# THE LOCAL THEORY OF ELLIPTIC OPERATORS AND THE HODGE THEOREM

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Abstract. In this paper, we develop the local theory of elliptic operators with a mind to proving the Hodge Decomposition Theorem. We then deduce a few of its corollaries including, for compact, oriented manifolds, Poincaré Duality and finite-dimensionality of the de Rham cohomology groups.

## Contents



## 1. Introduction

We will begin by putting an $L_2$ inner product on the space $E(M)$ of differential forms on a manifold $M$. Questions about the kernel of the exterior derivative and its adjoint relative to this inner product, as well as of the existence of energy minimizers with respect to this inner product in de Rham cohomology classes, prompt the introduction of a Laplacian-type operator– the Laplace-Hodge operator– on differential forms. The Laplace-Hodge operator will allow us to leverage the analytic side of differential forms to produce a decomposition of $E(M)$ that gives considerable topological insight. Toward this end, it will be necessary to develop the theory of partial differential operators, in particular elliptic operators, a class of operators that includes the Laplace-Hodge operator and generalizes the standard Laplacian. The tools and setting for all this will be Fourier series and Sobolev spaces, respectively. The approach, briefly, is, starting with a partial differential operator on $C^k$ functions, to fatten this space of functions to get a space of functions– a Sobolev space– that is easier to do functional analysis on, get information in this space, and then bring the information back to the initial function space. The three main results we need to implement this are the Sobolev Lemma, the Rellich Compactness Lemma, and Gårding's Theorem. They, together with elementary Hilbert space methods, are all we will need to prove the Hodge Decomposition of $E(M)$. We conclude by presenting a few of the topological results that come out of the Hodge Decomposition.

## 2. Some Preliminaries

Let $V$ be an n-dimensional real vector space, and let $A(V)$ (resp. $A_p(V)$) denote the space of alternating multilinear functions on $V$ (resp. of degree $p$.) Given an inner product $\langle , \rangle$ on $V$, we define one on $A(V)$ as follows. An element of $A(V)$ is homogeneous of degree $k$ if it has the form

$$v_1^* \wedge ... \wedge v_k^*,$$

where $v_i^*$ is dual to $v_i \in V$. We define elements homogeneous of different degrees to be orthogonal, and for those of the same degree we set

$$\langle v_1^* \wedge ... \wedge v_k^*, w_1^* \wedge ... \wedge w_k^* \rangle = \det\langle v_i, w_j \rangle,$$

and then extend linearly to all of $A(V)$. Positive-definiteness follows from the fact that

$$\langle v_i, v_j \rangle = \det(V)\det(V^T) = (\det(V))^2,$$

where $V$ is a matrix obtained by choosing an orthonormal basis for the subspace spanned by the $v_i$ and expressing each $v_i$ in terms of this basis, and the other properties of an inner product are easily verified.

The geometry of this inner product is as follows. For simplicity, suppose we are in $\mathbb{R}^3$. Then a 2-form acts on two vectors by scaling the parallelogram given by the two vectors by some factor, projecting onto some plane, and then taking the area of the projected parallelogram. Ignoring the last part of the previous, we can view 2-forms as acting by scaling and then projecting onto a plane. With this interpretation, we can compose two 2-forms, by first performing the scaling and projection for one, and then for the other. The inner product we have defined measures the scaled closeness of two differential forms by the effect their composition has on the areas of parallelograms. For instance, $dx \wedge dy$ acts by projecting onto the $xy$ plane and $dx \wedge dz$ by projecting onto the $xz$ plane. The composition of these takes parallelograms to degenerate parallelograms, and accordingly $dx \wedge dy$ and $dx \wedge dz$ are perpendicular.[1]

**Definition 2.1.** An *orientation* on $V$ is a choice of component of $A_n(V) - \{0\}$, or equivalently a choice of basis of $V$. For a given orientation, a linear transformation $*$, called the *Hodge Star*, is defined by the requirements that, for any $\{dx_1, ..., dx_n\}$ dual to an orthonormal basis $\{e_1, ..., e_n\}$ for $V$,

$$*(1) = \pm dx_1 \wedge ... \wedge dx_n,$$
$$*(dx_1 \wedge ... \wedge dx_n) = \pm 1, \text{ and}$$
$$*(dx_1 \wedge ... \wedge dx_k) = \pm dx_{k+1} \wedge ... \wedge dx_n,$$

where we take "+" in the above if $dx_1 \wedge ... \wedge dx_n$ lies in the chosen component of $A_n(V) - \{0\}$ and "−" otherwise. This is well defined, and we observe that since

$$*(dx_{i_1} \wedge ... \wedge dx_{i_k} \wedge *(dx_{i_1} \wedge ... \wedge dx_{i_k})) = 1 = \det\langle e_{i_r}, e_{i_s} \rangle,$$

we can write

$$\langle \alpha, \beta \rangle = *(\alpha \wedge *\beta).$$

[1]Credit to Nate Sauder for this interpretation of the inner product.

It is also clear that on $A_p(V)$ the Hodge Star satisfies

$$** = (-1)^{(p)(n-p)}.$$

**Definition 2.2.** For a compact, oriented, Riemannian manifold $M$, let $E(M)$ (resp. $E_p(M)$) denote the space of differential forms on $M$ (resp. of degree $p$.) For any vector bundle $E$ with an inner product over a compact, oriented manifold $N$, we can define an inner product on sections of $E$ by integrating the pointwise inner product over $N$. Choose a volume form $dV$. Then in the case of differential forms this translates to

$$\langle \alpha, \beta \rangle = \int_M \langle \alpha, \beta \rangle dV = \int_M *(\alpha \wedge *\beta) dV.$$

We define $L_2\Lambda(M)$ to be the completion of $E(M)$ with respect to the norm induced by the inner product.

**Definition 2.3.** For a p-form $\omega$, define the *codifferential* $\delta$ by

$$\delta\omega = (-1)^{n(p+1)+1} * d * \omega.$$

Let $\alpha$ be a (p-1)-form and $\beta$ a p-form. Then

$$\begin{aligned} d(\alpha \wedge *\beta) &= d\alpha \wedge *\beta + (-1)^{p-1} \alpha \wedge d * \beta \\ &= d\alpha \wedge *\beta + (-1)^{(p-1)+(p+1)(n-p-1)+n(p+1)+1} \alpha \wedge *\delta\beta \\ &= d\alpha \wedge *\beta - \alpha \wedge *\delta\beta. \end{aligned}$$

By Stokes theorem and the fact that $M$ is a manifold without boundary, we therefore have

$$0 = \int_M (d(\alpha \wedge *\beta)) = \int_M (d\alpha \wedge *\beta) - \int_M (\alpha \wedge *\delta\beta) = \langle d\alpha, \beta \rangle - \langle \alpha, \delta\beta \rangle,$$

or

$$\langle d\alpha, \beta \rangle = \langle \alpha, \delta\beta \rangle.$$

Thus $\delta$ is the adjoint of $d$ relative to the inner product we have defined.

