CONVERGENCE OF FOURIER SERIES IN L^p SPACE

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Abstract. The convergence of Fourier series of trigonometric functions is easy to see, but the same question for general functions is not simple to answer. We study the convergence of Fourier series in L^p spaces. This result gives us a criterion that determines whether certain partial differential equations have solutions or not.We will follow closely the ideas from Schlag and Muscalu's Classical and Multilinear Harmonic Analysis.

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1. Fourier Series, Partial Sums, and Dirichlet Kernel

Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the one-dimensional torus (in other words, the circle). We consider various function spaces on the torus $\mathbb{T}$, namely the space of continuous functions $C(\mathbb{T})$, the space of Hölder continuous functions $C^\alpha(\mathbb{T})$ where $0 < \alpha \leq 1$, and the Lebesgue spaces $L^p(\mathbb{T})$ where $1 \leq p \leq \infty$. Let f be an L^1 function. Then the associated Fourier series of f is

$$f(x) \sim \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n x} \tag{1.1}$$

and

$$\hat{f}(n) = \int_0^1 f(x) e^{-2\pi i n x}\, dx. \tag{1.2}$$

The symbol $\sim$ in (1.1) is formal, and simply means that the series on the right-hand side is associated with f. One interesting and important question is when f equals the right-hand side in (1.1). Note that if we start from a trigonometric polynomial

$$f(x) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n x}, \tag{1.3}$$

Date: DEADLINES: Draft AUGUST 18 and Final version AUGUST 29, 2013.

where all but finitely many a_n are zero, then

(1.4)
$$\hat{f}(n) = \int_0^1 \left(\sum_{m=-\infty}^{\infty} a_m e^{2\pi imx} \right) e^{-2\pi inx}\, dx = \sum_{m=-\infty}^{\infty} \int_0^1 a_m e^{2\pi imx} e^{-2\pi inx}\, dx.$$

We can interchange summation and integration here because only finitely many a_n's are nonzero. Note that the integral of the last term of (1.4) is 0 if $n \neq m$ and 1 if $n = m$. Thus, we get $\hat{f}(n) = a_n$ for all $n \in \mathbb{Z}$. The $\{a_n\}_{n\in\mathbb{Z}}$ are also called *Fourier coefficients.*

Therefore, we see that $f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{2\pi inx}$ when only finite many Fourier coefficents of f are nonzero. In general, we want to know when the Fourier series of a function will converge to the original function. We will specify the sense of convergence of the infinite series later in this paper.

It is natural to start from the most basic notion of convergence, namely that of pointwise convergence. The partial sums of $f \in L^1(\mathbb{T})$ are defined as

$$\begin{aligned} S_N f(x) &= \sum_{n=-N}^{N} \hat{f}(n)e^{2\pi inx} = \sum_{n=-N}^{N} \int_{\mathbb{T}} e^{-2\pi iny} f(y)\, dy\ e^{2\pi inx} \\ &= \int_{\mathbb{T}} \sum_{n=-N}^{N} e^{2\pi in(x-y)} f(y)\, dy = \int_{\mathbb{T}} D_N(x-y) f(y)\, dy, \end{aligned} \tag{1.5}$$

where $D_N(x) = \sum_{n=-N}^{N} e^{2\pi inx}$ is called the *Dirichlet kernel.* If we simplify and bound the Dirichlet kernel, we get

(1.6)
$$\begin{aligned} D_N(x) &= \sum_{n=-N}^{N} e^{2\pi inx} = \frac{e^{2\pi i(N+1)x} - e^{-2\pi iNx}}{e^{2\pi ix} - 1} \\ &= \frac{\cos(2\pi(N+1)x) - \cos(2\pi Nx) + i[\sin(2\pi(N+1)x) + \sin(2\pi Nx)]}{\cos(2\pi x) - 1 + i\sin(2\pi x)} \\ &= \frac{(\cos(2\pi x) - 1)(\cos(2\pi(N+1)x) - \cos(2\pi Nx)) + \sin(2\pi x)(\sin(2\pi(N+1)x) + \sin(2\pi Nx))}{2 - 2\cos(2\pi x)} \\ &+ \frac{i[(\cos(2\pi x) - 1)(\sin(2\pi(N+1)x) + \sin(2\pi Nx)) - \sin(2\pi x)(\cos(2\pi(N+1)x) - \cos(2\pi Nx))]}{2 - 2\cos(2\pi x)} \\ &= \frac{-2\sin((2N+1)\pi x)\sin(\pi x)(-2)\sin^2(\pi x) + \sin(2\pi x)2\sin((2N+1)\pi x)\cos(\pi x)}{4\sin^2(\pi x)} \\ &= \sin((2N+1)\pi x)\sin(\pi x) + \frac{\sin((2N+1)\pi x)\cos^2(\pi x)}{\sin(\pi x)} \\ &= \frac{\sin((2N+1)\pi x)}{\sin(\pi x)}. \end{aligned}$$

Looking at Figure 1, we see that D_N is very large at $x = 0$ and as N increases, the value at $x = 0$ will become larger. We have $D_{N\to\infty}(0) = \infty$. Also D_N is bounded by $\frac{1}{|x|}$ away from zero. When $\pi x \leq \frac{\pi}{2}$, $x \leq \frac{1}{2}$. Our previously defined domain is $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, which is the same as periodic function with period 1. Thus,

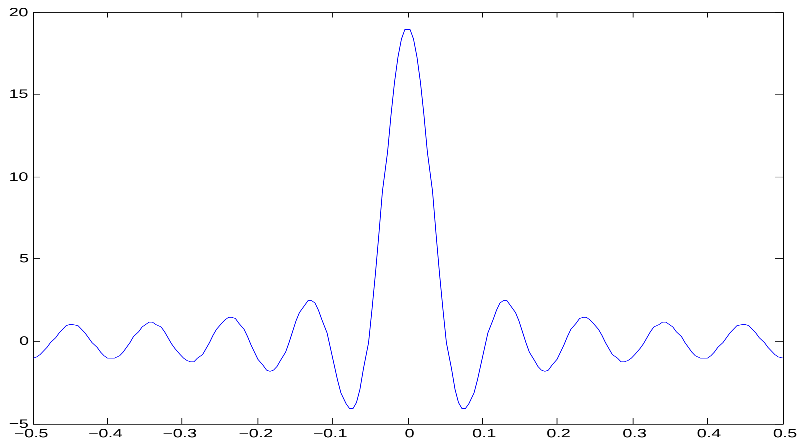

FIGURE 1. The Dirichlet kernel D_N for $N = 9$

we can translate the domain to $[\frac{-1}{2}, \frac{1}{2}]$. Now let's bound the Dirichlet kernel for each N. Since $|\sin n\alpha| \leq n|\sin\alpha|$, we know that $|D_N(x)| \leq 2N+1 \leq CN$ where C is some finite constant. Also note that when $0 < x \leq \dfrac{\pi}{2}$, $|\sin x| \geq \dfrac{2|x|}{\pi}$. Therefore, we have $|D_N(x)| = \dfrac{|\sin((2N+1)\pi x)|}{|\sin(\pi x)|} \leq \dfrac{1}{2\pi|x|/\pi} = \dfrac{1}{2|x|}$. To sum up, we have

$$|D_N(x)| \leq C \min\left(N, \frac{1}{|x|}\right), \tag{1.7}$$

for all $N \geq 1$ and some absolute constant C. Next, we try to find a bound of the L^1 norm of D_N. From (1.7) we know that

$$\begin{aligned}\int_0^1 |D_N(x)|\,dx &\leq \int_{|x|>\frac{1}{N}} \frac{1}{|x|}\,dx + \int_{|x|\leq\frac{1}{N}} N\,dx \\ &= 2\left(\log\left(\frac{1}{2}\right) - \log\left(\frac{1}{N}\right) + 1\right) \\ &\leq 2(\log N + 1) \leq C_1 \log N.\end{aligned} \tag{1.8}$$

