|_{F_i}} \pi_1(F_i,x) \xrightarrow{\phi_i} G$$

But $H|_{F_i} \circ J_{F_i} = 1_{F_i}$, so the composition of this diagram yields simply ϕ_i, which is what we wanted to verify. Finally, Φ is unique due to the uniqueness of ζ.

□

Lemma: (van Kampen Theorem) Let $\mathcal{O}$ be a cover of a connected space X by path connected open subsets, with $x \in O_i$, $\forall O_i \in \mathcal{O}$. Let $\mathcal{O}$ be closed under finite intersection. We can look at $\mathcal{O}$ as a category with morphisms simply inclusions between subsets. Then $\pi_1(X,x) \simeq \operatorname{colim}_{O_i \in \mathcal{O}}(\pi_1(O_i,x))$.

pf. Let $\mathcal{O}$ be an open cover of a space X. Let $\mathcal{F}$ be the set of finite subsets of $\mathcal{O}$ that are closed under finite intersections. For $F \in \mathcal{F}$, let $U_F := \bigcup_{U \in F} U$. Note that we can apply Lemma 4.2 to the set U_F with the cover F. So $\operatorname{colim}_{U \in F}(\pi_1(U,x)) \simeq \pi_1(U_F,x)$. We can regard $\mathcal{F}$ as a category with morphisms only the identity and inclusions, as above. We claim that $\operatorname{colim}_{F \in \mathcal{F}}(\pi_1(U_F,x) \simeq \pi_1(X,x)$. We can use a similar construction as before to show that any loop or homotopy in (X,x) gives rise to a corresponding loop or homotopy in some (U_F,x), and then the universal property is satisfied (with the obvious inclusions as the maps in the co-cone to $\pi_1(X,x)$). This shows the isomorphism claimed above. Now we claim that $\operatorname{colim}_{U \in \mathcal{O}}(\pi_1(U,x)) \simeq \operatorname{colim}_{F \in \mathcal{F}}(\pi_1(U_F,x)$. Looking closer, we can substitute the fundamental groups on the right for colimits, yielding $\operatorname{colim}_{F \in \mathcal{F}}(\pi_1(U_F,x) \simeq A := \operatorname{colim}_{F \in \mathcal{F}}(\operatorname{colim}_{U \in F}(\pi_1(U,x))$. Now let us look at $B := \operatorname{colim}_{(U,F) \in (\mathcal{O},\mathcal{F})}(\pi_1(U,x))$, where (U,F) is an object of $(\mathcal{O},\mathcal{F})$ if $U \in \mathcal{O}$ and $U \in F$, and there is a morphism $(U,F) \longrightarrow (U',F')$ if $U \subset U'$ and $F \subset F'$. If there is a unique homomorphism $\phi : B \longrightarrow G$ for some group G that restricts appropriately, we can use this information to construct a unique homomorphism $\zeta : A \longrightarrow G$, because ϕ commutes with respect to pairs of inclusions, whereas ζ would commute first among a U shaped diagram, and then an F shaped diagram. Thus, we can simply let $\zeta = \phi$, showing that A and B are really the same colimit. But it should be obvious that $\operatorname{colim}_{U \in \mathcal{O}}(\pi_1(U,x)) \simeq \operatorname{colim}_{(U,F) \in (\mathcal{O},\mathcal{F})}(\pi_1(U,x))$, as they are the same diagram, but whereas if $\pi_1(U,x) \longrightarrow \pi_1(U',x)$ appears once in the colimit on the left, it must appear at least once in the colimit on the left (because $\mathcal{F}$ covers $\mathcal{O}$). However, that morphism could appear more than once on the right, due to it being included in multiple F. It should be obvious, however, that this multiplicity will not contribute to the colimit.

□

Proposition 4.4 Let X be the one point union of path connected spaces X_i where each of the X_i has a contractible neighborhood V_i of the base point. Then $\pi_1(X, x)$ is isomorphic to the free product of the $\pi_1(X_i, x)$.

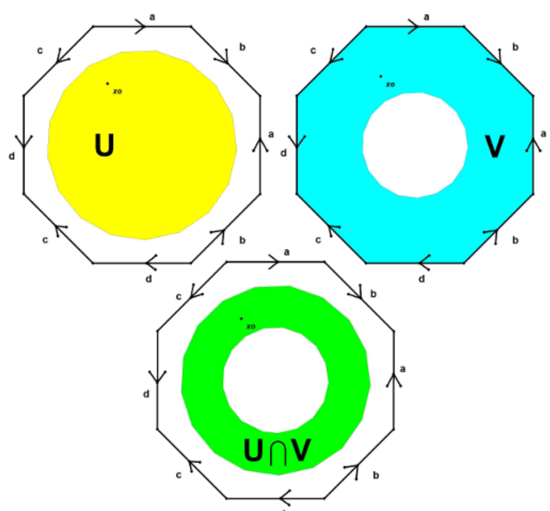
□

pf. Let F_i be the union of X_i and all of the V_j for $i \neq j$. Let $\mathcal{O}$ be the F_i and all finite intersections of F_i. Apply the van Kampen theorem to show that $\pi_1(X, x)$ is the free product with amalgamation over the $\pi_1(X_i, x)$. However, each morphism from the intersections is trivial, as the intersections are all contractible, so no relation is added to the free product, which proves the proposition.

Corollary 4.5 The fundamental group of n circles connected at a point is the free group on n generators.

Calculation 4.6 The fundamental group of the double torus $T\#T$.

calc. Decompose the double torus like so (image credit Christopher Walken):



$\pi_1(U) = \{1\}$, and $\pi_1(U \cap V) = \mathbb{Z}$. We must calculate $\pi_1(V)$. By collapsing the edges to each other, it becomes apparent that V deformation retracts to a bouquet of 4 circles, so $\pi_1(V)$ is the free group on four letters. Now we must calculate the kernels of the homomorphisms induced by inclusions from $U \cap V$ into U and V. Any homomorphism to $\pi_1(U)$ can only be trivial, so the kernel is likewise trivial. For the other inclusion, we simply go around the edges, starting at a, to obtain $\pi_1(1)(0) = aba^{-1}b^{-1}cdc^{-1}d^{-1}$, so let N be the normal subgroup generated by that element. By the theorem, the fundamental group of the whole space is simply $(\mathbb{Z} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z} * \{1\})/N$.

□

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