**Definition 2.4.** A *non-degenerate pairing* of two vector spaces $V$ and $W$ is a bilinear map $B$ on $V \times W$ such that for every non-zero $v \in V$ there exists a $w \in W$ for which $B(v, w) \neq 0$ and for every non-zero $w \in W$ there exists a $v \in V$ such $B(v, w) \neq 0$. Consider the maps

$$v \to B(v, w) \quad \text{and} \quad w \to B(v, w)$$

from $V$ and $W$ to $W*$ and $V*$ respectively. These have null kernels by the definition of $B$ and are therefore injections, so we see that $V$ and $W$ have the same dimension, and a non-degenerate pairing of $V$ and $W$ induces isomorphisms between $V$ and $W^*$ and $W$ and $V^*$.

**Example 2.5.** The bilinear map $B$ on $A_p \times A_{n-p}$ defined by

$$B(\alpha, \beta) = *(\alpha \wedge \beta)$$

is non-degenerate, since

$$B(\alpha, *\alpha) = \langle \alpha, \alpha \rangle \qquad \text{and} \quad B(*\beta, \beta) = \pm\langle *\beta, \beta \rangle.$$

This is one way to see that $A_p$ and $A_{n-p}$ have the same dimension and are isomorphic via $*$. It follows that $*$ also gives an isomorphism between $E^p(M)$ and $E^{n-p}(M)$.

**Definition 2.6.** For a manifold M, we define the pth *de Rham cohomology group* $H^p_{\mathrm{dR}}$ to be the vector space of closed forms modulo exact forms. In symbols, if

$$Z^p(M) = \mathrm{Ker}[d : \mathrm{A}^p(\mathrm{M}) \to \mathrm{A}^{p+1}(\mathrm{M})] \quad \text{and} \quad B^p(M) = \mathrm{Im}[d : \mathrm{A}^{p-1}(\mathrm{M}) \to \mathrm{A}^p(\mathrm{M})],$$

then

$$H^p_{\mathrm{dR}} = \frac{Z^p(M)}{B^p(M)}.$$

The de Rham cohomology groups are diffeomorphism invariants (as they have to be if we are to study them in differential geometry) because pullbacks commute with exterior differentiation. In fact, something much stronger is true.

**Theorem 2.7.** *If $M$ and $N$ are homotopy equivalent manifolds, then $H^p_{Dr}(M) \cong H^p_{Dr}(N)$ for each $p$.*

In particular, de Rham cohomology groups are topological invariants, a rather surprising fact on the face of it. Let $H^p(M)$ be the *pth* smooth singular homology group of $M$. The connection between de Rham cohomology and the topology of a manifold is made explicit by the following theorem of de Rham.

**Theorem 2.8.** *For any manifold $M$, the map $H^p_{dR} \mapsto Hom(H^p(M), \mathbb{R})$ defined to be, for any $[\omega] \in H^p_{dR}$, the element of $Hom(H^p(M), \mathbb{R})$ that maps the equivalence class of a chain $c$ to*

$$\int_c \omega$$

*is an isomomorphism.*

For a reference on the material of the previous two theorems, we refer the reader to [2].

Because $*$ defines an isomorphism between $E^p(M)$ and $E^{n-p}(M)$, we might guess that it also defines isomorphisms between $H^p_{dR}$ and $H^{n-p}_{dR}$. It does, in fact, but this is more difficult to prove. Proving it will involve singling out representative elements of each de Rham cohomology class. Differential forms being most basically things we integrate, we would expect a way of choosing representative elements of de Rham cohomology classes to have something to do with integration. The following lemma will clarify the picture to an extent.

**Lemma 2.9.** *Let $M$ be a compact, oriented Riemannian manifold of dimension $n$. Then if $\omega$ is a closed k-form, $\delta\omega = 0$ if and only if $\omega$ is the unique smooth form in its de Rham cohomology class with minimum norm.*

*Proof.* Suppose $\delta\omega = 0$, and denote by $[\omega]$ the de Rham cohomology class of $\omega$. Then for another element of $[\omega]$, $\omega + d\eta$, we have

$$\begin{aligned}\langle \omega + d\eta, \omega + d\eta \rangle &= \langle \omega, \omega \rangle + 2\langle \omega, d\eta \rangle + \langle d\eta, d\eta \rangle, \\ &= ||\omega||^2 + 2\langle \delta\omega, \eta \rangle + ||d\eta||^2, \\ &= ||\omega||^2 + ||d\eta||^2 > ||\omega||^2,\end{aligned}$$

and $\omega$ is the unique element with minimum norm.

Now assume $\omega$ is the element of its de Rham cohomology class with minimum norm, but $\delta\omega \neq 0$. Using the fact that $\delta\omega \neq 0$, we will show it is possible to shift $\omega$ slightly to get an element of smaller norm. Define

$$f(t) = ||\omega + d(\delta t)||^2.$$

Then

$$\begin{aligned}
f'(0) &= \lim_{t\to 0} \frac{1}{t} \left( \langle \omega + d(\delta t), \omega + d(\delta t) \rangle - ||\omega||^2 \right) \\
&= \lim_{t\to 0} \frac{1}{t} \left( 2\langle \omega, d\delta t\omega \rangle + \langle d\delta t\omega, d\delta t\omega \rangle \right) \\
&= \lim_{t\to 0} \left( 2\langle \delta\omega, \delta\omega \rangle + t\langle d\delta\omega, d\delta\omega \rangle \right) \\
&= 2\langle \delta\omega, \delta\omega \rangle \neq 0,
\end{aligned}$$

so $f$ cannot assume a minimum at 0 and $\omega$ cannot be the minimum. □

The proof above showed that for a closed form $\omega$ with $\delta\omega \neq 0$ and for small $t > 0$, $\omega - td\delta\omega$ will be an element of the same de Rham cohomology class as $\omega$, but with smaller norm. Although the previous theorem does not tell us that there exists an element $\omega$ of each cohomology class with $\delta\omega = 0$ and minimal norm, we might expect that choosing an $\omega$ from a cohomology class, subtracting $td\delta\omega$ for small $t$, and repeating, will in the limit get us to a unique "energy minimizing" element in that cohomology class.

To find energy minimizing elements, it makes sense to introduce a Laplacian type operator.

