THE TYPE PROBLEM: EFFECTIVE RESISTANCE AND RANDOM WALKS ON GRAPHS

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Abstract. The question of recurrence or transience – the so-called *type problem* – is a central one in the theory of random walks. We consider edge-weighted random walks on locally finite graphs. The effective resistance of such weighted graphs is defined electrically and shown to be infinite if and only if the weighted graph is recurrent. We then introduce the Moore-Penrose pseudoinverse of the Laplacian as an easier method for calculating this effective resistance. Finally, we discuss the Nash-Williams test and use it to prove recurrence of simple random walks on the one- and two-dimensional integer lattices.

Contents

1. Introduction

Pólya's theorem on random walks on the d-dimensional integer lattice $\mathcal{L}^d$ – namely, that such walks are recurrent for $d = 1, 2$ and transient for $d \geq 3$ – suggests what will be the first motivating question of this paper: *Where, exactly, is the tipping point between recurrence and transience of random walks on graphs?* The classical proof [9, pp. 1-2] of this theorem provides little help. This combinatorial approach proves recurrence of walks on $\mathcal{L}^d$ precisely for those d for which $\sum n^{-d/2}$ diverges. The problem is that this proof, while firmly establishing such a tipping point at $d = 2$, does not seem to admit a ready generalization beyond the lattices.

We wish to generalize, not only to more complex graphs, but also to more complex random walks. Pólya's proof is for *simple random walks*, defined as random walks with the following transition probabilities p_{ij} of moving from state i to state j in one step:

$$p_{ij} = \begin{cases} \frac{1}{\deg i} & i \sim j, \\ 0 & i \not\sim j. \end{cases}$$

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As usual, the notation $i \sim j$ (respectively, $i \not\sim j$) means that the vertices i and j are (respectively, are not) adjacent. We use standard graph-theoretic notation throughout, following [1], and pause to recall the following three definitions.

Definition 1.1. Per Feller [4, p. 340], a *(time-homogeneous) Markov Chain Z on state space* $X = \{E_1, E_2, \ldots\}$ consists of

- a sequence of trials with possible outcomes $E_1, E_2, \ldots$,
- a set of fixed conditional probabilities p_{jk} that the present trial will result in E_k, given that the preceding trial resulted in E_j, and
- a probability distribution $\{a_k\}$ where a_j is the probability that the trial numbered zero results in E_j.
- The conditional probabilities and probability distribution must satisfy the following probabilities for sample sequences:
$$\mathbb{P}(E_{j_0}, E_{j_1}, \ldots, E_{j_n}) = a_{j_0} p_{j_0 j_1} p_{j_1 j_2} \cdots p_{j_{n-2} j_{n-1}} p_{j_{n-1} j_n}.$$

If the result of the nth trial is E_k, then we will say that the Markov chain Z is *in state k at time n* and write $Z_n = E_k$. An initial distribution and the transition probabilities p_{ij}, the latter organized in a so-called *transition matrix* $P = (p_{ij})$, fully determine a Markov chain.

Markov chains are often visualized as edge-weighted digraphs, with the state space used as a vertex set and arcs with weight p_{jk} going from j to k added when p_{jk} is strictly positive.[1]

It is well known [4, pp. 347-349] that the conditional probability $p_{ij}^{(n)}$ that the system is in state j at any time x, given that it was in state i at time $x - n$, is the ij-th entry of the matrix P^n under usual matrix multiplication.

Definition 1.2. In a Markov chain Z, a state j is said to be *accessible* from a state i if $p_{ij}^{(n)} > 0$ for some n. A Markov chain is said to be *irreducible* if every state is accessible from every other state. (Equivalently, the associated digraph is strongly connected).

Definition 1.3. A Markov chain Z on state space X is said to be *reversible* if there exists a function $\pi : X \to \mathbb{R}$ satisfying the following condition for all pairs $(i, j) \in X \times X$:

$$\pi_i p_{ij} = \pi_j p_{ji}. \tag{1.4}$$

Reversible Markov chains are interesting because they can be represented by an undirected graph instead of a digraph. To create this undirected graph, we replace weighted arcs $e_1 = (i, j)$ and $e_2 = (j, i)$ with a single undirected edge e with weight $\pi_i p_{ij} = \pi_j p_{ji}$, for all states i and j. (This is only useful if we have the function π, so that the Markov chain can be recovered from the undirected graph.)

With this motivation, we may consider random walks on *networks*, or tuples $[G, c]$ where G is an undirected graph and c is a (usually positive) real-valued function defined on the edges of G. The standard transition probabilities for random walks on networks are as follows:

$$p_{ij} = \begin{cases} \frac{c_{ij}}{\sum_{\ell \sim i} c_{i\ell}} & i \sim j, \\ 0 & i \not\sim j. \end{cases} \tag{1.5}$$

[1] An initial distribution is required in addition to the digraph to determine the Markov chain.

(For ease, we define $C_i \equiv \sum_{\ell \sim i} c_{i\ell}$.) A simple random walk is then a random walk on a network with unit edge weights.

More precisely, a random walk on a network $[G, c]$ is a Markov chain on state space $V(G)$ with transition probabilities given by Equation 1.5. We will restrict our attention to networks $[G, c]$ where G is connected and locally finite.[2] Connectedness of G implies irreducibility of the associated Markov chain, and one can check that the function $\pi : V(G) \to \mathbb{R}$ defined as

$$\pi(i) \equiv \frac{C_i}{\sum_{j \in V(G)} C_j}$$

satisfies the reversibility condition 1.4. We will assume these conditions on every network and Markov chain discussed hereafter.

Definition 1.6. A Markov chain is said to be *recurrent* at a state i if the following condition holds:

$$\mathbb{P}(Z_n = i \text{ for some } n > 0 \mid Z_0 = i) = 1.$$

The chain is called *transient* at i if this probability is strictly less than one.

It is well known [4, p. 355] that the states of an irreducible Markov chain are all recurrent or all transient; in this spirit, we say that the Markov chain itself is recurrent or transient. A network is called *recurrent* (respectively, *transient*) if its associated Markov chain is recurrent (respectively, transient). Accordingly, the initial distribution $\{a_k\}$ of the Markov chain is irrelevant; we will henceforth discuss a Markov chain, its associated (di)graph, and its transition matrix interchangeably.

The problem of ascertaining whether a random walk is recurrent or transient is called the *type problem*. To summarize: We are looking for the properties of a network that help solve the type problem on that network.

Such properties emerge from the study of graph-theoretic networks as electric circuits, in which graph-theoretic vertices are electrical nodes and graph-theoretic edges e are electrical resistors with with conductance c_e (resistance $\frac{1}{c_e}$). After defining a graph-theoretic "effective resistance," we will show that the recurrent networks are precisely those with infinite effective resistance. These connections between graph theory and the circuit theory were originally explored by C. St. J. A. Nash-Williams [8], and collected into an excellent monograph by P. G. Doyle and J. L. Snell [3]. In Section 2, we formally introduce this electrical viewpoint for finite networks. Section 3 abstracts to infinite networks, using Thomson's principle and Rayleigh's monotonicity law to prove the probabilistic implications of effective resistance. In Section 4, we discuss a spectral method that uses the pseudoinverse of the Laplacian to compute effective resistance. Section 5 introduces the pioneering result of Nash-Williams [8], which we use to provide simple and useful bounds for the effective resistance of "nice" networks. Since the integer lattice is "nice," Section 6 uses the Nash-Williams test to easily prove Pólya's theorem in one and two dimensions; dimensions three and above are left to the reader.

2. Effective Resistance (finite network)

Standing Assumptions: All networks $[G, c]$ are taken to consist of connected, locally finite graphs with positive edge weights. All Markov Chains Z are taken to

[2] The network itself is called connected, infinite, or locally finite if G has these properties.