The last inequality is true if $N \geq 2$. Also, if we only integrate the part where $|\sin(2N+1)\pi x| \geq \dfrac{1}{2}$, we get

$$\int_0^1 |D_N(x)|\,dx \geq \frac{1}{2}\int_0^1 \frac{1}{|\sin\pi x|} \geq C_2 \log N. \tag{1.9}$$

Since $|\sin\pi x| \leq 1$, we have $\frac{1}{|\sin\pi x|} \geq 1$. We are only integrating over the range of x where $|\sin(2N+1)\pi x| \geq \frac{1}{2}$. The length of this range has a $1/N$ factor. This is how the $\log N$ comes up in the last inequality above. Thus, we proved that

$$C_3^{-1} \log N \leq \|D_N\|_{L^1(\mathbb{T})} \leq C_3 \log N \tag{1.10}$$

for all $N \geq 2$ where C_3 is another absolute constant.

2. Convolution

The convolution of functions arises naturally in the discussion of the Dirichlet kernel. We define and discuss further properties of this operation here.

Definition 2.1. If f, g are two functions on $\mathbb{T}$, and both f, g are continuous, then

$$(f * g)(x) := \int_{\mathbb{T}} f(x-y)g(y)\,dy = \int_{\mathbb{T}} g(x-y)f(y)\,dy \tag{2.2}$$

is the convolution of f and g.

It is helpful to think of g or f as dirac delta functions which are generalized functions, or distributions, on the real number line that are zero everywhere except at zero, with an integral of one over the entire real line. Here we list and prove some useful lemmas about convolution.

Lemma 2.3. *(1) Let $f, g \in L^1(\mathbb{T})$. Then $\|f * g\|_r \leq \|f\|_p \|g\|_q$ for all $1 \leq r$, $p, q \leq \infty$, $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, $f \in L^p$, $g \in L^q$. This is called the Young's inequality.*
(2) For $f, g \in L^1(\mathbb{T})$ one has that for all $n \in \mathbb{Z}$, $\widehat{f * g}(n) = \hat{f}(n)\hat{g}(n)$.

Proof. (1) By Fubini's theorem, since $f(x-y)g(y)$ is jointly measurable on $\mathbb{T} \times \mathbb{T}$ and belongs to $L^1(\mathbb{T} \times \mathbb{T})$, we know that $\|f * g\|_1 \leq \|f\|_1 \|g\|_1 < \infty$. The details of the proof of the Young's inequality are shown in the appendix.

(2) is a consequence of Fubini's theorem and the homomorphism property of the exponentials $e^{2\pi in(x+y)} = e^{2\pi inx} e^{2\pi iny}$:

$$\begin{aligned} \widehat{f * g}(n) &= \int_{\mathbb{T}} \left(\int_{\mathbb{T}} f(x-y)g(y)\,dy \right) e^{-2\pi inx}\,dx \\ &= \left(\int_{\mathbb{T}} f(w)e^{-2\pi inw}\,dw \right) \left(\int_{\mathbb{T}} g(t)e^{-2\pi int}\,dt \right) \\ &= \hat{f}(n)\hat{g}(n). \end{aligned} \tag{2.4}$$

□

One of the most basic as well as oldest results on the pointwise convergence of Fourier series is the following theorem. Note that it requires a function to be Hölder continuous, instead of just continuous.

Theorem 2.5. *If $f \in C^\alpha(\mathbb{T})$ with $0 < \alpha \leq 1$, then $\|S_N f - f\|_\infty \to 0$ as $N \to \infty$.*

Proof. One has, with $\delta \in (0, \frac{1}{2})$ to be determined,

$$\begin{aligned} S_N f(x) - f(x) &= \int_0^1 (f(x-y) - f(x))D_N(y)\,dy \\ &= \int_{|y| \leq \delta} (f(x-y) - f(x))D_N(y)\,dy \\ &\quad + \int_{\frac{1}{2} > |y| > \delta} (f(x-y) - f(x))D_N(y)\,dy. \end{aligned} \tag{2.6}$$

Equation (2.6) is true because $\int_0^1 D_N(y)\,dy = 1$, since the integration of $e^{2\pi inx}$ over $[0,1]$ is 0 except when $n=0$. Also since $D_N(x)$ is periodic with period 1, integration over $[0,1]$ is the same as that over $[-\frac{1}{2},\frac{1}{2}]$. We now use the bound from (1.8), i.e.,

$$|D_N(y)| \le C\min\left(N, \frac{1}{|y|}\right). \tag{2.7}$$

Here and in what follows, C will denote a numerical constant that can change from line to line. The first integral in (2.6), which we denote by A, can be estimated as follows

$$A \le C\int_{|y|\le\delta} |f(x)-f(x-y)|\frac{1}{|y|}\,dy \le C[f]_\alpha \int_{|y|\le\delta}|y|^{\alpha-1}\,dy \le C[f]_\alpha\delta^\alpha, \tag{2.8}$$

with the usual C^α semi-norm:

$$[f]_\alpha = \sup_{x,y}\frac{|f(x)-f(x-y)|}{|y|^\alpha}. \tag{2.9}$$

To bound the second integral in (2.6), which we denote by B, we need to use the oscillation of $D_N(y)$. We can rewrite the expression as

$$\begin{aligned} B &= \int_{\frac{1}{2}>|y|>\delta}(f(x-y)-f(x))D_N(y)\,dy \\ &= \int_{\frac{1}{2}>|y|>\delta} h_x(y)\sin((2N+1)\pi y)\,dy \\ &= -\int_{\frac{1}{2}>|y|>\delta} h_x(y)\sin\left((2N+1)\pi\left(y+\frac{1}{2N+1}\right)\right)\,dy \\ &= -\int_{|y-\frac{1}{2N+1}|>\delta} h_x\left(y-\frac{1}{2N+1}\right)\sin((2N+1)\pi y)\,dy, \end{aligned} \tag{2.10}$$

where $h_x(y) = \frac{f(x-y)-f(x)}{\sin(\pi y)}$. Therefore, with all integrals being understood to be in the interval $(-\frac{1}{2},\frac{1}{2})$, we get

$$\begin{aligned} 2B &= \int_{|y|>\delta} h_x(y)\sin((2N+1)\pi y)\,dy \\ &\quad - \int_{|y-\frac{1}{2N+1}|>\delta} h_x\left(y-\frac{1}{2N+1}\right)\sin((2N+1)\pi y)\,dy \\ &= \int_{|y|>\delta}\left(h_x(y)-h_x\left(y-\frac{1}{2N+1}\right)\right)\sin((2N+1)\pi y)\,dy \\ &\quad - \int_{[-\delta,-\delta+\frac{1}{2N+1}]} h_x\left(y-\frac{1}{2N+1}\right)\sin((2N+1)\pi y)\,dy \\ &\quad + \int_{[\delta,\delta+\frac{1}{2N+1}]} h_x\left(y-\frac{1}{2N+1}\right)\sin((2N+1)\pi y)\,dy. \end{aligned} \tag{2.11}$$