**Example 2.10.** For real functions defined in a region $U \subset \mathbb{R}^n$, consider the Dirichlet energy functional

$$E(f) = \frac{1}{2} \int_U ||\nabla f||^2.$$

Taking the variation of $E$ and applying Green's first identity, we have

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} E(f + \varepsilon u) = \int_U \nabla f \cdot \nabla u = -\int_U \Delta f u.$$

So if $\Delta f = 0$, the Dirichlet energy functional is stationary at $f$. On the other hand, the fundamental lemma of calculus of variations states that functions $u$ vanishing on $\partial U$ separate continuous functions, so for any stationary point $f$ of $E$, $\Delta f = 0$. Viewing $f$ as a differential 0-form, the vanishing of $\Delta f$ is equivalent to the vanishing of

$$\delta df = \Delta f.$$

The kernel of the operator we want to define should contain precisely those $\omega$ for which $d\omega$ and $\delta\omega$ vanish and, by the previous example, a good guess is that it includes a $\delta d$ term. Since as a Laplacian-type operator it should also be self-adjoint, defining

$$\Delta = d\delta + \delta d$$

does not seem like a bad bet at all. Indeed, if $\Delta\omega = 0$, then

$$0 = \langle \Delta\omega, \omega \rangle = \langle \delta\omega, \delta\omega \rangle + \langle d\omega, d\omega \rangle,$$

so $d\omega = \delta\omega = 0$, while if $d\omega = \delta\omega = 0$, then $\Delta\omega = 0$, so the vanishing of $d\omega$ and $\delta\omega$ is equivalent to the vanishing of $\Delta\omega$. We also note that on smooth real functions $\Delta = \delta d = -\sum \frac{\partial^2}{\partial x_i^2}$, which is just the normal Laplacian.

**Definition 2.11.** The operator $\Delta$ is called the *Laplace-Hodge Operator*, or just the Laplacian for short. We define the space $H^p$ of *Harmonic forms* of degree p to be

$$\{\omega \in E^p(M) : \Delta\omega = 0\}.$$

Note that harmonic forms are just our energy minimizing forms of before, and therefore proving each de Rham cohomology class contains a unique energy minimizing form is the same as proving each de Rham cohomology class contains a unique harmonic form. In addition to determining whether each de Rham cohomology class contains a unique harmonic form, we would also like to say something about the range of $\Delta$, as well as a way of projecting differential forms onto de Rham cohomology classes. The Hodge Decomposition Theorem addresses all three of these matters.

**Theorem 2.12** (Hodge)**.** *For each compact, oriented, Riemannian manifold $M$ of dimension $n$, and each integer $p$ with $0 \leq p \leq n$, the space of Harmonic p-forms is finite dimensional, and we have the following orthogonal direct sum decompositions of the space $E^p(M)$ of smooth p-forms on $M$:*

$$E^p(M) = \Delta(E^p) \oplus H^p = d(E^{p-1}) \oplus \delta(E^{p+1}) \oplus H^p, \tag{2.13}$$

*where $H^p$ is the space of Harmonic p-forms.*

To prove Theorem 2.12, it suffices to show the equation

$$\Delta\omega = \alpha \text{ on M} \tag{2.14}$$

has a solution $\omega$ for all $\alpha$ in $(H^p)^{\perp}$.

The inner product structure we have put on $E(M)$ will allow us to bring methods from functional analysis into the picture. If $\Delta\omega = \alpha$ for a p-form $\alpha$, then for all $\beta \in E^p(M)$ we have

$$\langle \Delta\omega, \beta \rangle = \langle \alpha, \beta \rangle,$$

or

$$\langle \omega, \Delta\beta \rangle = \langle \alpha, \beta \rangle.$$

A solution to (2.14) consequently determines a bounded linear functional $l$ satisfying

$$l(\Delta\beta) = \langle \alpha, \beta \rangle. \tag{2.15}$$

A bounded linear functional on $E^p(M)$ satisfying Equation (2.15) is called a *weak solution* of $\Delta\omega = \alpha$. One approach to solving Equation (2.14) involves demonstrating the existence of a weak solution and using properties of $\Delta$ and techniques from functional analysis to show the weak solution corresponds to an actual solution. An approach along these lines can be found in [4]. The approach we take will involve the operator $d+\delta$, the "square root" of the Laplacian $(d+\delta)(d+\delta) = d\delta + \delta d$. The Laplacian is a partial differential operator, and the crucial property we will use is its ellipticity, which we will define later in the paper. We first need to develop some of the machinery needed for dealing with partial differential equations in general.

## 3. Fourier Series

Our basic framework for Sobolev spaces will be Fourier series. First, some definitions and notational points. If $\alpha = (\alpha_1, ..., \alpha_n)$, where the $\alpha_i$ are integers, then we define

$$[\alpha] = \alpha_1 + ... + \alpha_n.$$

If the $\alpha_i$ are non-negative, we define $D^\alpha$ to be

$$\left(\frac{1}{i}\right)^{[\alpha]} \frac{\partial^{[\alpha]}}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}}.$$

If $\eta$ is another n-tuple of integers, we set

$$\eta^\alpha = \eta_1^{\alpha_1}...\eta_n^{\alpha_n}.$$

Let $\mathscr{P}$ denote the space of smooth, $2\pi$-periodic functions from $\mathbb{R}^n$ to $\mathbb{C}^m$ and let

$$Q = \{x \in \mathbb{R}^n \mid 0 < x_i < 2\pi, i = 1, ..., n\}.$$

For $\phi$ and $\psi$ in $\mathscr{P}$, the standard $L_2$ inner product is then

$$\langle \phi, \psi \rangle = \int_Q \phi \cdot \psi \, dx,$$

where $\psi \cdot \phi$ is the Hermitian inner product and $dx$ is Lebesgue measure divided by $1/(2\pi)^n$ . If $\psi \in \mathscr{P}$ and $\xi \in \mathbb{Z}^n$, the $\xi$th Fourier coefficient of $\psi$, $\hat{\psi}(\xi)$, is defined to be

$$\langle \psi, e^{i\xi\cdot x} \rangle.$$

**Proposition 3.1.** *For all $\psi \in \mathscr{P}$, the sum*

$$\sum_{\xi \in \mathbb{Z}^n} \hat{\psi}(\xi) e^{i\xi\cdot x} \tag{3.2}$$

*converges uniformly to $\psi$.*

*Proof.* We begin by showing that $\sum_{\xi \in \mathbb{Z}^n} \hat{\psi}(\xi) e^{i\xi\cdot x}$ converges uniformly. For any positive integer $k$, integrating (3.2) by parts repeatedly and noting that the boundary terms cancel, we see that for some $c'_k$ depending on the derivatives of $\psi$ to order $2nk$,

$$|\hat{\psi}(\xi)| \le \frac{c'_k}{(\xi_1)^{2k}...(\xi_n)^{2k}}. \tag{3.3}$$

It's not difficult to see by induction that

$$1 + |\xi|^2 < \text{const}\, \xi_1^2 ... \xi_n^2.$$

So, by (3.3), we can write

$$|\hat{\psi}(\xi)| \le \frac{c_k}{(1 + |\xi|^2)^k}$$

for some $c_k$. Showing uniform convergence thus reduces to showing convergence of

$$\sum_{\xi \in \mathbb{Z}^n} \frac{1}{(1 + |\xi|^2)^k} \tag{3.4}$$

for sufficiently large k. Applying an integral test, this is the same as the convergence of

$$\int_{\mathbb{R}^n} \frac{1}{(1 + |x|^2)^k} \, dx.$$

In polar coordinates, this is

$$\text{const} \int_{-\infty}^{\infty} \frac{r^{n-1}}{(1 + r^2)^k} \, dr,$$

which converges when $2k \ge n + 1$, or $k \ge \lfloor \frac{n}{2} \rfloor + 1$. So the sum (3.2) converges uniformly to a continuous function $\phi$ for $k \ge \lfloor \frac{n}{2} \rfloor + 1$. Note that we are only using the fact that $\psi$ is $C^k$ for $k \ge \lfloor \frac{n}{2} \rfloor + 1$.