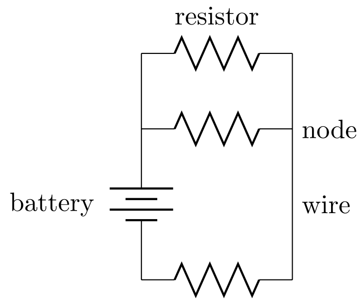

Figure 1. A sample electrical circuit, with various terms labeled.

be time-homogeneous, irreducible, and reversible, and to have transition matrices P with only finitely many non-zero entries in each row.

In addition, all networks discussed in this section are understood to be finite. The following two definitions will be useful throughout the paper.

Definition 2.1. The *standard (geodesic) graph metric* defines the distance $d(a, b)$ between vertices a and b to be the length (i.e., number of edges) of a minimum-length path connecting a and b, regardless of edge weight. Such a minimum-length path is called a *geodesic*. If a and b have no path connecting them, we define $d(a, b) \equiv \infty$.

Definition 2.2. The *weighted geodesic graph metric*, with respect to an edge-weighting c, defines the distance $d_c(a, b)$ between vertices a and b to be the total weight (i.e., sum of weights over all edges) of a minimum-weight path connecting a and b. Such a minimum-weight path is called a *weighted geodesic*. If a and b have no path connecting them, we define $d_c(a, b) \equiv \infty$. (The weighted geodesic graph metric, with respect to a unit edge-weighting, reduces to the standard graph metric.)

2.1. **Physical preliminaries.**

Of course, we cannot model graphs as electrical circuits without first understanding the circuits themselves. We recall several relevant facts [10] about electrical circuits, emphasizing that these are physical laws, not mathematical ones.

- **Basics:** A basic direct-current electric circuit consists of a battery and one or more circuit elements connected by wires. As in Figure 1, we will restrict our considerations to circuits in which every circuit element is a *resistor*. A resistor between two points a and b has an associated (undirected, symmetric) quantity $\mathsf{r}_{a,b}$ called *resistance*. The *conductance* $c_{a,b}$ of a resistor between points a and b is defined to be the multiplicative inverse of its resistance:

$$c_{a,b} \equiv \frac{1}{\mathsf{r}_{a,b}}. \tag{2.3}$$

- **Ohm's Law:** In addition to resistance/conductance, the study of networks relies heavily on two directed antisymmetric quantities, each defined on an edge e between points a and b: the *current* $\mathsf{i}_{a,b}$ and the *potential difference* $\phi_{a,b}$. All three quantities are subject to the following relation, called *Ohm's*

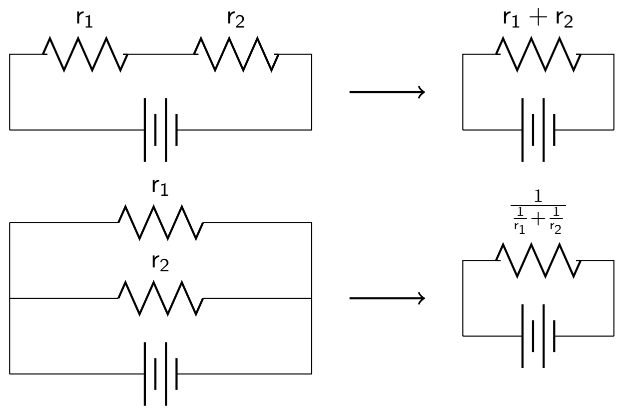

FIGURE 2. Computing $\mathsf{r}_{\mathsf{eff}}$ for resistors with resistance r_1 and r_2 in series and in parallel, respectively.

Law, that allows us to determine the third given any two:

$$\phi_{a,b} = \mathsf{i}_{a,b}\mathsf{r}_{a,b} \tag{2.4}$$

(Although quantities such as resistance and current are defined on edges, our notation will often use their endpoints instead: $\mathsf{i}_{a,b}$, not i_e. Within a single equation, we fix one edge among parallel edges if necessary.)

- **Kirchhoff Laws:** Physical networks are also subject to the two Kirchhoff laws, which state that the net potential difference around a cycle is zero and that the total currents flowing into and out of a node not connected to the battery are equal. These are called *Kirchhoff's potential law* and *Kirchhoff's current law*, respectively.

$$\sum_{j\sim i}^{\text{cycle}} \phi_{i,j} = 0, \tag{2.5}$$

$$\sum_{b:b\sim a} \mathsf{i}_{a,b} = 0, \qquad a \text{ not connected to the battery.} \tag{2.6}$$

The first law is equivalent to the existence of a so-called *potential function* $\phi : V(G) \to \mathbb{R}$ satisfying $\phi_{a,b} = \phi_b - \phi_a$.

- **Effective resistance:** The most important tool in circuit analysis for our purposes is the idea of effective resistance. A collection of resistors between points a and b may be replaced with a single resistor between a and b that acts "effectively" like the collection (i.e., that changes neither the current through a and b nor the potential difference between a and b). The resistance of this single resistor is called the *effective resistance* between a and b and is denoted $\mathsf{r}_{\mathsf{eff}}(a, b)$. When the source and sink are clear, this notation will be abbreviated to $\mathsf{r}_{\mathsf{eff}}$. Figure 2 indicates effective resistance for resistors in two common configurations, respectively called *series* and *parallel*.

All of the above observations were physical, not graph-theoretic. In abstracting to graphs, we have several choices for which physical observations to use as mathematical definitions (say, Ohm's Law and the Kirchoff Laws) and which as

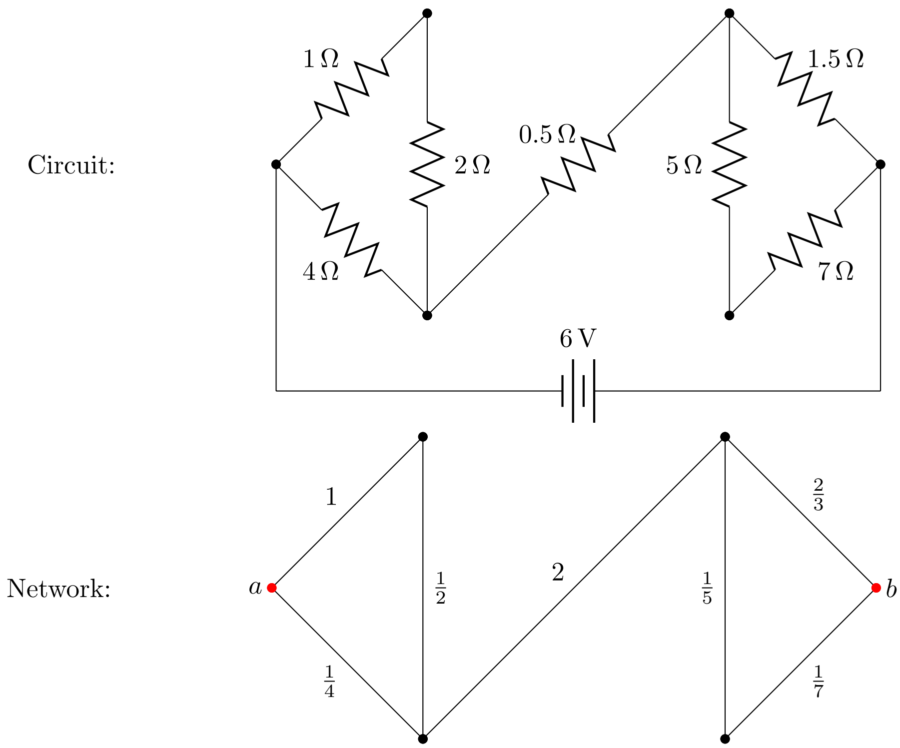

FIGURE 3. Structural analogies between circuits, above, and networks, below. Note that the vertices in the source-set $\{a, b\}$, colored red in the network, need not be adjacent. We do not add an edge between them, even though the corresponding electrical network has one.

mathematical consequences (say, the rules for effective resistance in series and parallel). Generally, these are motivated by the idea of building the graph as a physical network and applying a battery across two vertices. It does not seem efficient here to mathematically derive some physical laws from others; we will simply discuss networks in which all hold true. Real-world electrical circuits should allay any concerns about the existence of such networks.