We bound first the last two integrals. Note that we have

$$|h_x(z)| = \left|\frac{f(x-z)-f(x)}{\sin(\pi z)}\right| \le \frac{2\|f\|_\infty}{|\sin(\pi z)|} \le \frac{C\|f\|_\infty}{|z|}. \tag{2.12}$$

The last inequality is true by Jordan's inequality $|\sin z| \ge \frac{2|z|}{\pi}$ when $|z| \le \frac{\pi}{2}$. Provided that $\delta > \frac{2}{2N+1}$, it follows that $|z| > C|\delta|$ for all z in the intervals

$[-\delta, -\delta + \frac{1}{2N+1}]$ and $[\delta, \delta + \frac{1}{2N+1}]$. Therefore, we obtain the following inequality:

$$|h_x(z)| \leq \frac{C\|f\|_\infty}{z} \leq \frac{C\|f\|_\infty}{|\delta|}.$$

This inequality allows us to bound each of the last two inequalities in (2.11) by $\frac{C\|f\|_\infty}{\delta(2N+1)}$. Now we bound the remaining integral:

$$\begin{aligned} |h_x(y) - h_x(y+\tau)| &= \left| \frac{f(x-y) - f(x)}{\sin(\pi y)} - \frac{f(x-y-\tau) - f(x)}{\sin(\pi(y+\tau))} \right| \\ &\leq \left| \frac{f(x-y) - f(x-y-\tau)}{\sin(\pi y)} \right| \\ &\quad + \left| (f(x-y-\tau) - f(x)) \left(\frac{1}{\sin(\pi y)} - \frac{1}{\sin(\pi(y+\tau))} \right) \right| \\ &\leq \left| \frac{(f(x-y) - f(x)) - (f(x-y-\tau) - f(x))}{\sin(\pi y)} \right| \\ &\quad + \left| (f(x-y-\tau) - f(x)) \left(\frac{1}{\sin(\pi y)} - \frac{1}{\sin(\pi(y+\tau))} \right) \right|. \end{aligned} \tag{2.13}$$

If $|y| > \delta > 2\tau$. Using similar ideas, we bound the first term by

$$\frac{[f]_\alpha |\tau|^\alpha}{|y|} \leq \frac{[f]_\alpha |\tau|^\alpha}{|\delta|}.$$

Using similar arguments, we bound the second term by

$$\frac{C\|f\|_\infty \pi\tau}{\sin(\pi y)\sin(\pi(y+\tau))} \leq \frac{C'\|f\|_\infty \pi\tau}{|\delta|^2}.$$

Letting $\tau < 1/N$, we get

$$|B| \leq C\left(\frac{N^{-\alpha}[f]_\alpha}{\delta} + \frac{N^{-1}\|f\|_\infty}{\delta^2}\right). \tag{2.14}$$

Choosing $\delta = N^{-\alpha/2}$ we can conclude from (2.8) and (2.14) that

$$|(S_N f)(x) - f(x)| \leq C(N^{-\alpha^2/2} + N^{-\alpha/2} + N^{-1+\alpha}). \tag{2.15}$$

Thus, as $N \to \infty$ we have $|(S_N f)(x) - f(x)| \to 0$, which means that $\|S_N f - f\|_\infty \to 0$. □

3. Fejér kernel and Approximate identity

The Dirichlet kernel is not ideal in the sense that its L^1 norm is not uniformly bounded as shown in (1.10). Why this unboundedness of the Dirichlet kernel is a problem that will be seen later in the paper. Thus, we introduce another form of partial sum, namely the average of the partial sum operator:

$$\sigma_N f := \frac{1}{N} \sum_{n=0}^{N-1} S_n f. \tag{3.1}$$

Setting $K_N = \frac{1}{N} \sum_{n=0}^{N-1} D_n$, which is called the *Fejér kernel*, we therefore have $\sigma_N f = K_N * f$. Here we list some useful properties of the Fejér kernel.

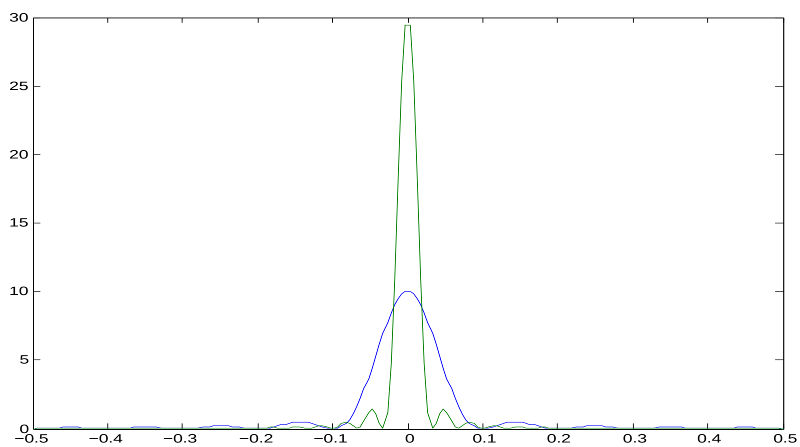

FIGURE 2. The Fejér kernel K_N for $N = 10$ (blue), and $N = 30$ (green)

Proposition 3.2. *(1)* $\hat{K}_N(n) = \left(1 - \frac{|n|}{N}\right)^+$*, where* f^+ *means the positive part of the function* f*,*
(2) $K_N(x) = \frac{1}{N}\left(\frac{\sin(N\pi x)}{\sin(\pi x)}\right)^2$*,*
(3) $0 \leq K_N(x) \leq CN^{-1}\min(N^2, x^{-2})$.

Proof. (1) By definition

$$\begin{aligned}\hat{K}_N(n) &= \int_0^1 \frac{1}{N}\sum_{m=0}^{N-1} D_m e^{-2\pi i n x}\,dx \\ &= \int_0^1 \frac{1}{N}\sum_{m=0}^{N-1}\sum_{k=-m}^{m} e^{2\pi i k x}e^{-2\pi i n x}\,dx.\end{aligned} \tag{3.3}$$

Note that $\int_{\mathbb{T}} e^{2\pi ikx}e^{-2\pi inx}\,dx = 1$ if and only if $k = n$. Property (1) follows from this fact. Using formula for summation of geometric sequence, we can get (2). By Jordan's inequality for $x \in [-\frac{1}{2}, \frac{1}{2}]$ and the inequality $|\sin nx| \leq n|\sin x|$, we get (3). □

We can see from Figure 2 that as N increases, the graph of K_N "squeezes up" around $x = 0$.The above property of K_N ensures that K_N are what we call an *approximate identity.*

Definition 3.4. We say that the family $\{\Phi_N\}_{N=1}^{\infty} \subset L^\infty(\mathbb{T})$ form an approximate identity provided
A1) $\int_0^1 \Phi_N(x)\,dx = 1$ for all N;
A2) $\sup_N \int_0^1 |\Phi_N(x)|\,dx < \infty$;
A3) for all $\delta > 0$ one has $\int_{|x|>\delta} |\Phi_N(x)|\,dx \to 0$ as $N \to 0$.