Now let $\varphi = \psi - \phi$. By the completeness of the trigonometric system, for any $\epsilon > 0$ we can find a trigonometric polynomial $P$ for which

$$||\varphi - P|| < \epsilon.$$

Then, because $\varphi$'s Fourier coefficients all vanish, $\langle \varphi, P\rangle = 0$. We therefore have

$$||\varphi||^2 = \langle \varphi, \varphi\rangle = \langle \varphi, \varphi - P\rangle \leq ||\varphi|| ||\varphi - P|| < ||\varphi||\epsilon,$$

or

$$||\varphi|| < \epsilon.$$

Since $\varphi$ was continuous, it must be identically zero, and $\psi = \phi$. $\square$

## 4. Sobolev Spaces

We have shown that a sufficiently differentiable function will have a Fourier series converging to it. One of the goals of the next part of the paper will be to give conditions under which a sequence of candidate Fourier coefficients will correspond to an actual function with a certain number of derivatives, in some sense reversing the process of the proof of the last section. The natural environment for this question will be the Sobolev spaces. These are function spaces equipped with norms measuring the regularity of a function and its derivatives to a certain degree, and they have all the nice functional analytic properties we could ask for.

**Definition 4.1.** For $\psi \in \mathscr{P}$ and non-negative integer $s$, we define a norm $||\cdot||_s$ to be the square root of the sum of the $L_2$ norms of the derivatives of $\psi$ to order $s$:

$$||\psi||_s^2 = \sum_{[\alpha]\leq s} \int_Q |D^\alpha \psi(x)|^2 dx. \tag{4.2}$$

The Sobolev space $H_s$ is defined to be the completion of $\mathscr{P}$ relative to this norm. Recall from the previous section that for n-tuples of integers $\xi$ and $\alpha$,

$$\xi^\alpha = \xi_1^{\alpha_1}...\xi_n^{\alpha_n}.$$

By Proposition 3.1, any $\psi \in \mathscr{P}$ is equal to its Fourier series. Hence, (4.2) is just

$$||\psi||_s^2 = \sum_{\xi\in\mathbb{Z}^n}\sum_{[\alpha]\leq s} |\widehat{D^\alpha\psi}(\xi)|^2 = \sum_{\xi\in\mathbb{Z}^n}\sum_{[\alpha]\leq s} (\xi^\alpha)^2|\hat{\psi}(\xi)|^2. \tag{4.3}$$

There exists a $c > 0$ depending only on $s$ and $n$ for which

$$c\sum_{\xi\in\mathbb{Z}^n}(1+(|\xi_1|^2+...+|\xi_n|^2))^s < \sum_{\xi\in\mathbb{Z}^n}\sum_{[\alpha]\leq s}(\xi^\alpha)^2|\hat{\psi}(\xi)|^2 < \sum_{\xi\in\mathbb{Z}^n}(1+(|\xi_1|^2+...+|\xi_n|^2))^s.$$

Therefore, our norm in (4.2) can also be written as

$$||\psi||_s^2 = \sum_{\xi\in\mathbb{Z}^n}(1 + (|\xi_1|^2 + ... + |\xi_n|^2))^s\hat{\psi}(\xi) = \sum_{\xi\in\mathbb{Z}^n}(1 + |\xi|^2)^s\hat{\psi}(\xi). \tag{4.4}$$

Although what we have in mind in Definition 4.1 is relatively clear, it's not so apparent what $H_s$, as an abstract completion, looks like. We might guess by (4.4) that $||\cdot||_s$ takes a more tractable form in the frequency domain, and this is indeed the case. Let $\mathscr{S}$ denote the space of all $\mathbb{C}^m$ sequences indexed by $\mathbb{Z}^n$. For real $s$, we define a new $H_s$ to contain all $\sigma \in \mathscr{S}$ for which

$$\sum_{\xi\in\mathbb{Z}^n}(1 + |\xi|^2)^s|\sigma_\xi|^2 < \infty.$$

This new $H_s$ will soon be seen to be the same as the old one in the case of non-negative integer $s$. For $\sigma$ and $\tau$ in $H_s$, we set their inner product

$$\langle \sigma, \tau\rangle_s = \sum_{\xi\in\mathbb{Z}^n}(1 + |\xi|^2)^s(\sigma_\xi \cdot \tau_\xi), \tag{4.5}$$

and denote the corresponding norm by $||\cdot||_s$.

The previous sum is finite, since, by Cauchy-Schwartz,

$$\left|\sum_{\xi\in\mathbb{Z}^n}(1+|\xi|^2)^{\frac{s+t}{2}}\sigma\cdot\tau\right|^2 \leq \sum_{\xi\in\mathbb{Z}^n}(1+|\xi|^2)^s|\sigma_\xi|^2\sum_{\xi\in\mathbb{Z}^n}(1+|\xi|^2)^t|\tau_\xi|^2. \tag{4.6}$$

This can also be stated as

$$|\langle\sigma,\tau\rangle_s| \leq ||\sigma||_{s+t}||\tau||_{s-t}.$$

Note that when $t<s$, $||\sigma||_s \geq ||\sigma||_t$, so $H_s\subset H_t$. By Proposition 3.1, sequences of Fourier coefficients of elements of $\mathscr{P}$ form a subspace of each $H_s$, which we identify with $\mathscr{P}$. Since every element of $H_s$ with only finitely many nonzero terms corresponds to a trigonometric polynomial and member of $\mathscr{P}$, $\mathscr{P}$ is dense in $H_s$, as for any element $\sigma$ of $H_s$ the sequence $\{\sigma_n\}$ obtained by taking $\sigma_n$ to include only the finitely many terms $\sigma_\xi$ of $\sigma$ with $|\xi|<n$ converges to $\sigma$ in $H_s$. Thus, by the uniqueness of completions, our new $H_s$ is equivalent to the old $H_s$ for non-negative integer $s$. The new $H_s$ is very simple from the standpoint of functional analysis. It is just an $l^2$ space with respect to the counting measure weighted by $(1+|\xi|^2)^s$, and therefore a Hilbert space.