When considering a network $[G, c]$ electrically, we will take the vertices to be nodes and the edges $e = xy$ to be resistors with resistance $\frac{1}{c_{x,y}}$, current $\mathsf{i}_{x,y}$, and potential difference $\phi_{x,y}$. Our "battery" is two distinguished (distinct) vertices $\{a, b\}$, collectively called the *source-set*. One of these two vertices is a *source*, corresponding to the positive terminal of our "battery"; the other, corresponding to the negative terminal, is a *sink*. The analogies between networks and circuits are displayed in Figure 3.

2.2. **Rayleigh Monotonicity.**

One of the most important tools in computing and approximating effective resistance is Rayleigh's Monotonicity Law, which states that the effective resistance of a network varies directly with the resistance of each resistor (i.e., inversely with the edge weights).

To prove this, we will need the concept of a *flow*, which can be imagined as the generalization of a current.

Definition 2.7. Let a network $[G, c]$ be given, along with a source-set $\{a, b\}$. An *a/b-flow* j is a real-valued function j on ordered pairs of distinct vertices in $V = V(G)$ satisfying the properties listed below. For convenience, we define $j_{v,w} \equiv j(v, w)$ and $J_v \equiv \sum_{w\sim v} j_{v,w}$.

- $j_{v,w} = -j_{w,v}$,
- $j_{v,w} = 0$ for $v \not\sim w$,
- $J_v = 0$ for $v \notin \{a, b\}$.

Before defining the size of a flow, we must show that the total flow out of a is equal to the total flow into b, up to units:

$$J_a + J_b = \sum_{v\in V} J_v = \sum_{v\in V}\sum_{w\sim v} j_{v,w} = \sum_{v\in V}\sum_{w\in V} j_{v,w} = \frac{1}{2}\sum_{v\in V}\sum_{w\in V} j_{v,w} + j_{w,v} = 0.$$

This allows us to define the *size* $|j|$ of the flow j to be $|J_a| = |J_b|$. A *unit flow* is a flow of size one. A flow is *physical* if it satisfies the Kirchhoff laws when taken as an assignment of current to edges.[3]

We can use the concept of a flow to formally define effective resistance.

Definition 2.8. Let a network $[G, c]$ and a physical flow i be given. Define the effective resistance $\mathsf{r}_{\mathsf{eff}}(a, b)$ between a source a and a sink b as the resistance in an (imagined) edge ab with current $\mathsf{i}_{a,b} = \mathsf{I}_a$:

$$\mathsf{r}_{\mathsf{eff}}(a, b) \equiv \frac{\phi_b - \phi_a}{\mathsf{I}_a}.$$

(Since $\mathsf{I}_a = -\mathsf{I}_b$, we see that $\mathsf{r}_{\mathsf{eff}}$ is symmetric, as we would expect it to be.)

Definition 2.9. Given a network $[G, c]$ and an a/b-flow j, define the *energy* $E(j)$ of j by the following equation:

$$E(j) \equiv \frac{1}{2}\sum_{u,v\in V} \frac{(j_{u,v})^2}{c_{u,v}}.$$

To save space, we only sketch the following two proofs:

Lemma 2.10 (Thomson's Principle)**.** *Let $[G, c]$ be a network with strictly positive edge-weights and a source-set $\{a, b\}$. Then the (unique) physical unit flow has minimal energy among unit flows.*

Proof ([3, pp. 50-51]*;* [5, pp. 8-10]*).* This proof sets the physical unit flow i, another unit flow j, and aims to show $E(j) = E(\mathsf{i}) + \alpha$ for some $\alpha \geq 0$. To do this, it defines

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[3] By Ohm's Law (Equation 2.4), Kirchhoff's potential law 2.5 can be written using currents:

$$\sum_{j\sim i}^{\text{cycle}} \phi_{i,j} = 0 \iff \sum_{j\sim i}^{\text{cycle}} \frac{\mathsf{i}_{i,j}}{c_{i,j}} = 0.$$

A flow on a network $[G, c]$ must satisfy this latter condition (and the Kirchhoff current law 2.6) to be called physical.

a flow $k = j - \mathsf{i}$ of size zero and uses Ohm's Law (2.4):

$$\begin{aligned} E(j) &= \frac{1}{2}\sum_{u,v\in V} \frac{(j_{u,v})^2}{c_{u,v}} \\ &= \frac{1}{2}\sum_{u,v\in V} \frac{(\mathsf{i}_{u,v}+k_{u,v})^2}{c_{u,v}} \\ &= E(\mathsf{i}) + E(k) + \frac{1}{2}\sum_{u,v\in V}\frac{\mathsf{i}_{u,v}k_{u,v}}{c_{u,v}} \\ &= E(\mathsf{i}) + E(k) + \frac{1}{2}\sum_{u,v\in V}[\phi(v)-\phi(u)]k_{u,v}. \end{aligned}$$

We know $E(k) \geq 0$, and the last term disappears after proving that

$$\frac{1}{2}\sum_{u,v\in V}[\phi(v)-\phi(u)]k_{u,v} = [\phi(b)-\phi(a)]K_a. \tag{2.11}$$

□

Lemma 2.12. *Let a network $[G, c]$ with source-set $\{a, b\}$ be given. Then the effective resistance from a to b is equal to the energy of a unit a/b-flow.*

Proof ([5, pp. 8-9]). This is obtained from Equation 2.11 (proved for a physical unit flow), the definition of effective resistance (Definition 2.8), and a little algebra. □

With these lemmas in hand, we are ready for the main result of this subsection.

Theorem 2.13 (Rayleigh's Monotonicity Law)**.** *The effective resistance $\mathsf{r}_{\mathsf{eff}}$ between any two points of a network varies monotonically with individual resistances.*

What follows appears to be the classical proof, found in both [3] and [5]. It relies on an application of Thomson's Principle (2.10), noting that a flow and its size are defined on any graph, but its energy and whether or not it is physical require reference to a specific edge-weighting. This means that a unit flow j on a graph G may be physical with respect to one edge-weighting c; with respect to a different edge weighting $\tilde{c}$, though, it is still of unit size but may not be physical, and thus has greater energy (with respect to $\tilde{c}$) than does the unit flow $\tilde{j}$ that is physical with respect to $\tilde{c}$.