The name "approximate identity" derives from the fact that $\Phi_N * f \to f$ as $N \to \infty$ in many reasonable senses, which we will prove soon. Note that $\{D_N\}_{N\geq 1}$ are not an approximate identity since A2) does not hold. Also note that Definition 3.4 has nothing to do with the torus $\mathbb{T}$. It applies equally well to the line $\mathbb{R}$, tori $\mathbb{T}^d$, or Euclidean spaces $\mathbb{R}^d$. Next, we verify that the function K_N belong to this class.

Lemma 3.5. *The Fejér kernels $\{K_N\}_{N=1}^{\infty}$ form an approximate identity.*

Proof. Since $\int_0^1 e^{2\pi inx} = 1$ if $n = 0$ and 0 otherwise, we clearly have $\int_0^1 K_N(x)\,dx = 1$. Since $K_N(x) \geq 0$ by Proposition 3.2, we have $\int_0^1 |K_N(x)|\,dx = 1$ for all N, so A2) holds. By part (3) of Proposition 3.2 we have

$$\int_{|x|>\delta} |K_N(x)|\,dx \leq \int_{|x|>\delta} CN^{-1}x^{-2}\,dx \leq C'(\delta N)^{-1}, \tag{3.6}$$

where C and C' are constants. Then clearly as $N \to \infty$, the integral goes to 0. All of these prove the result. □

Now we establish the basic property of an approximate identity.

Proposition 3.7. *For any approximate identity $\{\Phi_N\}_{N=1}^{\infty}$ one has*
(1) If $f \in C(\mathbb{T})$, then $\|\Phi_N * f - f\|_\infty \to 0$ as $N \to \infty$.
(2) If $f \in L^p(\mathbb{T})$ where $1 \leq p < \infty$, then $\|\Phi_N * f - f\|_p \to 0$ as $N \to \infty$.

Proof. We begin with the uniform convergence. Since $\mathbb{T}$ is compact, f is uniformly continuous. Given $\epsilon > 0$, let $\delta > 0$ be such that

$$\sup_{|y|<\delta} |f(x-y) - f(x)| < \epsilon, \tag{3.8}$$

for all $x \in \mathbb{T}$. Then, by the definition of approximate identity we have

(3.9)
$$\begin{aligned} |(\Phi_N * f)(x) - f(x)| &= \left| \int_{\mathbb{T}} (f(x-y) - f(x))\Phi_N(y)\,dy \right| \\ &\leq \epsilon |f(x-y) - f(x)| \int_{\mathbb{T}} |\Phi_N(y)|\,dy + \int_{|y|\geq\delta} |\Phi_N(y)| 2\|f\|_\infty\,dy \\ &< C\epsilon \end{aligned}$$

when N is large enough so that the second term in the second to last line of (3.9) is less than $2\|f\|_\infty \epsilon$. Thus, we proved (1). To prove (2), fix any $f \in L^p(\mathbb{T})$ and let $g \in C(\mathbb{T})$ be such that $\|f - g\|_p < \epsilon$. Then

$$\|\Phi_N * f - f\|_p \leq \|\Phi_N * (f-g)\|_p + \|f - g\|_p + \|\Phi_N * g - g\|_p. \tag{3.10}$$

By Young's inequality (Theorem A.1 in the appendix) we have

$$\|\Phi_N * (f-g)\|_p \leq \|\Phi_N\|_1 \|f - g\|_p. \tag{3.11}$$

Using the fact that $\sup_N \|\Phi_N\|_1 < \infty$ and (1) we get $\|\Phi_N * f - f\|_p \to 0$ as $N \to \infty$. □

Here we also list some basic and useful theorems about Fourier transform that will be used to prove the L^p convergence of Fourier series later.

Corollary 3.12. *(1) For any $f \in L^2(\mathbb{T})$, we have*

$$\|f\|_2^2 = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2. \tag{3.13}$$

This is called Plancherel's theorem.

(2) For $f, g \in L^2(\mathbb{T})$ we have Parseval's identity

$$\int_{\mathbb{T}} f(x)\overline{g}(x)\,dx = \sum_{n\in\mathbb{Z}} \hat{f}(n)\overline{\hat{g}(n)}. \tag{3.14}$$

Proof. For (1) we have

$$\begin{aligned} \|f\|_2{}^2 &= \int_0^1 |f|^2\,dx = \int_0^1 \sum_{n\in\mathbb{Z}} |\hat{f}(n)e^{2\pi inx}|^2\,dx \\ &= \int_0^1 \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2\,dx = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2. \end{aligned} \tag{3.15}$$

The second equality is due to the orthogonality of $\{e^{2\pi inx}\}_{n\in\mathbb{Z}}$. The last equality is true because $\hat{f}(n)$ is independent of x for all n.

For (2) the proof is similar:

$$\begin{aligned} \int_{\mathbb{T}} f(x)\overline{g}(x)\,dx &= \int_{\mathbb{T}} \left(\sum_{n\in\mathbb{Z}} \hat{f}(n)e^{2\pi inx}\right)(\overline{\hat{g}(m)e^{2\pi imx}})\,dx \\ &= \int_{\mathbb{T}} \left(\sum_{n\in\mathbb{Z}} \hat{f}(n)e^{2\pi inx}\right)(\overline{\hat{g}(m)}e^{-2\pi imx})\,dx \\ &= \sum_{n\in\mathbb{Z}} \hat{f}(n)\overline{\hat{g}(n)}. \end{aligned} \tag{3.16}$$

The last equality results from the orthogonality of the basis. □

4. L^p CONVERGENCE OF PARTIAL SUMS

We now turn our attention to the convergence of the partial sums $S_N f$ in $L^p(\mathbb{T})$ or $C(\mathbb{T})$ (substituted for $L^\infty(\mathbb{T})$).

Proposition 4.1. *The following statements are equivalent for any $1 \le p \le \infty$:*

(a) For every $f \in L^p(\mathbb{T})$ (or $f \in C(\mathbb{T})$ if $p = \infty$) we have

$$\|S_N f - f\|_p \to 0 \text{ as } N \to \infty \tag{4.2}$$

(b) We have $\sup_N \|S_N\|_{p\to p} < \infty$ where $\|S_N\|_{p\to p} = \sup_{\|f\|_p=1} \|S_N f\|_p$.

Proof. "⇒" Assume$S_N f$ converges to f in L^p. Then we have $\|S_N f\|_p \to \|f\|_p$ by the triangle inequality. Since $f \in L^p(\mathbb{T})$, we know that $\|f\|_p < \infty$. Thus, $\sup_N \|S_N f\|_p < \infty$. By the Uniform Boundedness Principle (see the appendix), we have $\sup_N \|S_N\|_{p\to p} < \infty$.