Much of the utility of Sobolev spaces comes from the following lemma.

**Lemma 4.7** (Sobolev)**.** *If $t\geq\lfloor\frac{n}{2}\rfloor+1$ and $\sigma$ is in $H_t$, then the sum*

$$\sum_\xi \sigma_\xi e^{ix\cdot\xi} \tag{4.8}$$

*converges uniformly.*

*Proof.* It is enough to show $\sum_\xi|\sigma_\xi|$ converges. For positive integer $N$ and by the Cauchy-Schwartz inequality,

$$\begin{aligned}\sum_{|\xi|<N}|\sigma_\xi| &= \sum_{|\xi|<N}(1+|\xi|^2)^{-t/2}(1+|\xi|^2)^{t/2}|\sigma_\xi|\\ &\leq \sum_{|\xi|<N}\frac{1}{(1+|\xi|^2)^t}\sum_{|\xi|<N}(1+|\xi|^2)^t|\sigma_\xi|^2\\ &\leq \left(\sum_{|\xi|<N}\frac{1}{(1+|\xi|^2)^t}\right)||\sigma||_t^2.\end{aligned}$$

In the proof of Proposition 3.1, we showed that

$$\sum_\xi\frac{1}{(1+|\xi|^2)^t}<\infty, \tag{4.9}$$

so we have absolute convergence in (4.8). □

**Corollary 4.10.** *If $t\geq\lfloor\frac{n}{2}\rfloor+1$ and $\sigma$ is in $H_t$, then there is a constant $c$ such that*

$$||\sigma||_\infty\leq c.$$

*Proof.* This follows from the proof of the Sobolev lemma and the inequality

$$||\sigma||_\infty\leq\sum_\xi|\sigma_\xi|.$$

□

It is clear that the inclusion mapping $H_s \mapsto H_t$ for $t < s$ is continuous. The next lemma states that it is also compact.

**Lemma 4.11** (Rellich)**.** *Let $\{\sigma_i\}$ be a sequence of elements in $H_s$ with $||\sigma_i||_s \leq 1$. Then if $t < s$, $\{\sigma_i\}$ has a subsequence that converges in $H_t$.*

*Proof.* For any $\xi$, the sequence $\{(1+|\xi|^2)^{t/2}(\sigma_i)_\xi\}$ is bounded and has a convergent subsequence. A Cantor diagonal process gives a subsequence $\{\sigma_{i_k}\}$ such that for each $\xi$, the sequence

$$\{(1+|\xi|^2)^{t/2}(\sigma_{i_k})_\xi\}$$

converges. We claim $\{\sigma_{i_k}\}$ is Cauchy and therefore convergent in $H_t$. For a positive integer $N$, we write $||\sigma_{i_r} - \sigma_{i_s}||_t^2$ as

$$\sum_\xi (1+|\xi|^2)^t |(\sigma_{i_r})_\xi - (\sigma_{i_s})_\xi|^2$$

$$= \sum_{|\xi|<N} (1+|\xi|^2)^t |(\sigma_{i_r})_\xi - (\sigma_{i_s})_\xi|^2 + \sum_{|\xi|\geq N} (1+|\xi|^2)^{t-s}(1+|\xi|^2)^s |(\sigma_{i_r})_\xi - (\sigma_{i_s})_\xi|^2$$

$$\leq \sum_{|\xi|<N} (1+|\xi|^2)^t |(\sigma_{i_r})_\xi - (\sigma_{i_s})_\xi|^2 + \frac{1}{(1+N^2)^{s-t}} \sum_{|\xi|\geq N} (1+|\xi|^2)^s |(\sigma_{i_r})_\xi|^2 + 2|(\sigma_{i_r})_\xi||(\sigma_{i_s})_\xi| + |(\sigma_{i_s})_\xi|^2$$

$$\leq \sum_{|\xi|<N} (1+|\xi|^2)^t |(\sigma_{i_r})_\xi - (\sigma_{i_s})_\xi|^2 + 4N^{t-s},$$

where to get the last inequality we're using the fact that $||\sigma_i||_s \leq 1$. We can make $4N^{t-s}$ as small as we want by choosing large $N$, and since

$$\sum_{|\xi|<N} (1+|\xi|^2)^t |(\sigma_{i_r})_\xi - (\sigma_{i_s})_\xi|^2 \tag{4.12}$$

has only finitely many terms, the convergence of the sequences $\{(1+|\xi|^2)^{t/2}(\sigma_i)_\xi\}$ allows us to make the quantity (4.12) arbitrarily small as well. The sequence $\{\sigma_{i_k}\}$ is thus Cauchy, and the proof is complete. ☐

The following estimate will be useful.

**Proposition 4.13** (The "Peter-Paul" Inequality)**.** *For $s < t < u$ and $\epsilon > 0$, we can find a constant $c$ for which*

$$||\sigma||_t^2 < \epsilon ||\sigma||_u^2 + c||\sigma||_s^2 \tag{4.14}$$

*for all $\sigma \in H_u$.*

*Proof.* For sufficiently large $N$ and all $|\xi| > N$ we have

$$(1+|\xi|^2)^t < \epsilon(1+|\xi|^2)^u.$$

Therefore, $||\sigma||_s^2$ has to compensate for only finitely many terms, which it can if $c$ is made large enough. ☐

Since we are trying to get from the generalized functions of $H_s$ back to actual functions with a certain degree of differentiability, we would like a notion of formal differentiation on $H_s$.

**Definition 4.15.** We define the map

$$D^\alpha : H_s \mapsto H_{s-[\alpha]}$$

to be the extension of the usual differentiation map on $\mathscr{P}$ by density. More concretely,

$$(D^\alpha(\sigma))_\xi = (\xi)^\alpha \sigma_\xi.$$

**Proposition 4.16.** *For $\sigma \in H_s$,*

$$||D^\alpha \sigma||_{s-[\alpha]} \leq ||\sigma||_s.$$

*Proof.* This follows from

$$|(D^\alpha \sigma)_\xi|^2 = |\xi^\alpha \sigma_\xi|^2 \leq (1 + |\xi|^2)^{[\alpha]} |\sigma_\xi|^2.$$

□

Proposition 4.16 and the Sobolev lemma give us the following proposition.