Proof. Let a graph G be given with two sets of resistances $\{\mathsf{r}_{\mathsf{v,w}} \mid v \sim w\}$ and $\{\widetilde{\mathsf{r}_{v,w}} \mid v \sim w\}$ satisfying $\mathsf{r}_{v,w} \leq \widetilde{\mathsf{r}_{v,w}}$ for all $v \sim w$. (This is equivalent to two functions c and $\tilde{c}$ on the edges satisfying $c \geq \tilde{c}$.) Suppose that the source-set is $\{a, b\}$, and let i and $\tilde{\mathsf{i}}$ be the physical unit a/b-flows on $[G, c]$ and $[G, \tilde{c}]$, respectively. For this proof, ${}_cE(j)$ will denote the energy of a flow j with respect to an edge-weighting c. By Definition 2.9, Thomson's Principle (2.10), and Lemma 2.12, the following hold, where $\mathsf{r}_{\mathsf{eff}}$ and $\widetilde{\mathsf{r}_{\mathsf{eff}}}$ are taken from a to b:

$$\mathsf{r}_{\mathsf{eff}} = {}_cE(\mathsf{i}) \leq {}_cE(\tilde{\mathsf{i}}) = \frac{1}{2}\sum_{u,v\in V}(\widetilde{\mathsf{i}_{u,v}})^2\mathsf{r}_{u,v} \leq \frac{1}{2}\sum_{u,v\in V}(\widetilde{\mathsf{i}_{u,v}})^2\widetilde{\mathsf{r}_{u,v}} = {}_{\tilde{c}}E(\tilde{\mathsf{i}}) = \widetilde{\mathsf{r}_{\mathsf{eff}}}.$$

□

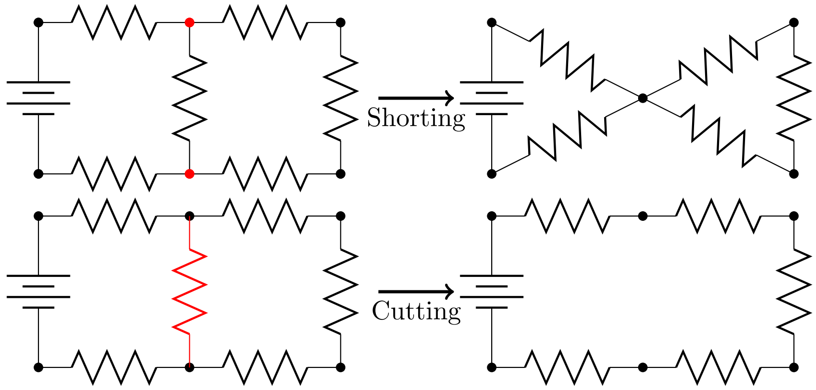


FIGURE 4. The same network (drawn electrically) before and after shorting and cutting, respectively. The vertices to be shorted together are in red, as is the resistor to be cut.

2.3. **Shorting and cutting.** Rayleigh's Monotonicity Law allows for several useful tools for bounding the effective resistance of a network. *Shorting* is electrical vertex identification.[4] *Cutting* is electrical edge deletion. The two are illustrated electrically in Figure 4.

Per definition 1.5 of transition probabilities, shorting vertices v and w together is practically the same as adding an edge of infinite conductance (zero resistance) between them (or increasing the conductance of the existing edge to infinity, depending).[5] Similarly, cutting an edge between vertices v and w is practically the same as decreasing the conductance of $c_{v,w}$ to zero (increasing $\mathsf{r}_{v,w}$ to infinity). These two observations, combined with Rayleigh's Monotonicity Law (2.13), lead to the following vital facts:

Facts 2.14.

- Shorting a network can only decrease its effective resistance.
- Cutting a network can only increase its effective resistance.

Remark 2.15. The two facts in 2.14 are individually equivalent to Rayleigh's Monotonicity Law, Theorem 2.13 [3, p. 76].

3. Effective Resistance (infinite network)

The previous section assumed network finiteness; we now wish to extend our results to infinite networks, the primary focus of this paper.

[4]Recall that *vertex identification* is the process of combining vertices $v_1, \ldots, v_n$ into a new single vertex v^* such that every edge incident with v_i for some $1 \leq i \leq n$ is, instead, incident with v^*. Sometimes the literature [1] uses "vertex identification" only when the vertices to be identified are pairwise nonadjacent, preferring "edge contraction" when the edge(s) between v_i and v_j must be deleted before identification. For simplicity's sake, we use "identification" for all vertices we wish to group together, with the understanding that edges are to be deleted as necessary.

[5]"Practically the same" for our purposes, at least, since we are only concerned about long-term behavior. The electrical interpretation would not be "practically the same" if we were interested in, say, hitting times. For example, given a simple random walk on the path $P_3 = v_1 \sim v_2 \sim v_3 \not\sim v_1$, vertex identification of v_2 and v_3 would return a hitting time of one between v_1 and v_3, while increasing c_{v_2,v_3} to infinity would return a hitting time of two.

We might worry that the machinery we have developed is not strong enough. After all, we have only defined effective resistances of finite networks with a single source and a single sink. The single-source/single-sink problem is solved by shorting: We take a neighborhood around an origin vertex and short its boundary into a single point, computing the effective resistance of the network thus obtained. The jump from the finite to the infinite is then addressed, as always, with limits. Intuitively, large neighborhoods passed to the limit look like the entire network, no matter which origin point we select. This seems, then, like a reasonable way to define effective resistance of an infinite network. We shall see that it is a useful way, as well.

The definition of network effective resistance and the proof of Theorem 3.9 are due to Grimmett [5]. In the following, we will assume that a network $[G, c]$ with a distinguished vertex 0 has been given.

Notations 3.1. With respect to the standard graph metric $d(a, b)$ (Definition 2.1), write Λ_n for the neighborhood of radius n around 0 and $\partial\Lambda_n$ for its boundary. That is,

$$\Lambda_n \equiv \{v \in V(G) \mid d(0, v) \leq n\},$$
$$\partial\Lambda_n \equiv \{v \in V(G) \mid d(0, v) = n\}.$$

Assume that n has been fixed in the following three definitions.

Definition 3.2. Define the graph G_n to be the subgraph of G whose vertex set is Λ_n and whose edge set is all those edges of G between points in Λ_n.

Definition 3.3. Define the graph $\overline{G_n}$ to be the graph obtained from G_n by shorting the vertices in $\partial\Lambda_n$ into a single vertex, which we will call I_n.

Definition 3.4. Using the restriction $\overline{c}$ of c to $\overline{G_n}$, define $\mathsf{r_{eff}}(n)$ to be the effective resistance of the network $[\overline{G_n}, \overline{c}]$ with source 0 and sink I_n. This creates a sequence $\{\mathsf{r_{eff}}(n)\}$ in n.

Proposition 3.5. *The sequence $\{\mathsf{r_{eff}}(n)\}$ is non-decreasing in n.*

Proof. The network $[\overline{G_{n-1}}, \overline{c}]$ can be obtained from the network $[\overline{G_n}, \overline{c}]$ by shorting together I_n and all the points in $\partial\Lambda_{n-1}$, as in Figure 5. The first fact in 2.14 allows us to conclude $\mathsf{r_{eff}}(n-1) \leq \mathsf{r_{eff}}(n)$. □

Definition 3.6. The effective resistance $\mathsf{r_{eff}}$ of an infinite network is defined as follows:

$$\mathsf{r_{eff}} \equiv \lim_{n\to\infty} \mathsf{r_{eff}}(n).$$

Proposition 3.7. *Whether network effective resistance is finite or infinite does not depend on starting vertex* 0.

Proof. For the duration of this proof, we will introduce the following notation changes: We will write $G_{k,v}$ for the relevant graph centered at v. The graph $\overline{G_{k,v}}$ is defined analogously, with the boundary vertices shorted into a point we will call $I_{k,v}$. The effective resistance of this latter graph, with source v and sink $I_{k,v}$, will be written $\mathsf{r_{eff}}(k, v)$. The limit of the sequence thus formed willl be written $\mathsf{r_{eff}}(v)$.

