THE ZETA FUNCTION AND THE RIEMANN HYPOTHESIS

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ABSTRACT. The zeta function has been studied for centuries but mathematicians are still learning about it. In this paper, I will discuss some of the zeta function's properties and introduce the Riemann hypothesis, an important open question.

CONTENTS

1. HISTORY

The zeta function, usually referred to as the Riemann zeta function today, has been studied in many different forms for centuries. The harmonic series, $\zeta(1)$, was proven to be divergent as far back as the 14th century [1]. In the 18th century, the Swiss mathematician Leonhard Euler found a closed form expression for the sum of the reciprocals of the squared integers i.e. $\zeta(2)$. He also generalized this result and found a closed form expression for $\zeta(2n)$ for $n \in \mathbb{N}$ [2]. In the 19th century, the German mathematician Bernhard Riemann considered ζ as a complex function. He published his work in the 1859 paper "On the Number of Primes Less Than a Given Magnitude", which is one of the most influential works of modern mathematics [5]. In this paper, he conjectured that all non-trivial zeros of ζ have real part $\frac{1}{2}$, a conjecture that has become known as the Riemann hypothesis and whose proof is perhaps the most sought after in all of mathematics. The Riemann hypothesis remains unproven and its resolution, either in the negative or affirmative, would have enormous consequences across many areas of mathematics and science [3].

2. DEFINITION AND BASIC PROPERTIES

Definition (Zeta function). $\zeta(x) = \sum\limits_{n=1}^{\infty} \frac{1}{n^x} = \sum\limits_{n=1}^{\infty} n^{-x}$

In particular, $\zeta(1)$ is the harmonic series and $\zeta(2)$ is the subject of the Basel problem. Before considering the domain of ζ, note that if x is in $\mathbb{N}$ and a, b are in

$\mathbb{R}$, and $a < b$, then $x^a < x^b$. Therefore, for any i in $\mathbb{N}$, the i-th term in $\sum_{n=1}^{\infty} \frac{1}{n^a} = \zeta(a)$ is greater than the i-th term in $\sum_{n=1}^{\infty} \frac{1}{n^b} = \zeta(b)$, so if both sums converge, the former is greater than the latter. Therefore, in $\mathbb{R}$, ζ is decreasing wherever it is defined.

Proposition (Domain of ζ)**.** *The domain of ζ, considered as a complex function, is $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > 1\}$.*

Proof. First, we consider ζ as a real-valued function. We use the integral test for convergence of infinite sums, which tells us that $\sum_{n=1}^{\infty} \frac{1}{n^x} = \zeta(x)$ converges if and only if $\int_1^{\infty} \frac{dn}{n^x}$ exists. But $\int_1^{\infty} \frac{dn}{n^x}$ exists if and only if $x > 1$ so the domain of ζ is $\{x \in \mathbb{R} \mid x > 1\}$. Now, we broaden our attention from $\mathbb{R}$ to $\mathbb{C}$. Consider s, an arbitrary element of $\mathbb{C}$. Then $|\frac{1}{n^s}| = |\frac{1}{e^{s \log n}}| = \frac{1}{e^{\mathrm{Re}(s) \log n}} = \frac{1}{n^{\mathrm{Re}(s)}}$. As shown previously, $\sum_{n=1}^{\infty} \frac{1}{n^{\mathrm{Re}(s)}}$ converges if and only if $\mathrm{Re}(s) > 1$ so the domain of ζ, considered now as a complex function, is $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > 1\}$. □

3. Euler product

Euler gave this identity:

$$\zeta(s) = \prod_{k=1}^{\infty} \frac{1}{1 - p_k^{-s}}$$

where p_k is the k-th prime number.

Proof. $\prod_{k=1}^{\infty} \frac{1}{1-p_k^{-x}} = \prod_{k=1}^{\infty} \left(\sum_{n=0}^{\infty} \left(\frac{1}{p_k^x} \right)^n \right)$

$= \left(1 + \frac{1}{p_1^x} + \frac{1}{p_1^{2x}} + \frac{1}{p_1^{3x}} + \dots \right) \left(1 + \frac{1}{p_2^x} + \frac{1}{p_2^{2x}} + \frac{1}{p_2^{3x}} + \dots \right) \dots$

$= 1 + \sum_{i=1}^{\infty} \frac{1}{p_i^x} + \sum_{i,j=1}^{\infty} \frac{1}{p_i^x p_j^x} + \sum_{i,j,k=1}^{\infty} \frac{1}{p_i^x p_j^x p_k^x} + \dots = 1 + \frac{1}{2^x} + \frac{1}{3^x} + \dots = \zeta(x)$ □

Using this identity, we prove the infinitude of the prime numbers.

Proof. Note that $\zeta(1)$ is the harmonic series. So $\zeta(1) = 1 + \frac{1}{2} + \frac{1}{3} + \dots = \prod_{k=1}^{\infty} \frac{1}{1-\frac{1}{p_k}}$ but we know that the harmonic series diverges, so this product must also diverge. But for a product of positive numbers to diverge, the product must have an infinite number of terms, so we conclude that there are an infinite number of prime numbers. □

While the Euler product is beautiful and surprising, it is difficult to evaluate an infinite product over all prime numbers, so mathematicians typically study ζ using the summation definition and related tools, and use this to learn about the Euler product formula.