**Proposition 4.17.** *If $\sigma \in H_t$ for $t \geq \lfloor \frac{n}{2} \rfloor + m + 1$, then the sum*

$$\sum_\xi (D^\alpha \sigma)_\xi = \sum_\xi \xi^\alpha \sigma_\xi e^{ix\cdot\xi}$$

*converges uniformly for $[\alpha] \leq m$. Hence, $\sigma$ corresponds to an actual function with derivatives to order $m$. Furthermore, there is a constant $c$ for which*

$$||D^\alpha \sigma||_\infty \leq c||\sigma||_s.$$

**Definition 4.18.** For $\sigma \in H_s$, we define

$$(K_t(\sigma))_\xi = (1 + |\xi|^2)^t \sigma_\xi.$$

It's clear that

$$||\sigma||_s = ||K_t \sigma||_{s-2t}$$

and that $K_t : H_s \mapsto H_{s-2t}$ is an isometry. The operator $K_t$ will thus permit us to transport information between different Sobolev spaces. We also have

$$\langle \sigma, \tau \rangle_s = \langle \sigma, K_t \tau \rangle_{s-t}.$$

Note that as a partial differential operator,

$$K_t = \left( 1 - \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} \right)^t.$$

It will be useful later on to know how multiplication by a complex function affects the $H_s$ norm.

**Proposition 4.19.** *Let $\omega$ be a smooth, complex-valued periodic function on $\mathbb{R}^n$. Then for any integer $s$, we can find constants $c$ and $c'$, depending on $s$, $n$, and the derivatives of $\omega$ to order $s$, for which for all $\psi$ in $\mathscr{P}$,*

$$||\omega\psi||_s \leq c||\psi||_{s-1} + c'||\omega||_\infty ||\psi||_s. \tag{4.20}$$

**Corollary 4.21.** *There is a constant $c$ depending on $s$, $n$, and the derivatives of $\omega$ to order $s$, such that*

$$||\omega\psi||_s \leq c||\psi||_s.$$

*Proof.* We first handle the case of $s \geq 0$. By equation (4.4) in Definition 4.1,

$$||\omega\psi||_s \leq \text{const} \sum_{[\alpha]\leq s} ||D^\alpha(\omega\psi)||. \tag{4.22}$$

The key observation is that $D^\alpha\omega\psi - \omega D^\alpha\psi$ is a partial differential operator of order $[\alpha]-1$. Forcing this term into (4.22), we get

$$\begin{aligned}\sum_{[\alpha]\leq s} ||D^\alpha(\omega\psi)|| &\leq \sum_{[\alpha]\leq s} ||D^\alpha(\omega\psi) - \omega D^\alpha\psi|| + \sum_{[\alpha]\leq s} ||\omega D^\alpha\psi|| \\ &\leq \text{const} \sum_{[\alpha]\leq s-1} ||D^\alpha(\psi)|| + \text{const } ||\omega||_\infty \sum_{[\alpha]\leq s} ||D^\alpha\psi|| \\ &\leq \text{const}||\psi||_{s-1} + \text{const}||\omega||_\infty ||\psi||_s.\end{aligned}$$

For $s < 0$, we use Cauchy-Schwartz and the operator $K_t$ to reduce ourselves to the positive case:

$$\begin{aligned}||\omega\psi||_s^2 &= \langle \omega K_{-s}K_s\psi, K_s\omega\psi\rangle_0 \\ &= \langle K_{-s}\omega K_s\psi, K_s\omega\psi\rangle_0 + \langle(\omega K_{-s} - K_{-s}\omega)K_s\psi, K_s\omega\psi\rangle_0 \\ &\leq |\langle K_{-s}\omega K_s\psi, K_s\omega\psi\rangle_0| + \sum_{[\alpha]\leq -2s-1} |\langle c_\alpha D^\alpha K_s\psi, K_s\omega\psi\rangle_0|,\end{aligned}$$

by the fact that $\omega K_{-s} - K_{-s}\omega$ is a partial differential operator of order $-s+1$ and where the $c_\alpha$ depend on the derivatives of $\omega$ to order $-s+1$. By the positive case, for the first term we have

$$\begin{aligned}|\langle K_{-s}\omega K_s\psi, K_s\omega\psi\rangle_0| = |\langle \omega K_s\psi, K_s\omega\psi\rangle_{-s}| &\leq ||\omega K_s\psi||_{-s}||K_s\omega\psi||_{-s} \\ &\leq (c||\omega||_\infty ||K_s\psi||_{-s} + c'||K_s\psi||_{-s-1})||K_s\omega\psi||_{-s} \\ &= (c||\omega||_\infty ||\psi||_s + c'||\psi||_{s-1})||\omega\psi||_s.\end{aligned}$$

For the other term,

$$\begin{aligned}\sum_{[\alpha]\leq -2s-1} |\langle c_\alpha D^\alpha K_s\psi, K_s\omega\psi\rangle_0| &\leq \text{const} \sum_{[\alpha]\leq -2s-1} ||D^\alpha K_s\psi||_s ||K_s\omega\psi||_{-s} \\ &\leq \text{const} \sum_{[\alpha]\leq -2s-1} ||K_s\psi||_{s+[\alpha]} ||K_s\omega\psi||_{-s} \\ &\leq \text{const}||K_s\psi||_{-s-1}||K_s\omega\psi||_{-s} \\ &\leq \text{const}||\psi||_{s-1}||\omega\psi||_s.\end{aligned}$$

Dividing by $||\omega\psi||_s$ gives the inequality for $s < 0$. □

**Definition 4.23.** We define the *translate of* $\sigma$ by $h$ to be

$$T_h(\sigma)_\xi = e^{ih\cdot\xi}\sigma_\xi$$

and the *difference quotient* $\sigma^h$ for a nonzero $h$ to be

$$\frac{T_h(\sigma) - \sigma}{|h|}.$$

As the next proposition shows, uniform boundedness of different quotients in $H_s$ corresponds to an additional degree of formal differentiability.

**Proposition 4.24.** *For $\sigma$ in $H_s$, if for some $M$ and all nonzero $h$*

$$||\sigma^h||_s \leq M,$$

*then $\sigma^h$ is in $H_{s+1}$.*

*Proof.* We want to use the fact that

$$||\sigma^h||_s^2 = \sum_\xi \left| \frac{e^{ih\cdot\xi} - 1}{|h|} \right|^2 (1 + |\xi|^2)^s |\sigma_\xi|^2 < M^2 \tag{4.25}$$

to bound

$$\sum_\xi (1 + |\xi|^2)(1 + |\xi|^2)^s |\sigma_\xi|^2.$$

Fix an orthonormal basis $\{e_i\}$ and a positive integer $N$. Then,

$$|\xi_i|^2 = \lim_{t\to 0} \left| \frac{e^{it(e_i\cdot\xi)} - 1}{t} \right|^2 .$$

Taking only the finitely many terms with $|\xi| < N$ in (4.25) and sending $t$ to 0 gives

$$\sum_{|\xi|<N} |\xi_i|^2 (1 + |\xi|^2)^s |\sigma_\xi|^2 \leq M^2,$$

so that

$$\sum_{|\xi|<N} (1 + |\xi|^2)(1 + |\xi|^2)^s |\sigma_\xi|^2 \leq nM^2 + ||\sigma||_s^2.$$