We will also require the following results, due to Klein and Randić [6]: For a fixed connected network $[G, c]$, the function $\Omega : V(G) \times V(G) \to \mathbb{R}$ that maps a pair

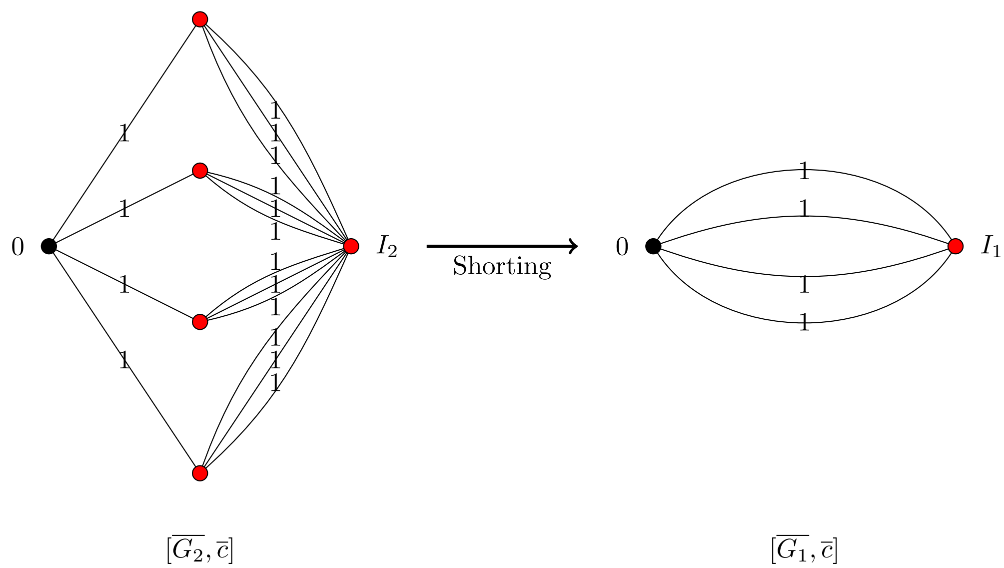


FIGURE 5. The network $[\overline{G_n}, \overline{c}]$ can be shorted into the network $[\overline{G_{n-1}}, \overline{c}]$. This figure shows $[\overline{G_2}, \overline{c}]$ and $[\overline{G_1}, \overline{c}]$ for the two-dimensional integer lattice under unit edge-weighting. Red indicates the points to be shorted in the former and the point thus formed in the latter. These are I_2 and the points in $\partial\Lambda_1$ and then I_1, respectively.

(a, b) of vertices to the effective resistance of $[G, c]$ with source-set $\{a, b\}$ is a metric. This so-called *resistance distance* is bounded above by the weighted geodesic graph metric (Definition 2.2). That is, for all vertices $a, b \in V(G)$, we have the following:

$$\Omega(a, b) \leq d_c(a, b). \tag{3.8}$$

Fix $K \in \mathbb{N}$, and let a network $[G, c]$ with distinct starting vertices 0 and $0'$ be given. Define $n \equiv d(0, 0')$. Since the network $[\overline{G_{K,0'}}, \overline{c}]$ can be obtained from $[\overline{G_{n+K,0}}, \overline{c}]$ by shorting together all vertices w satisfying $d(w, 0') \geq K$, the shorting rule introduced in Fact 2.14 allows us to conclude the following, where $\Omega(0', I_{n+K,0})$ refers to effective resistance in the network $[\overline{G_{n+K,0}}, \overline{c}]$:

$$\mathsf{r}_{\mathsf{eff}}(K, 0') \leq \Omega(0', I_{n+K,0}).$$

This fact and the triangle inequality of the metric Ω on the network $[\overline{G_{n+K,0}}, \overline{c}]$ give us the following:

$$\begin{aligned} \mathsf{r}_{\mathsf{eff}}(K, 0') &\leq \Omega(0', I_{n+K,0}) \\ &\leq \Omega(0, 0') + \Omega(0, I_{n+K,0}) \\ &= \Omega(0, 0') + \mathsf{r}_{\mathsf{eff}}(n + K, 0). \end{aligned}$$

The value of $\Omega(0, 0')$ (taken with respect to the network $[\overline{G_{n+k,0}}, \overline{c}]$) may vary with k. For each such network, the resistance distance is bounded above by $d_c(0, 0')$, which may itself depend on k (Equation 3.8). However, those k-dependent bounds $d_c(0, 0')$ are themselves eventually bounded above by the original value $d_c(0, 0')$ for the fixed K above, since the weighted geodesic in $[\overline{G_{n+K,0}}, \overline{c}]$ whose sum weight is $d_c(0,0')$ is still a path connecting 0 and $0'$ in the network $[\overline{G_{n+k,0}}, \overline{c}]$ for k large enough. If we define

$$\alpha \equiv d_c(0,0') \qquad (\text{w.r.t. the network } [\overline{G_{n+K,0}}, \overline{c}]),$$

then we obtain the much nicer

$$\mathsf{r}_{\mathsf{eff}}(k,0') \leq \alpha + \mathsf{r}_{\mathsf{eff}}(n+k,0), \qquad k \geq K.$$

A similar argument on the network $[\overline{G_{2n+K,0'}}, \overline{c}]$ gives the following inequality:

$$\mathsf{r}_{\mathsf{eff}}(n+k,0) \leq \alpha' + \mathsf{r}_{\mathsf{eff}}(2n+k,0'), \qquad k \geq K,$$

where

$$\alpha' \equiv d_c(0,0') \qquad (\text{w.r.t. the network } [\overline{G_{2n+K,0'}}, \overline{c}]).$$

We can then combine these into

$$\mathsf{r}_{\mathsf{eff}}(k,0') - \alpha \leq \mathsf{r}_{\mathsf{eff}}(n+k,0) \leq \mathsf{r}_{\mathsf{eff}}(2n+k,0') + \alpha', \qquad k \geq K,$$

and pass k to the limit to obtain

$$\mathsf{r}_{\mathsf{eff}}(0') - \alpha \leq \mathsf{r}_{\mathsf{eff}}(0) \leq \mathsf{r}_{\mathsf{eff}}(0') + \alpha',$$

which proves the proposition, since α and α' are finite.

We could, if desired, prove that $\mathsf{r}_{\mathsf{eff}}$ is completely independent of starting vertex, but this weaker proposition will suffice, since this paper only tests whether the effective resistance is infinite or finite. □

The main result of this section is the following theorem.

Theorem 3.9. *A network is recurrent if and only if its effective resistance is infinite.*

Before proving this, we need several new concepts and lemmas. In the interest of space, we will only sketch the proofs of these lemmas.

Definition 3.10. Let a Markov chain Z with transition matrix $P = (p_{ij})$ be given. A real function defined on the vertex set V of the underlying weighted graph is said to be *harmonic* on $U \subset V$ if the following holds:

$$f(u) = \sum_{v \in V} p_{uv} f(v), \qquad u \in U.$$

In general, the values of f on $V \setminus U$ will be called *boundary values*, and the points of $V \setminus U$ *boundary points*; analogous on U are the *interior values* and *interior points*. The problem of finding a harmonic function given boundary points and boundary values is called the *Dirichlet problem*; it is explored at length in [3].

Lemma 3.11. *Once a Markov chain, state space, boundary points, and boundary values are fixed, then a harmonic function on the interior points is unique.*

Proof. If f and g are two harmonic functions under the same above-listed constraints, then this follows from the Maximum Principle (which states that a harmonic function takes its maximum value on the boundary) and analogous Minimum Principle applied to the harmonic function $h \equiv f - g$. □

Lemma 3.12. *Let a network $[G, c]$ and a source-set $\{a, b\}$ be given. Then a potential function on $V(G)$ is harmonic on $V(G) \setminus \{a, b\}$.*

Proof. This proof, per [5], uses Ohm's Law (2.4) and Kirchhoff's Current Law (2.6) to obtain

$$\sum_{v\in V} c_{u,v}(\phi(v)-\phi(u)) = 0, \qquad u \neq a, b.$$

The lemma follows from a little algebraic manipulation. □

Definition 3.13. Let a Markov chain Z and underlying network $[G, c]$ be given. For two disjoint subsets $A, B \subset V(G)$, the so-called *hitting function* $h : V(G) \to [0, 1]$ that maps a vertex to its *hitting probability* is defined on the set of vertices as follows:

$$h(v) = \mathbb{P}(\exists k \text{ such that } Z_k \in B \text{ and } Z_\ell \notin A\, \forall\, 1 \le \ell \le k \mid Z_0 = v).$$

For readability, this function will often be written with more words than symbols, following the notation of [5]:

$$h(v) = \mathbb{P}(Z \text{ hits } B \text{ before } A \mid Z_0 = v).$$

Lemma 3.14. *The hitting function for a network $[G, c]$ with respect to $A, B \subset V(G)$ is harmonic on $V(G) \setminus (A \cup B)$.*

Proof. This follows from the general probabilistic fact [3, p. 5] that, if A is an event and B and C mutually exclusive events, then

$$\mathbb{P}(A) = \mathbb{P}(B) \cdot \mathbb{P}(A \mid B) + \mathbb{P}(C) \cdot \mathbb{P}(A \mid C).$$

The different possible results of a certain trial in a Markov chain are, of course, mutually exclusive. □

We are now ready to prove Theorem 3.9. The essential idea of the following proof is creating conditions such that the hitting function and the potential function are equal to one another. The hitting function is then related probabilistically to the return probability, while the potential function is related electrically to the effective resistance.