"$\Leftarrow$" Let $K_N(x)$ be the Fejér kernel. We have

$$\begin{aligned}\|S_N f - f\|_p &= \|S_N(f - K_N * f) + (S_N(K_N * f) - f)\|_p \\ &= \|S_N(f - K_N * f) + (K_N * f - f)\|_p \\ &\le \|S_N(f - K_N * f)\|_p + \|K_N * f - f\|_p \\ &\le \|S_N\|_{p\to p}\|f - K_N * f\|_p + \|K_N * f - f\|_p \\ &= (\|S_N\|_{p\to p} + 1)\|f - K_N * f\|_p \\ &\le (M+1)\epsilon,\end{aligned} \tag{4.3}$$

where $M = \sup_N \|S_N\|_{p\to p} < \infty$ by assumption and ϵ can be made arbitrarily small as $N \to \infty$ since the Fejér kernel forms an approximate identity. This finishes the proof. □

With the above proposition, we can now settle the question of the convergence of Fourier series in L^p space for $p = 1$ and $p = \infty$.

Corollary 4.4. *Fourier series do not converge in $C(\mathbb{T})$ and $L^1(\mathbb{T})$, i.e., there exists $f \in C(\mathbb{T})$ so that $\|S_N f - f\|_\infty \nrightarrow 0$ and $g \in L^1(\mathbb{T})$ so that $\|S_N g - g\|_1 \nrightarrow 0$ as $N \to \infty$.*

Proof. By Proposition 4.1 it suffices to verify the limits

$$\sup_N \|S_N\|_{\infty\to\infty} = \infty$$

$$\sup_N \|S_N\|_{1\to 1} = \infty.$$

Recall that $\|D_N\|_1 \to \infty$ as $N \to \infty$ by (1.9). Notice that

$$\begin{aligned}\|S_N\|_{\infty\to\infty} &= \sup_{\|f\|_\infty = 1} \|D_N * f\|_\infty \\ &\ge \sup_{\|f\|_\infty = 1} |(D_N * f)(0)| \\ &\ge \sup_{\|f\|_\infty = 1} \int_{\mathbb{T}} D_N(-y) f(y)\, dy \\ &= \int_{\mathbb{T}} |D_N(y)|\, dy = \|D_N\|_1.\end{aligned} \tag{4.5}$$

The second to last inequality is true because $\|f\|_\infty = 1$, so $D_N(-y)f(y) \le |D_N(-y)|$ for all y. We can then let f be a continuous approximation (in L^1) of $\operatorname{sign} D_N(y)$. Thus, Fourier series do not converge in $C(\mathbb{T})$.

Recall that we proved that for the Fejér kernel, we have $\int_{\mathbb{T}} |K_M(x)|\, dx = 1$ for all M. Similar to (4.5), we have

$$\|S_N\|_{1\to 1} \ge \|D_N * K_M\|_1 \to \|D_N\|_1 \tag{4.6}$$

as $M \to \infty$. Thus, Fourier series do not converge in $L^1(\mathbb{T})$. □

Now our goal is to prove that for any p such that $1 < p < \infty$ we have that the Fourier series do converge in $L^p(\mathbb{T})$. The case $p = 2$ is relatively easy. Recall the Plancherel theorem we proved earlier which states

$$\|f\|_2^2 = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2. \tag{4.7}$$

Thus, by the Plancherel theorem, we have

$$\|S_N f - f\|_2 = \left\| \sum_{|n|>N} \hat{f}(n) e^{2\pi i n x} \right\|_2 = \left(\sum_{|n|>N} |\hat{f}(n)|^2 \right)^{1/2}, \tag{4.8}$$

for all $f \in L^2(\mathbb{T})$ which goes to 0 as $N \to \infty$.

For $1 < p < \infty$ and $p \neq 2$, we prove it in the following way which is a bit tricky. We start by defining the *conjugate function*.

Definition 4.9. For f a trigonometric function, define the conjugate function $\tilde{f}$ by

$$\tilde{f}(x) = -i \sum_{m \in \mathbb{Z}} sgn(m) \hat{f}(m) e^{2\pi i m x}, \tag{4.10}$$

Also define the *Riesz projections* P_+ and P_- by

$$\begin{aligned} P_+(f)(x) &= \sum_{m=1}^{\infty} \hat{f}(m) e^{2\pi i m x} \\ P_-(f)(x) &= \sum_{m=-\infty}^{-1} \hat{f}(m) e^{2\pi i m x} \end{aligned} \tag{4.11}$$

Thus, we have that $f = P_+(f) + P_-(f) + \hat{f}(0)$, where $\tilde{f} = -iP_+(f) + iP_-(f)$, when f is a trigonometric function. The following is a consequence of Proposition 4.1.

Proposition 4.12. *Let $1 < p < \infty$. Then the expression $S_N(f) = D_N * f$ converges to f in $L^p(\mathbb{T})$ as $N \to \infty$ if and only if there exists a constant $C_p > 0$ such that $\|\tilde{f}\|_{L^p(\mathbb{T})} \leq C_p \|f\|_{L^p(\mathbb{T})}$ for all trigonometric polynomials.*

Proof. By the previous definition, it is easy to show that

$$P_+(f) = \frac{1}{2}\left(f + i\tilde{f}\right) - \frac{1}{2}\hat{f}(0). \tag{4.13}$$

Therefore, the L^p boundedness of the operation $f \mapsto \tilde{f}$ is equivalent to that of the operator $f \mapsto P_+(f)$.

Next, we need to show that

$$e^{-2\pi i N x} \sum_{m=0}^{2N} (\widehat{f(x) e^{2\pi i N x}})(m) e^{2\pi i m x} = \sum_{m=-N}^{N} \hat{f}(m) e^{2\pi i m x}. \tag{4.14}$$

To prove this, note that

$$\begin{aligned} \left(\widehat{f(x) e^{2\pi i N x}}\right)(m) &= \int_{\mathbb{T}} f(x) e^{2\pi i N x} e^{-2\pi i m x}\, dx = \int_{\mathbb{T}} f(x) e^{-2\pi i (m-N)x}\, dx \\ &= \hat{f}(m-N) \end{aligned} \tag{4.15}$$

Thus, the left hand side of (4.14) is equal to $\sum_{m=0}^{2N} \hat{f}(m-N) e^{2\pi i (m-N)x} = \sum_{m=-N}^{N} \hat{f}(m) e^{2\pi i m x}$, which is the right hand side of (4.14).

Since multiplication by exponentials does not affect L^p norms, (4.15) implies that the norm operator $S_N(f) = D_N * f$ from L^p to L^p is equal to that of the operator

$$S'_N(g)(x) = \sum_{m=0}^{2N} \hat{g}(m) e^{2\pi i m x} \tag{4.16}$$

from L^p to L^p. Therefore, we have the following equivalence:

$$\sup_{N\geq 0} \|S_N\|_{L^p\to L^p} < \infty \Leftrightarrow \sup_{N\geq 0} \|S'_N\|_{L^p\to L^p} < \infty. \tag{4.17}$$

"$\Rightarrow$ ". Suppose now that for all $f \in L^p(\mathbb{T})$, we have $S_N(f) \to f$ as $N \to \infty$. Proposition 4.1 then yields $\sup\|S_N\|_{L^p\to L^p} < \infty$ and thus $\sup\|S'_N\|_{L^p\to L^p} < \infty$ by (4.17). Let $A(f) = P_+(f) + \hat{f}(0)$. We know that $A(f)$ is bounded on $L^p(\mathbb{T})$. Hence so is P_+. Thus, $\tilde{f}$ is also bounded.