Since this holds for all positive integers $N$,

$$\sum_\xi (1 + |\xi|^2)(1 + |\xi|^2)^s |\sigma_\xi|^2 < \infty,$$

and $\sigma$ is in $H_{s+1}$. □

## 5. Partial Differential Operators and Ellipticity

**Definition 5.1.** A *linear partial differential operator $P$ of order $p$* on smooth $\mathbb{C}^m$-valued functions on $\mathbb{R}^n$ is an $m \times n$ matrix of the form

$$P_{i,j} = \sum_{[\alpha]\leq p} a_{i,j}^\alpha D^\alpha,$$

where the $a_{i,j}^\alpha$ are smooth complex-valued functions on $\mathbb{R}^n$ and at least one $a_{i,j}^\alpha$ with $[\alpha] = p$ is nonzero. This operator is said to be periodic if the $a_{i,j}^\alpha$ are periodic. Define the *formal adjoint* of a periodic partial differential operator $P$ by

$$p_{i,j}^* = \sum_\alpha D^\alpha \overline{a_{j,i}^\alpha}.$$

Then, integrating by parts, one can show

$$\langle P\psi, \phi\rangle_0 = \langle \psi, P^*\phi\rangle_0. \tag{5.2}$$

We use the word "formal" because $P^*$ is not the adjoint of $P$ in the sense of Hilbert space Hermitian adjoints, but just in that it satisfies (5.2).

**Proposition 5.3.** *If $P$ is a periodic partial differential operator of order $p$, then there are constants $c$, $M$, and $c'$, where $c$ depends on $n$, $m$, $l$, and $s$, $M = \max_{[\alpha]=p} |a^{\alpha}_{i,j}|$ where the $a^{\alpha}_{i,j}$ are the coefficients $P$, and $c'$ depends on $n$, $m$, $l$, $s$ and the derivatives of the coefficients of $P$ to order $l$, such that for all $\psi$ in $\mathscr{P}$,*

$$||P\psi||_s \leq cM||\psi||_{s+l} + c'||\psi||_{s+l-1}. \tag{5.4}$$

*A consequence is that*

$$||P\psi||_s \leq const\,||\psi||_{s+[\alpha]},$$

*and we can extend $P$ to a bounded operator $H_{s+[\alpha]} \mapsto H_s$.*

*Proof.* For the case $m = 1$, this follows immediately from Proposition 4.19, which stated that we had an inequality of the form

$$||\omega\psi||_s \leq c||\psi||_{s-1} + c'||\omega||_\infty ||\psi||_s,$$

and the other cases follow from the fact that

$$||P\psi||_s \leq \text{const} \sum_{i,j} ||P_{i,j}\psi_j||_s,$$

with the above constant depending only on $m$. □

Noting that

$$\left(\left(\sum_\alpha a_\alpha D^\alpha\right)\sigma\right)_\xi = a_\alpha \xi^\alpha \sigma_\xi,$$

if $P_l$ is the highest order part of a partial differential operator $P$ on complex-valued functions, we might expect the multilinear form

$$S(\xi) = a_\alpha \xi^\alpha$$

to be related to $P$.

**Definition 5.5.** The *symbol* $S : \mathbb{R}^n \mapsto M_n(\mathbb{R})$ of a partial differential operator $P$ is defined by

$$S(\xi)_{i,j} = \sum_\alpha a^\alpha_{i,j}\xi^\alpha.$$

A partial differential operator is *elliptic* if its symbol $S$ is everywhere non-singular.

**Example 5.6.** The symbol of a second order partial differential operator $P = \sum_{i,j} a_{i,j} \frac{\partial^2}{\partial x_i \partial x_j}$ with $a_{i,j} = a_{j,i}$ is the bilinear form

$$\sum_{i,j} a_{i,j}\xi_i\xi_j.$$

P is elliptic at a point exactly when this form is definite and the matrix $A_{i,j} = a_{i,j}$ has exclusively positive or negative eigenvalues. If this is the case, then we can diagonalize $P$ by some change of coordinates to get a multiple of the Laplacian.

Gårding's inequality will be the main tool in our work with elliptic operators. Before proving it, we need a lemma.

**Lemma 5.7.** *If $s$ is an integer, $\omega$ is a real-valued smooth periodic function, and $\psi$ and $\phi$ are in $\mathscr{P}$, then*

$$\langle \omega\psi, \phi\rangle_s \leq |\langle \psi, \omega\phi\rangle_s| + const||\psi||_s||\phi||_{s-1}. \tag{5.8}$$

*Proof.* For $s = 0$, $\langle \omega\psi, \phi\rangle_s = \langle \psi, \omega\phi\rangle_s$, and (5.8) is clear. Now suppose $s < 0$. To get the $\omega$ on the other side of the inner product, we put ourselves in $L_2$ and then go back:

$$\begin{aligned}\langle \omega\psi, \phi\rangle_s &= \langle \omega\psi, K_s\phi\rangle_0 \\ &= \langle K_{-s}K_s\psi, \omega K_s\phi\rangle_0 = \langle K_s\psi, K_{-s}(\omega K_s\psi)\rangle_0 \\ &\leq \langle \psi, \omega\phi\rangle_s + \langle \psi, (K_{-s}\omega - \omega K_{-s})(K_s\psi)\rangle_s.\end{aligned}$$

Observing that $(K_{-s}\omega - \omega K_{-s})$ is a partial differential operator of order $s-1$ and applying Cauchy-Schwartz and Proposition 4.16 gives the inequality. The argument is the same for $s > 0$, except we instead force $(K_s\omega - \omega K_s)$ into our expression. □

**Theorem 5.9** (Gårding's Inequality)**.** *Let $P$ be a periodic elliptic operator of order $p$. Then for every integer $s$ there is a constant $c$ such that for all $\sigma$ in $H_s$*

$$||\sigma||_{s+p} \leq c(||\sigma||_s + ||P\sigma||_s). \tag{5.10}$$

*Proof.* First assume $P$ has constant coefficients and only terms $D^\alpha$ of order $p$, and let $S$ be the symbol of $P$. $S$ is everywhere non-singular, so because the unit sphere is compact, there is a constant $c > 0$ such that for $|\xi| = |v| = 1$,

$$|S(\xi)v|^2 > c.$$

This means that for all $\xi$ and $v$, we have

$$|S(\xi)v|^2 > c|\xi|^{2p}|v|^2.$$

This implies

$$||P\sigma||_s^2 = \sum_\xi |S(\xi)\sigma_\xi|^2(1+|\xi|^2)^s \geq \text{const}\sum_\xi |\xi|^{2p}|\sigma_\xi|^2(1+|\xi|^2)^s,$$

and

$$\begin{aligned}(||P\sigma||_s + ||\sigma||_s)^2 &\geq ||\sigma||_s^2 + ||P\sigma||_s^2 \\ &\geq \sum_\xi |\sigma_\xi|^2(1+|\xi|^2)^s + \text{const}\sum_\xi |\xi|^{2p}|\sigma_\xi|^2(1+|\xi|^2)^s \\ &= \sum_\xi |\sigma_\xi|^2(1+|\xi|^2)^s(1+\text{const}|\xi|^{2p}) \\ &\geq \text{const}\sum_\xi |\sigma_\xi|^2(1+|\xi|^2)^{s+p} = \text{const}||\sigma||_{s+p}^2.\end{aligned}$$