Proof of Theorem 3.9 ([5, pp. 11-13]). We wish to prove the following equation:

$$\mathbb{P}(Z_n = 0 \text{ for some } n \ge 1 \mid Z_0 = 0) \stackrel{?}{=} 1 - \frac{1}{C_0 \mathsf{r}_{\mathsf{eff}}}. \tag{3.15}$$

Define the following hitting functions on G_n and $\overline{G_n}$, respectively:

$$\begin{aligned} g_n(v) &\equiv \mathbb{P}(Z \text{ hits } \partial\Lambda_n \text{ before } 0 \mid Z_0 = v), && v \in G_n, \\ \overline{g_n}(v) &\equiv \mathbb{P}(Z \text{ hits } I_n \text{ before } 0 \mid Z_0 = v), && v \in \overline{G_n}. \end{aligned}$$

By Lemma 3.14, $\overline{g_n}$ is harmonic on $\overline{G_n}$ with boundary conditions $\overline{g_n}(0) = 0$, $\overline{g_n}(I_n) = 1$. Since we can apply a unit voltage across $\overline{G_n}$ such that $\phi(0) = 0$ and $\phi(I_n) = 1$, and since Lemma 3.12 tells us that the resulting potential function will be harmonic, Lemma 3.11 lets us conclude that $\overline{g_n}$ is a potential function on $\overline{G_n}$. Of course, this means that $\overline{g_n}$ will satisfy Ohm's Law (2.4) on $\overline{G_n}$. Since $\overline{g_n}$ and g_n evidently agree on $\tilde{G_n} \equiv G_n - \partial\Lambda_n = \overline{G_n} - I_n$, this means that g_n, too, will satisfy Ohm's Law on $\tilde{G_n}$. (To be able to use the probabilistic and electrical properties of g_n, we need $\tilde{G_n}$ to be non-trivial; accordingly, we assume $n \ge 2$ for the rest of the proof.)

The following holds by definition of g_n and the fact (Lemma 3.14) that g_n is harmonic:

$$\begin{aligned}
1 &= \mathbb{P}(Z_n \text{ hits } 0 \text{ before } \partial\Lambda_n \mid Z_0 = 0) + \mathbb{P}(Z_n \text{ hits } \partial\Lambda_n \text{ before } 0 \mid Z_0 = 0) \\
&= \mathbb{P}(Z_n \text{ hits } 0 \text{ before } \partial\Lambda_n \mid Z_0 = 0) + g_n(0) \\
&= \mathbb{P}(Z_n \text{ hits } 0 \text{ before } \partial\Lambda_n \mid Z_0 = 0) + \sum_{v\sim 0} p_{0v} g_n(v).
\end{aligned} \tag{3.16}$$

In applying a unit voltage across $\overline{G_n}$, we fixed ourselves a physical $0/I_n$-flow $\mathsf{i}(n)$ on $\overline{G_n}$. For fixed n, define $\mathsf{I}_0(n) \equiv \sum_{v\sim 0} \mathsf{i}_{0,v}(n)$. Then we have the following:

$$\mathsf{r}_{\mathsf{eff}}(n) = \mathsf{r}_{\mathsf{eff}}(0, I_n) = \frac{\phi(I_n) - \phi(0)}{\mathsf{I}_0(n)} = \frac{1}{\mathsf{I}_0(n)}. \tag{3.17}$$

After rearranging 3.16, we use Definition 1.5 of transition probabilities, Ohm's Law (Equation 2.4), and Equation 3.17 to obtain the following:

$$\begin{aligned}
\mathbb{P}(Z_n \text{ hits } 0 \text{ before } \partial\Lambda_n \mid Z_0 = 0) &= 1 - \sum_{v\sim 0} p_{0v} g_n(v) \\
&= 1 - \sum_{v\sim 0} \frac{c_{0,v}}{C_0}[g_n(v) - g_n(0)] \\
&= 1 - \frac{1}{C_0} \sum_{v\sim 0} \mathsf{i}_{0,v}(n) \\
&= 1 - \frac{\mathsf{I}_0(n)}{C_0} \\
&= 1 - \frac{1}{C_0 \mathsf{r}_{\mathsf{eff}}(n)}.
\end{aligned}$$

Since $\mathbb{P}(Z_n \text{ hits } 0 \text{ before } \partial\Lambda_n \mid Z_0 = 0)$ approaches $\mathbb{P}(Z_n = 0 \text{ for some } n > 0 \mid Z_0 = 0)$ as n approaches infinity, this proves equation 3.15. Since we are assuming local finiteness, we know $C_0 < \infty$, and the statement of the theorem follows. □

This completes our original goal of finding a necessary and sufficient condition for the recurrence of a random walk on $[G, c]$. However, the computational option currently available to us is unsatisfactory. For a network like Pólya's three-dimensional lattice, computing the effective resistance of $\overline{G_n}$ for large n using only electrical methods is not an easy task. The rest of the paper is devoted to the development of easier computational methods.

4. The Pseudoinverse of the Laplacian

Like many other geometrically-defined computations, the computation of effective resistance can be greatly simplified by spectral graph theory. In particular, the effective resistance between two points can be computed using the Moore-Penrose pseudoinverse of the weighted discrete Laplacian.

Fix a network $[G, c]$ and let $\mathbb{R}^V$ be the set of all functions $f : V(G) \to \mathbb{R}$. This is a vector space over $\mathbb{R}$ that comes equipped with an inner product

$$\langle f, g\rangle \equiv \sum_{v\in V} f(v)g(v)$$

and a norm

$$||f|| \equiv \langle f, f\rangle^{\frac{1}{2}}.$$

We will work throughout with the orthonormal basis $\underline{e} \equiv e_1, \ldots, e_n$ where e_i is the following function:[6]

$$e_i(v_j) = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases}$$

The definition of the Laplacian uses two important spectral matrices. The *admittance matrix* $A = (a_{ij})$ of $[G, c]$ is the $n \times n$ matrix satisfying the following:

$$a_{ij} \equiv \begin{cases} c_{ij} & i \sim j, \\ 0 & i \not\sim j. \end{cases} \tag{4.1}$$

The *weighted degree matrix* $D = (d_{ii})$ is the diagonal $n \times n$ matrix satisfying the following:

$$d_{ii} \equiv C_i. \tag{4.2}$$

The *weighted discrete Laplacian* $\mathscr{L}$ is defined as the difference of these two matrices:

$$\mathscr{L} \equiv D - A. \tag{4.3}$$

This Laplacian reduces to the traditional discrete Laplacian when each edge has unit conductance, in which case D is equal to the unweighted degree matrix and A to the adjacency matrix.