"$\Leftarrow$". Conversely, suppose that P_+ extends to a bounded operator from $L^p(\mathbb{T})$ to itself. For every trigonometric function f we then have

$$\begin{aligned} S'_N(f)(x) &= \sum_{m=0}^{\infty} \hat{f}(m) e^{2\pi i m x} - \sum_{m=2N+1}^{\infty} \hat{f}(m) e^{2\pi i m x} \\ &= \sum_{m=0}^{\infty} \hat{f}(m) e^{2\pi i m x} - e^{2\pi i (2N) x} \sum_{m=1}^{\infty} \hat{f}(m+2N) e^{2\pi i m x} \\ &= P_+(f)(x) - e^{2\pi i (2N) x} P_+(e^{-2\pi i (2N) x} f) + \hat{f}(0). \end{aligned} \tag{4.18}$$

(4.18) implies that

$$\sup_{N\geq 0} \|S'_N(f)\|_{L^p} \leq (2\|P_+\|_{L^p\to L^p} + 1)\|f\|_{L^p} \tag{4.19}$$

for all trigonometric polynomials, and, by density, for all $f \in L^p(\mathbb{T})$. Thus, estimate (4.19) also holds for S_N. By Proposition 4.1, we have that $S_N(f) \to f$ in L^p for all $f \in L^p(\mathbb{T})$. □

We now have that convergence of Fourier series in L^p is equivalent to the L^p boundedness of the conjugate function. We now establish that these operators are bounded on L^p.

Theorem 4.20. *Given $1 < p < \infty$, there is a constant $A_p > 0$ such that for all f in $L^\infty(\mathbb{T})$ we have*

$$\|\tilde{f}\|_{L^p} \leq A_p \|f\|_{L^p}. \tag{4.21}$$

Consequently, the Fourier series of an L^p function converge to the function in L^p for $1 < p < \infty$.

Proof. There is a short proof for this due to Bochner. Let $f(t)$ be a trigonometric polynomial on $\mathbb{T}$ with coefficients c_j. We write

$$f(t) = \sum_{j=-N}^{N} c_j e^{2\pi i j t} \tag{4.22}$$

Assume $f(t)$ is complex-valued. Then

$$\begin{aligned}\mathrm{Re}(f) &= \frac{f(t)+\overline{f(t)}}{2} \\ &= \frac{\sum_{j=-N}^{N} c_j e^{2\pi ijt} + \sum_{j=-N}^{N} \overline{c_j e^{2\pi ijt}}}{2} \\ &= \frac{\sum_{j=-N}^{N} c_j e^{2\pi ijt} + \sum_{j=-N}^{N} \overline{c_j} e^{-2\pi ijt}}{2} \\ &= \sum_{j=-N}^{N} \frac{c_j + \overline{c_{-j}}}{2} e^{2\pi ijt}.\end{aligned} \tag{4.23}$$

Similarly, the imaginary part can be expressed as

$$\begin{aligned}\mathrm{Im}(f) &= \frac{f(t)-\overline{f(t)}}{2i} \\ &= \frac{\sum_{j=-N}^{N} c_j e^{2\pi ijt} - \sum_{j=-N}^{N} \overline{c_j e^{2\pi ijt}}}{2i} \\ &= \frac{\sum_{j=-N}^{N} c_j e^{2\pi ijt} - \sum_{j=-N}^{N} \overline{c_j} e^{-2\pi ijt}}{2i} \\ &= \sum_{j=-N}^{N} \frac{c_j - \overline{c_{-j}}}{2i} e^{2\pi ijt}.\end{aligned} \tag{4.24}$$

Thus, we have

$$\tilde{f}(t) = \sum_{j=-N}^{N} c_j e^{2\pi ijt} = \left[\sum_{j=-N}^{N} \widetilde{\frac{c_j + \overline{c_{-j}}}{2}} e^{2\pi ijt}\right] + i\left[\sum_{j=-N}^{N} \widetilde{\frac{c_j - \overline{c_{-j}}}{2i}} e^{2\pi ijt}\right]. \tag{4.25}$$

Note that the expressions inside the square brackets are real-valued trigonometric polynomials. Since $\|f\|_P \leq \|\mathrm{Re} f\|_p + \|\mathrm{Im} f\|_p \leq 2\|f\|_p$, and $\|\hat{f}(0)\|_p = |\hat{f}(0)| \leq \|f\|_p$, we can assume that f is real-valued and $\hat{f}(0) = 0$. Since f is real-valued, we have that $\hat{f}(-m) = \overline{\hat{f}(m)}$ for all m, and since $\hat{f}(0) = 0$, we have

$$\hat{f}(t) = -i\sum_{m>0} \hat{f}(m)e^{2\pi imt} + i\sum_{m>0} \hat{f}(-m)e^{-2\pi imt} = 2\ \mathrm{Re}\ [-i\sum_{m>0} \hat{f}(m)e^{2\pi imt}], \tag{4.26}$$

which implies that $\tilde{f}$ is also real-valued. Note that

$$\begin{aligned}f(t) + i\tilde{f}(t) &= \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{2\pi int} + (-i^2)\sum_{n\in\mathbb{Z}} \mathrm{sgn}\ (n)\hat{f}(n)e^{2\pi inx} \\ &= \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{2\pi int} + \sum_{n>0} \hat{f}(n)e^{2\pi inx} - \sum_{n<0} \hat{f}(n)e^{2\pi inx} \\ &= \sum_{n>0} \hat{f}(n)e^{2\pi int}.\end{aligned} \tag{4.27}$$

Thus, $f+i\tilde{f}$ only has non-zero Fourier coefficients for $n>0$, which means the polynomial $f+i\tilde{f}$ contains only positive frequencies. Then we claim:

$$\int_{\mathbb{T}}(f(t)+i\tilde{f}(t))^{2k}\,dt=0. \tag{4.28}$$

To prove this, note that since $f+i\tilde{f}$ is a sum of trigonometric functions, $(f+i\tilde{f})^{2k}$ can also be a sum of trigonometric functions. Thus, we have

$$(f(t)+i\tilde{f}(t))^{2k}=\sum_{n\geq 0}\widehat{(f+i\tilde{f})}^{2k}(n)e^{2\pi int}. \tag{4.29}$$

The summation is over non-negative integers because $f+i\tilde{f}$ only has positive frequencies. Thus, we have

$$\int_{\mathbb{T}}(f(t)+i\tilde{f}(t))^{2k}\,dt=\int_{\mathbb{T}}\left(\sum_{n\geq 0}\widehat{(f+i\tilde{f})}^{2k}(n)e^{2\pi int}\right)\,dt=\widehat{(f+i\tilde{f})}^{2k}(0). \tag{4.30}$$