4. Evaluating zeta at particular points

Now we find some values of ζ. Numerical evaluation with a computer would be straightforward but finding exact values is more involved. First we introduce the Weierstrass factorization theorem, which will be useful for evaluating ζ.

Theorem (Weierstrass Factorization Theorem)**.** *Let f be an entire function and let $\{a_n\}$ be the non-zero zeros of f repeated according to multiplicity; suppose f has a zero at $z = 0$ of order $m \geq 0$. Then there is an entire function g and a sequence of integers $\{p_n\}$ such that*

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right)$$

[4, p170]

Example. *Sin is entire and has a zero at $z = 0$ of order 1 so we can apply the previous theorem. Doing so gives*

$$sin(\pi z) = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) \tag{4.1}$$

[4, p175]
Now we evaluate $\zeta(2)$ by using two different formulas for $\frac{\sin(x)}{x}$.

Proposition. $\zeta(2) = \frac{\pi^2}{6}$.

Proof. By the Taylor series of sin,

$$\frac{\sin(x)}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots$$

and by equation (4.1)

$$\frac{\sin(x)}{x} = \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right) = \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\dots$$

By expanding the product and equating coefficients for the x^2 term, we see

$$-\frac{1}{6} = -\frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Therefore, $\zeta(2) = \frac{\pi^2}{6}$. □

Continuing with this approach by equating coefficients of higher degree terms, as Euler did, gives

$$\zeta(2n) = \frac{|B_{2n}| 2^{2n-1} \pi^{2n}}{(2n)!} \text{ for } n \in \mathbb{N}$$

[5] where B_x denotes the x-th Bernoulli number, defined by

$$B_x = \sum_{k=0}^{x} \frac{1}{k+1} \sum_{r=0}^{k} (-1)^r \binom{k}{r} r^x$$

[9]. No closed form expression is known for the odd positive integer values of ζ, but some approximate values are given in the following table.

Table 1. Some values of ζ

x	$\zeta(x)$
2	$\frac{\pi^2}{6} = 1.6449340\ldots$
3	$1.2020569\ldots$
4	$\frac{\pi^4}{90} = 1.0823232\ldots$
5	$1.0369277\ldots$
6	$\frac{\pi^6}{945} = 1.0173430\ldots$

5. Analytic continuation

Definition (Analytic continuation)**.** *Suppose f is an analytic function on $U \subset \mathbb{C}$ and g is an analytic function on $V \subset \mathbb{C}$ where $V \supset U$ and V and U are open. Suppose further that $g(z) = f(z)$ for all $z \in U$. Then g is an analytic continuation of f.*

Intuitively, an analytic continuation of a function has a larger domain than the original function but agrees with the original function on the original domain. A powerful theorem of complex analysis states that if an analytic continuation exists, it is unique [6, p159]. As discussed earlier, the domain of the zeta function as defined above is $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > 1\}$. Now we will find the analytic continuation to $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > 0, s \neq 1\}$.

Definition (Alternating zeta function)**.** $Z(s) = \sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \ldots$

Proposition (Expanding the domain of ζ to $\mathrm{Re}(s) > 0, s \neq 1$)**.** $\frac{Z(s)}{1-2^{1-s}}$ *is an analytic continuation of ζ to $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > 0 \wedge s \neq 1\}$.*

Proof. If $\mathrm{Re}(s) > 1, \zeta(s) = Z(s) + 2\left(\frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \ldots\right) = Z(s) + \frac{\zeta(s)}{2^{s-1}}$. Solving for ζ gives $\zeta(s) = \frac{Z(s)}{1-2^{1-s}}$ [7]. But $Z(s)$ is analytic and converges for $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > 0\}$ so this new formula for ζ is an analytic continuation to $\{s \in \mathbb{C} \mid s \neq 1\}$. ☐

Using further techniques of complex analysis, ζ has been analytically continued to $\mathbb{C} \setminus \{1\}$ and the following functional equation has been proven [6, p373].

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s)$$

From this functional equation, it can be seen that $\zeta(-2n) = 0$ for all $n \in \mathbb{N}$ because *sin* vanishes and all other factors exist. The zeros at negative even integers are called trivial zeros [6, p375]. The analytic continuation of ζ gives surprising results such as $\zeta(0) = -\frac{1}{2}$ and $\zeta(-1) = -\frac{1}{12}$. $\zeta(-1)$, by the original definition of ζ, would be $\sum\limits_{n=1}^{\infty} \frac{1}{n^{-1}} = \sum\limits_{n=1}^{\infty} n = 1 + 2 + 3 + \ldots$ but this sum diverges, of course. -1 is not in the domain of ζ as originally defined but it is in the domain of the analytic continuation so evaluating $\zeta(-1)$ is not the same as evaluating the sum.

6. Riemann hypothesis

The Riemann Hypothesis is perhaps the greatest open question in mathematics. The claim is simple: all non-trivial zeros of ζ have real part of $\frac{1}{2}$ [6, p375]. This hypothesis, if found to be true, would have many powerful consequences, especially with regards to the distribution of prime numbers. Also, many theorems have been proven with the assumption that the Riemann hypothesis is true, so a proof of the hypothesis would validate the proofs of these theorems. For more information on the consequences of the Riemann hypothesis, see [3] and [8].

Acknowledgments. It is a pleasure to thank Professor May and my mentors, Yun Cheng and Preston Wake, for their input and guidance.

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