When G is connected, this weighted Laplacian (like its unweighted counterpart) has eigenvalue zero with respective $n \times 1$ eigenvector $\mathbf{1} \equiv \sum_{i=1}^{n} e_i$. This means, of course, that $\mathscr{L}$ is not invertible; however, it is invertible on the subspace $\mathbf{1}^{\perp}$ orthogonal to $\mathbf{1}$. This *Moore-Penrose pseudoinverse*, sometimes called the *generalized inverse*, will be denoted $\mathscr{L}^{\dagger}$.

Theorem 4.4. *Let a network $[G, c]$ be given. If $\mathscr{L}^{\dagger}$ is the Moore-Penrose pseudoinverse of the Laplacian, then the effective resistance $\mathsf{r}_{\mathsf{eff}}(a, b)$ between vertices a and b is given by the following:*

$$\begin{aligned} \mathsf{r}_{\mathsf{eff}}(a, b) &= (e_a - e_b)^t \mathscr{L}^{\dagger} (e_a - e_b) \\ &= (\mathscr{L}^{\dagger})_{aa} + (\mathscr{L}^{\dagger})_{bb} - (\mathscr{L}^{\dagger})_{ab} - (\mathscr{L}^{\dagger})_{ba}. \end{aligned} \tag{4.5}$$

The following proof is due to Klein and Randić [6]. Most of the proof is manipulation of basic electrical equations involving edge weights. Since the admittance and degree matrices capture these weights, the proof eventually abstracts to those matrices, and in turn to the Laplacian. Finally, the Laplacian is inverted under the necessary subspace restriction.

Proof. For a network $[G, c]$, let i be a current (i.e., a physical flow) with source a and sink b, and let $|\mathsf{i}|$ be its size. We wish to prove the following equation:

$$\mathsf{r}_{x,y} \stackrel{?}{=} \frac{|\mathsf{i}|}{\mathsf{i}_{x,y}} (e_x - e_y)^t \mathscr{L}^{\dagger} (e_a - e_b). \tag{4.6}$$

Using the Kronecker delta, the conditions on the sum of flows into and out of a vertex (see Definition 2.7) can be written as follows:

$$\sum_{y \sim x} \mathsf{i}_{x,y} = |\mathsf{i}| (\delta_{xa} - \delta_{xb}).$$

[6]As usual, $n = |V(G)|$.

Combined with Ohm's Law (Equation 2.4), this yields the following:

$$\sum_{y\sim x} c_{x,y}(\phi_x - \phi_y) = |\mathsf{i}|(\delta_{xa} - \delta_{xb}).$$

We can distribute and use Definitions 4.1 and 4.2 of the admittance matrix A and degree matrix D to obtain the following:

$$(D)_{xx}\phi_x - \sum_y (A)_{xy}\phi_y = |\mathsf{i}|(\delta_{xa} - \delta_{xb}).$$

We can rewrite this using the Laplacian (Definition 4.3):

$$\sum_y (\mathscr{L})_{xy}\phi_y = \sum_y (D-A)_{xy}\phi_y = |\mathsf{i}|(\delta_{xa} - \delta_{xb}).$$

Since this is true for all x, our orthonormal basis $\underline{e} = e_1, \ldots, e_n$ encodes all of this in matrices:

$$\mathscr{L}\sum_x \phi_x e_x = |\mathsf{i}|(e_a - e_b).$$

This is invertible on $\mathbf{1}^{\perp}$, where c is some constant:

$$\sum_x \phi_x e_x = |\mathsf{i}|\mathscr{L}^{\dagger}(e_a - e_b) + c\mathbf{1}.$$

In particular, the constant disappears when we consider the potential difference between two vertices x and y:

$$\phi_x - \phi_y = |\mathsf{i}|(e_x - e_x)^t\mathscr{L}^{\dagger}(e_a - e_b).$$

Ohm's Law (Equation 2.4) then implies the desired Equation 4.6. Since the imagined current $\mathsf{i}_{a,b}$ along an imagined edge is simply equal to the size $|\mathsf{i}|$ of the current, Equation 4.6 implies the first equality in the desired Equation 4.5. The second equality follows from simple matrix multiplication - its only purpose is to ease computation of $\mathsf{r}_{\mathsf{eff}}(a,b)$ once the matrix $\mathscr{L}^{\dagger}$ is found. □

The benefit of spectral computation is that we need not search the graph to find resistors we can simplify first, as we must in electrical computation. In fact, the pseudoinverse $\mathscr{L}^{\dagger}$ may even be feasible to compute: P. Courrieu [2, p. 27] has developed an algorithm for computing $\mathscr{L}^{\dagger}$ given $\mathscr{L}$ whose two main processes could run in as little as $O(k)$ and $O(\log r)$ time, where k is the number of vertices of the network in question (here, $[\overline{G_n}, \overline{c}]$) and r is the rank of $\mathscr{L}$. The problem is that this must be computed for each n.

Since they yield the effective resistance directly, spectral methods do not require us to guess whether a network is recurrent or transient. However, if our intuition suggests a solution to the type problem, we may turn to one of several confirmation tests that are significantly easier than the spectral method. The next section considers such a test for recurrence.

5. The Nash-Williams Test

Shorting and cutting, introduced in Section 2, were used Section 3 only to help define effective resistance in a limit case. The monotonicity properties introduced in 2.14 (namely, that shorting can only decrease effective resistance, while cutting can only increase it) make the two excellent tools in bounding effective resistance. Since we are primarily concerned about testing for finiteness, these bounds are

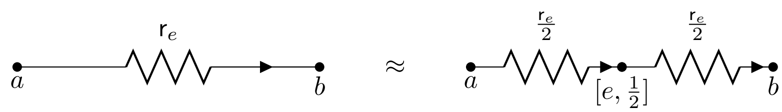


FIGURE 6. An edge e and its refinement have the same effective resistance, since we can add resistances in the refinement in series. (Nash-Williams uses digraphs for ease of definitions, but his results still hold for undirected graphs.)

useful: If a shorted network still has infinite effective resistance, then the original network is also recurrent; if a cut network still has finite resistance, then the original network is also transient. In a sense that will be made precise, the ability to short the network or its electrical equivalent (into a specific shape) that still has infinite effective resistance is not only a sufficient but also a necessary condition to declare the original network recurrent. This is the surprising finding of a 1959 paper by C. St. J. A. Nash-Williams [8], generally acknowledged [3, p. 2] to be the first modern application of electrical circuit theory to random walks on graphs.

We do not think that the reader would benefit from a rehashing of Nash-Williams's proof. We will, however, discuss several of the more important ideas in his paper in intuitive terms.

Notation 5.1. Resistance (respectively, effective resistance) with respect to the network $[G, c]$ will sometimes be written ${}_G\mathsf{r}$ (respectively, ${}_G\mathsf{r}_{\mathsf{eff}}$) when $[G, c]$ must be distinguished from another network.

Definition 5.2. A *refinement* of a network $[G, c]$ is a new network $[H, d]$, obtained (roughly speaking) by adding points to the edges of G and re-defining edge weights such that $[G, c]$ and $[H, d]$ are electrically "the same." (A network is its own so-called *trivial refinement*). Nash-Williams uses networks on digraphs so that it makes sense, as in Figure 6, to take an arc e with resistance r_e and add a so-called *division point* $[e, \frac{1}{2}]$ to it - bisecting it, intuitively - such that the two resulting edges $[e, 0, \frac{1}{2}]$ and $[e, \frac{1}{2}, 1]$ each have resistance $\frac{\mathsf{r}_e}{2}$.