By our assumption we know that $\widehat{f+i\tilde{f}}(0)=0$. Since $f+i\tilde{f}$ only has non-negative frequencies, we have $\widehat{f+i\tilde{f}}^{2k}(0)=((\widehat{f+i\tilde{f}})(0))^{2k}=0$. Therefore, we know that (4.28) is true. Expanding the $2k$ power and taking real parts, we obtain

$$\sum_{j=0}^{k}(-1)^{k-j}\binom{2k}{2j}\int_{\mathbb{T}}\tilde{f}(t)^{2k-2j}f(t)^{2j}\,dt=0, \tag{4.31}$$

where we used that f is real-valued. Separating the $j=0$ term from the rest in (4.31), we get

$$(-1)^k\int_{\mathbb{T}}\tilde{f}(t)^{2k}\,dt=-\sum_{j=1}^{k}(-1)^{k-j}\binom{2k}{2j}\int_{\mathbb{T}}\tilde{f}(t)^{2k-2j}f(t)^{2j}\,dt. \tag{4.32}$$

Then, we have

$$\int_{\mathbb{T}}\tilde{f}(t)^{2k}\,dt\leq\sum_{j=1}^{k}\binom{2k}{2j}\int_{\mathbb{T}}\tilde{f}(t)^{2k-2j}f(t)^{2j}\,dt. \tag{4.33}$$

Notice that the left hand side of (4.33) is just $\|\tilde{f}\|_{L^{2k}}^{2k}$. Now apply Hölder's inequality to the right hand side of (4.33) with exponents $\frac{2k}{2k-2j}$ and $\frac{2k}{2j}$. (The statement and proof of the Hölder's inequality can be found in the Appendix.) We have

$$\begin{aligned}\sum_{j=1}^{k}\binom{2k}{2j}\int_{\mathbb{T}}\tilde{f}(t)^{2k-2j}f(t)^{2j}\,dt&\leq\sum_{j=1}^{k}\binom{2k}{2j}\|\tilde{f}(t)^{2k-2j}\|_{\frac{2k}{2k-2j}}\|f(t)^{2j}\|_{\frac{2k}{2j}}\\&=\sum_{j=1}^{k}\binom{2k}{2j}\Big(\int_{\mathbb{T}}\tilde{f}(t)^{2k}\,dt\Big)^{\frac{2k-2j}{2k}}\Big(\int_{\mathbb{T}}f(t)^{2k}\,dt\Big)^{\frac{2j}{2k}}\\&=\sum_{j=1}^{k}\binom{2k}{2j}\|\tilde{f}\|_{L^{2k}}^{2k-2j}\|f\|_{L^{2k}}^{2j}.\end{aligned} \tag{4.34}$$

Thus, by (4.34) we have

$$\|\tilde{f}\|_{L^{2k}}^{2k} \leq \sum_{j=1}^{k} \binom{2k}{2j} \|\tilde{f}\|_{L^{2k}}^{2k-2j} \|f\|_{L^{2k}}^{2j}. \tag{4.35}$$

Dividing (4.35) by $\|f\|_{L^{2k}}^{2k}$ we get

$$R^{2k} \leq \sum_{j=1}^{2k} R^{2k-2j}, \tag{4.36}$$

where $R = \|\tilde{f}\|_{L^{2k}} / \|f\|_{L^{2k}}$. It is easy to see from (4.36) that there exists a positive C_{2k} such that $R \leq C_{2k}$. Therefore, we conclude that

$$\|\tilde{f}\|_{L^p} \leq C_p \|f\|_{L^p} \text{ when } p = 2k. \tag{4.37}$$

Now we want to generalize (4.37) to functions where $\hat{f}(0) \neq 0$. Apply (4.37) to $f - \hat{f}(0)$, and we know that the conjugate function of a constant is zero (since constant functions only have Fourier terms for $n = 0$). Since $|\hat{f}(0)| \leq \|f\|_1 \leq \|f\|_p$ (the second inequality is true by Hölder's inequality), we have that $\|\tilde{f}\|_p = \|\widetilde{(f - \hat{f}(0))} + \hat{f}(0)\|_p \leq \|\widetilde{f - \hat{f}(0)}\|_p + \|\widetilde{\hat{f}(0)}\|_p \leq C_p \|f\|_p + \|\hat{f}(0)\|_p \leq (C_p + 1)\|f\|_p$. Thus, (4.37) is true for $p = 2k$ and f real-valued trigonometric polynomial. Since a general trigonometric polynomial can be written as $P + iQ$ where P and Q are real-valued trigonometric polynomials, we have $\|\tilde{f}\|_{L^p} \leq 2(C_p + 1)\|f\|_p$. Since trigonometric polynomials are dense in L^p, (4.37) holds for all smooth functions when $p = 2k$.

Riesz-Thorin Interpolation implies that if the conjugate operator is bounded on L^p and L^q, where $p < q$, then it is bounded on L^r where $p < r < q$. Since every real number lies in an interval of the form $[2k, 2k+2]$, for some $k \in \mathbb{Z}^+$, we have that (4.37) is true for all p such that $2 \leq p < \infty$ when f is a trigonometric function. By density the same result is valid for all L^p functions when $p \geq 2$.

Finally, we need to prove that the result is valid for $1 < p < 2$. The L^p norm of a polynomial can also be computed in another way:

$$\|f\|_p = \sup_{g(t) \in L^q} \left\{ \left| \int_{\mathbb{T}} f(t) \overline{g(t)} \, dt \right| : \|g(t)\|_q = 1, \frac{1}{p} + \frac{1}{q} = 1 \right\}. \tag{4.38}$$

Let $p \in (1,2)$, so that $q > 2$. Then by Corollary 3.12, we have

$$\begin{aligned}
\int_{\mathbb{T}} \tilde{f}(t)\overline{g(t)}\,dt &= \sum_{n=-\infty}^{\infty} \widehat{\tilde{f}(n)}\overline{\hat{g}(n)} \\
&= \sum_{n=-\infty}^{\infty} \hat{f}(n)(-i\ \mathrm{sgn} n)\overline{\hat{g}(n)} \\
&= \sum_{n=-\infty}^{\infty} \hat{f}(n)\overline{i\ \mathrm{sgn}\ (n)\hat{g}(n)} \qquad (4.39)\\
&= \sum_{n=-\infty}^{\infty} \hat{f}(n)\overline{-\widehat{\tilde{g}(n)}} \\
&= \int_{\mathbb{T}} f(t)\overline{-\tilde{g}(t)}\,dt.
\end{aligned}$$

By (4.38) we have

$$\|\tilde{f}(t)\|_p = \sup_{g(t)\in L^q} \left\{ \left| \int_{\mathbb{T}} \tilde{f}(t)\overline{g(t)}\,dt \right| : \|g(t)\|_q = 1, \frac{1}{p} + \frac{1}{q} = 1 \right\}. \qquad (4.40)$$

Thus, by (4.39) we have

$$\begin{aligned}
\|\tilde{f}\|_{L^p} &= \sup_{g(t)\in L^q} \left\{ \left| \int_{\mathbb{T}} |f(t)\overline{\tilde{g}(t)}\,dt \right| : \|g(t)\|_q = 1, \frac{1}{p} + \frac{1}{q} = 1 \right\} \\
&\le \sup_{g\in L^q} \|f\|_{L^p}\|\tilde{g}\|_{L^q} \le \sup_{g\in L^q} \|f\|_{L^p} C_p \|g\|_{L^q} \qquad (4.41)\\
&\le \|f\|_{L^p} C_p.
\end{aligned}$$

The second inequality in (4.41) is true by Hölder's inequality. The third inequality is true because $q > 2$ and we proved the boundedness of the conjugate operator in L^q spaces for $q > 2$. The last inequality is true because we chose the g such that $\|g\|_{L^q} = 1$. Therefore, we proved the boundedness of the conjugate operator on L^p for all $1 < p < \infty$.

□

By Proposition 4.12 and Theorem 4.20 we conclude that for all p such that $1 < p < \infty$, the partial sum of the Fourier series of any L^p function converges to the original function.

Appendix A. Proofs of Theorems and Lemma

Theorem A.1. (Hölder's inequality) *Let (S, Σ, μ) be a measure space and let $p, q \in [1, \infty]$ with $1/p + 1/q = 1$. Then, for all measurable real- or complex- value functions f and g on S, we have $\|fg\|_1 \le \|f\|_p \|g\|_q$.*

Proof. If $\|f\|_p = 0$, then f is zero μ-almost everywhere, and the product fg is zero μ- almost everywhere. Hence the left hand side of Hölder's inequality is zero. The same is true if $\|f\|_q = 0$. Therefore, we may assume $\|f\|_p > 0$ and $\|g\|_q > 0$ in the following.

If $\|f\|_p = \infty$ or $\|g\|_q = \infty$, then the right hand side of Hölder's inequality is infinite. Therefore, we may assume that $\|f\|_p$ and $\|g\|_q$ are in $(0, \infty)$.

If $p = \infty$ and $q = 1$, then $|fg| \leq \|f\|_\infty |g|$ almost everywhere and Hölder's inequality follows from the monotonicity of the Lebesgue integral. Similarly for $p = 1$ and $q = \infty$. Therefore, we may also assume $p, q \in (1, \infty)$.

Dividing f and g by $\|f\|_p$ and $\|g\|_q$ respectively, we can assume that $\|f\|_p = \|g\|_q = 1$. We now use Young's inequality which states that

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \tag{A.2}$$

for all nonnegative a and b, where equality is achieved if and only if $a^p = b^q$. Hence

$$|f(s)g(s)| \leq \frac{|f(s)|^p}{p} + \frac{|g(s)|^q}{q} \tag{A.3}$$

for all $s \in S$. Integrating both sides gives

$$\|fg\|_1 \leq \frac{1}{p} + \frac{1}{q} = 1, \tag{A.4}$$

which proves the theorem. □

Theorem A.5. *If* $1 \leq p, q, r \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, $f \in L^p(\mathbb{T})$ *and* $g \in L^q(\mathbb{T})$, *then*

$$\|f * g\|_r \leq \|f\|_p \|g\|_q. \tag{A.6}$$

Proof. if $r = \infty$, then $\frac{1}{p} + \frac{1}{q} = 1$. Hence by Hölder's inequality and the translation invariance of Lebesgue measure, we have

$$\int |f(x-y)g(y)|\, dy \leq \|f\|_p \|g\|_q. \tag{A.7}$$

Thus, $f * g(x)$ exists for each x and $\|f * g\|_\infty \leq \|f\|_p \|g\|_q$.

Next suppose $1 \leq r < \infty$. Note that $p \leq r$ and $q \leq r$. Let $s = p(1-1/q) = 1-p/r$ and note $0 \leq s < 1$. Let $t = r/q$ and note that $1 \leq t < \infty$. Define q' by $1/q + 1/q' = 1$ and note $1 < q' \leq \infty$. Now let

$$h(x) = \int |f(x-y)g(y)|\, dy = \int |f(x-y)|^{1-s} |g(y)| |f(x-y)|^s \, dy. \tag{A.8}$$

By Hölder's inequality we have

$$h(x) \leq (\int |f(x-y)|^{(1-s)q} |g(y)|^q \, dy)^{1/q} \| |f|^s \|_{q'}. \tag{A.9}$$

If $s = 0$ then $q = 1$. If $s \neq 0$ then $sq' = p$. In either case taking the q^{th} powers we obtain

$$h(x)^q \leq \int |f(x-y)|^{(1-s)q} |g(y)|^q \, dy \|f\|_{q'}{}^{sq}. \tag{A.10}$$

Thus, by Minkowski's inequality we have

$$\begin{aligned} \|h\|_{qt}^t &= \|h^q\|_t \\ &\leq \|f\|_p{}^{sq} (\int (\int |f(x-y)|^{(1-s)q} |g(y)|^q \, dy)^t \, dx)^{1/t} \\ &\leq \|f\|_p{}^{sq} \int (\int |f(x-y)|^{(1-s)qt} |g(y)|^{qt} \, dx)^{1/t} \, dy \\ &= \|f\|_p{}^{sq} \|g\|_q{}^q \|f\|_{(1-s)qt}{}^{(1-s)q}. \end{aligned} \tag{A.11}$$

But $qt = r$ and $(1-s)r = p$. Taking the q^{th} roots we obtain the inequality in the conclusion of the theorem.

□

Theorem A.12. (Uniform Boundedness Principle). *Let X be a Banach space and Y be a normed vector space. Suppose that F is a collection of continuous linear operator from X to Y. If for all x in X we have*

$$\sup_{T\in F}\|T(x)\|_Y < \infty, \tag{A.13}$$

then

$$\sup_{T\in F}\|T\|_{B(X,Y)} < \infty. \tag{A.14}$$

Proof. Suppose that for every x in the Banach space X, we have:

$$\sup_{T\in F}\|T(x)\|_Y < \infty. \tag{A.15}$$

For every integer $n \in \mathbb{N}$, let

$$X_n = \{x \in X : \sup_{T\in F}\|T(x)\|_Y \le n\}. \tag{A.16}$$

The set X_n is a closed set and by the assumption,

$$\cup_{n\in\mathbb{N}} X_n = X \neq \emptyset. \tag{A.17}$$

By the Baire category theorem for a non-empty complete metric space X, there exist m such that X_m has non-empty interior, i.e., there exists $x_0 \in X_m$ and $\epsilon > 0$ such that

$$\overline{B_\epsilon(x_0)} := x \in X : \|x - x_0\| \le \epsilon \subset X_m. \tag{A.18}$$

Let $u \in X$ with $\|u\| \le 1$ and $T \in F$. we have:

$$\begin{aligned}\|T(u)\|_Y &= \epsilon^{-1}\|T(x_0+\epsilon u) - T(x_0)\| \\ &\le \epsilon^{-1}(\|T(x_0+\epsilon u)\|_Y + \|T(x_0)\|_Y) \\ &\le \epsilon^{-1}(m+m).\end{aligned} \tag{A.19}$$

Taking the supremum over u in the unit ball of X, it follows that

$$\sup_{T\in F}\|T\|_{B(X,Y)} \le 2\epsilon^{-1} m < \infty. \tag{A.20}$$

□

Acknowledgments. It is a pleasure to thank my mentors, Casey Rodriguez and Daniel Campos Salas, for leading me through the reading, answering all my questions, and giving advice on my talk. I would also like to thank Professor Peter May for giving me the opportunity for a talk and reading my paper.

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