Generally, an edge $[e, \lambda, \phi]$ in H $(0 \leq \lambda < \phi \leq 1)$ "cut out of" an edge e with resistance r_e in G will be assigned a resistance $(\phi - \lambda)\mathsf{r}_e$ in H. If $0 = \theta_0 < \theta_1 < \cdots < \theta_n = 1$ are such that an edge $e = ab$ in G is divided into edges $[e, \theta_0, \theta_1], [e, \theta_1, \theta_2], \ldots, [e, \theta_{n-1}, \theta_n]$ in H, then the following holds:

$$\begin{aligned} {}_H\mathsf{r}_{\mathsf{eff}}(a, b) &= \sum_{i=1}^{n} {}_H\mathsf{r}[e, \theta_{i-1}, \theta_i] \\ &= \sum_{i=1}^{n} (\theta_i - \theta_{i-1})\, {}_G\mathsf{r}_e \\ &= (\theta_n - \theta_0)\, {}_G\mathsf{r}_e \\ &= {}_G\mathsf{r}_e. \end{aligned}$$

That is, the effective resistance of the divided edge in the refinement is the same as that of the original edge in the original network, since all of the new edges just represent resistors in series and have resistances defined accordingly.

Nash-Williams also goes to lengths to make sure that the net current flowing through a division point is zero, as we expect at any non-node point along a wire.

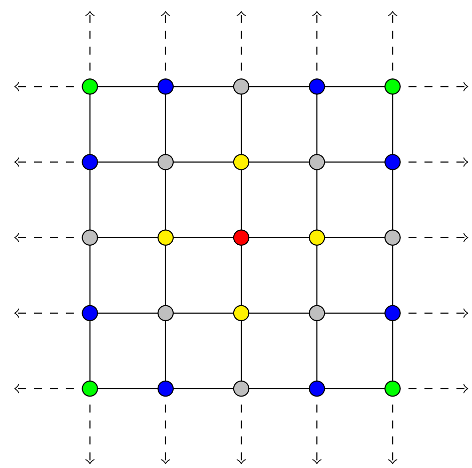


FIGURE 7. A picture illustrating (part of) a constriction of the two-dimensional integer lattice $\mathcal{L}^2$. This constriction groups vertices by standard geodesic distance from a selected origin. Vertices belonging to the same constricted subset of $V(\mathcal{L}^2)$ are drawn with the same color.

These two conditions suffice for us to think of a network and its refinement as being electrically "the same."

We are not very concerned with the technical role that refinements play in Nash-Williams's proof. Rather, we are concerned, first, with convincing the reader that they are electrically equivalent to the networks from which they are derived, and, second, with using them to show recurrence of several complicated networks with simple refinements.

For example, the graph on $\mathbb{Z}$ with vertices at the primes and resistance distance equal to Euclidean distance has the simple integer lattice on $\mathbb{Z}$ as its refinement. This latter graph, as will be discussed below, is significantly easier to work with.

Definition 5.3. A *constriction* of an infinite graph G is an infinite sequence y_0, y_1, y_2 of nonempty finite subsets of $V(G)$ satisfying the following conditions:

(i) The y_i form a partition of $V(G)$.
(ii) There are no edges between any subsets y_m and y_n whose indices satisfy $|m - n| \geq 2$.

Example 5.4. Although we will later use constrictions to short a network, the constriction itself is just a partition. Figure 7 displays a constriction of the two-dimensional integer lattice $\mathcal{L}^2$ that will be used in the following section. The constriction is defined as follows:

$$y_k \equiv \{v \in V(\mathcal{L}^2) \mid d(0, v) = k\}.$$

Definition 5.5. We can find a constriction of any infinite graph G, whether or not it is equipped with edge weights. If we do have edge weights and a resulting network $[G, c]$, however, more options are available to us. In particular, for fixed i we can short all the vertices in a single subset y_i into a point S_i. The resulting shorted network has finitely many resistors in parallel between S_i and S_{i+1} for $i = 0, 1, 2, \ldots$ and, once those are simplified, only an infinite chain of resistors in series. This means that the effective resistance of the shorted network is very, very easy to compute. If the effective resistance thus computed is infinite, the constriction is said to be *c-slowly-widening*. If the network $[G, c]$ has a c-slowly-widening constriction, then the network itself is said to be *slowly-widening*.

Theorem 5.6 (Nash-Williams [8])**.** *An infinite network is recurrent if and only if it has a slowly-widening refinement.*

The fact that recurrence implies the existence of a slowly-widening refinement is fascinating and deserves more attention than fits the scope of this paper.

The benefit of this Nash-Williams test is that it allows us to easily confirm that certain networks are recurrent. The catch is that we must either provide the correct constriction or prove that no such constriction can exist for any refinement. Proving non-existence is difficult. The networks for which Nash-Williams's criterion is most useful, then, are those which we suspect to be recurrent and which immediately suggest a certain constriction. In practice, highly symmetric networks often satisfy the latter.

6. Pólya's Lattices

Perhaps the most symmetric networks are the d-dimensional integer lattices $\mathcal{L}^d$. Pólya's famous theorem considers random walks on the network $[\mathcal{L}^d, c]$ where c assigns a unit resistance to every edge. (As a consequence, these random walks are simple.) Pólya proved that the network $[\mathcal{L}^d, c]$ is recurrent for $d = 1, 2$ and transient for $d \geq 3$.

As discussed in sections 3 and 4, computing the effective resistance of these networks electrically can be a nightmare, and the matrices used in the spectral method can become very large. Fortunately, recurrence for $d = 1, 2$ follows easily from the Nash-Williams test.

Theorem 6.1. *A simple random walk on the d-dimensional integer lattice $\mathcal{L}^d$ is recurrent for $d = 1, 2$.*

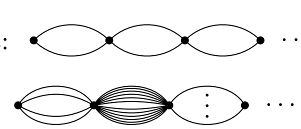
Proof. We wish to show that the trivial refinements of $[\mathcal{L}, c]$ and $[\mathcal{L}^2, c]$ (i.e., the networks themselves) are slowly-widening. To do this, we need two constrictions to short, one for each network. These constrictions, the latter of which is illustrated in Figure 7, both group together points by geodesic distance from the origin 0. Using the standard (geodesic) graph metric $d(a, b)$ (Definition 2.1), define

$$y_k \equiv \partial\Lambda_k = \{v \in V(\mathcal{L}^d) \mid d(0, v) = n\}.$$

The constriction is then $y_0, y_1, \ldots$. We have the following two pictures (the constrictions after shorting) and two sums (their infinite effective resistances):

$\mathcal{L}$:

$\mathcal{L}^2$:



$\sum_{j=1}^{\infty} \frac{1}{2} = \infty$

$\sum_{j=1}^{\infty} \frac{1}{8j-4} = \infty$

Figure 8. Shorted networks based on constrictions of $\mathcal{L}$ and $\mathcal{L}^2$, respectively. This figure is derived from one printed in [5, p. 14].

The statement of the theorem then follows from Nash-Williams's Theorem (5.6). □

We will not use the Nash-Williams test to prove transience for $d \geq 3$. Although beyond the scope of this paper, there is a constructive necessary and sufficient condition for transience, namely the existence of a finite-energy flow with a source but no sink; for the formalization of this, we refer readers to [7].

Acknowledgments. I would like to thank my mentor, Victoria Akin, for our conversations about graphs and electricity and random walks and for her regular suggestions of better, more coherent ways to think about the subject. This paper would not have come together without her detailed, helpful comments on multiple drafts, and I could not have proved Proposition 3.7 without her help. I would also like to thank Professor Peter May for running such an excellent program.

Annotations

The best introduction to electrical network theory of graphs, in this author's opinion, is found in **Doyle and Snell** [3]. The first chapter of **Grimmett** provides a higher-level, abridged version of the same [5]. The original paper of **Nash-Williams** is mathematically beautiful but provides little electrical motivation - it should not be a first introduction to the subject [8]. **Klein and Randic** present a singularly good idea, but the notation can be very confusing [6]. **Bondy and Murty** is an excellent reference for graph theory, but it only provides a short treatment of electrical networks [1].

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