Now let $P$ be a general elliptic operator of order $p$. The strategy is to prove the inequality locally, and then use partitions of unity and compactness to reduce to the local case. Consider a point $q$ in $\mathbb{R}^n$ and denote by $P'$ the constant coefficient operator determined by the highest order part of $P$ at $q$. Then, by the first case and for some constant $k$, we can write

$$||\sigma||_{s+p} \leq k(||\sigma||_s + ||P'\sigma||_s) \leq k(||\sigma||_s + ||P'\sigma - P\sigma||_s + |||P\sigma||_s).$$

Let $\tilde{P}$ be a periodic differential operator agreeing with $P' - P$ in a neighborhood $U$ of $q$ on which the coefficients of $P' - P$ are less than $1/2ck$ in magnitude for the $c$

given by Proposition 5.3. Then, for $\psi$ in $\mathscr{P}$ supported in $U$, we have by Proposition 5.3 that

$$\begin{aligned}
||\psi||_{s+p} &\le k(||\psi||_s + ||\tilde{P}\psi||_s + |||P\psi||_s) \\
&\le k||P\psi||_s + \frac{1}{2}||\psi||_{s+p} + \text{const}||\psi||_{s+p-1} + k||\psi||_s \\
&\le k||P\psi||_s + \frac{1}{2}||\psi||_{s+p} + \text{const}||\psi||_s + \frac{1}{4}||\psi||_{s+p} + \text{const}||\psi||_s,
\end{aligned}$$

where the last inequality comes from the Peter-Paul inequality. This proves (5.10) for these $\psi$. The neighborhoods $U$ corresponding to each point are an open cover of the torus $T^n$. Take a finite sub-cover, and a subordinate partition of unity $\{\omega_1', ..., \omega_m'\}$. Set

$$\omega_i = \frac{\omega_i'}{\sqrt{\sum_j \omega_j'^2}}.$$

Then the $\omega_i$ are smooth and

$$\sum \omega_i^2 = 1.$$

By Lemma 5.7 and the facts that multiplication by an $\omega_j$ is a bounded operator and there are only finitely many $\omega_j$,

$$\begin{aligned}
||\psi||_{s+p}^2 &= \sum \langle \omega_j^2 \psi, \psi \rangle_{s+p} \\
&\le \sum \langle \omega_j \psi, \omega_j \psi \rangle_s + \text{const}||\psi||_{s+p}||\psi||_{s+p-1} \\
&\le \text{const} \sum ||P\omega_j\psi||_s^2 + \text{const}||\psi||_s^2 + \text{const}||\psi||_{s+p}||\psi||_{s+p-1}.
\end{aligned}$$

To get rid of the $||P\omega_j\psi||_s^2$ terms, note that

$$\begin{aligned}
|\langle P\omega^2\psi, P\psi\rangle_s - ||P(\omega\psi)||_s^2| &\le |\langle P\omega^2\psi, P\psi\rangle_s - \langle \omega P\omega\psi, P\psi\rangle_s| + |\langle \omega P\omega\psi, P\psi\rangle_s - \langle P\omega\psi, P\omega\psi\rangle_s| \\
&= |\langle P\psi, P\omega^2\psi - \omega P\omega\psi\rangle_s| + |\langle P\omega\psi, \omega P\psi - P\omega\psi\rangle_s|.
\end{aligned}$$

Both $P\omega^2 u - \omega P\omega u$ and $\omega P u - P\omega u$ are differential operators of order $p-1$, so applying Cauchy-Schwartz gives

$$||P(\omega\psi)||_s^2 \le |\langle P\omega^2\psi, P\psi\rangle_s| + \text{const}||\psi||_{s+p}||\psi||_{s+p-1}. \tag{5.11}$$

Applying (5.11) and continuing where we left off,

$$\begin{aligned}
||\psi||_{s+p}^2 &\le \text{const} \sum \langle P\omega_j^2\psi, P\psi\rangle + \text{const}||\psi||_s^2 + \text{const}||\psi||_{s+p}||\psi||_{s+p-1} \\
&= \text{const}||P\psi||_s^2 + \text{const}||\psi||_s^2 + \text{const}||\psi||_{s+p}||\psi||_{s+p-1} \\
&\le \text{const}||P\psi||_s^2 + \text{const}||\psi||_s^2 + \frac{1}{2}||\psi||_{s+p}^2 + \text{const}||\psi||_{s+p-1}^2 \\
&\le \text{const}||P\psi||_s^2 + \text{const}||\psi||_s^2 + \frac{3}{4}||\psi||_{s+p}^2 + \text{const}||\psi||_s^2,
\end{aligned}$$

where in the last two steps we use first the arithmetic-geometric mean inequality and then the Peter-Paul inequality. Thus, (5.10) holds for $\psi$ in $\mathscr{P}$ and consequently all $\sigma$ in $H_{s+p}$. □

With a little more work, one can prove the regularity theorem for periodic elliptic operators. We will not need it in this paper, but we state it here for completeness.

**Theorem 5.12** (Elliptic Regularity). *If $P$ is a periodic elliptic operator of order $p$, $\sigma$ is in some $H_s$, $\tau$ is in $H_t$, and*

$$P\sigma = \tau,$$

*then $\sigma$ is in $H_{t+p}$.*

Thus if $\tau$ is smooth, $\sigma$ must also be. We also note that the corresponding result for elliptic operators with compact support or defined on $\mathbb{R}^n$ is not difficult to deduce from the periodic case.

**Example 5.13.** Consider the Cauchy-Riemann operator $\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}$. This is elliptic, and so every holomorphic function is smooth.

## 6. A Proof of the Hodge Theorem

The first thing is to lift our Sobolev Space technology to manifolds.

**Definition 6.1.** Cover a compact, oriented Riemannian Manifold $M$ with finitely many coordinate patches $(\varphi_i, U_i)$ with each $U_i$ contained in $T^n$, and let $\phi_i$ be a partition of unity subordinate to this cover. Define a norm $||\cdot||_s$ on the space of $C^\infty$ real functions on $M$ by

$$||f||_s = \sum_i ||\phi_i f \circ \varphi_i||_s. \tag{6.2}$$

$H_s(M)$ is defined to be the completion of this space. One can show that different choices of covers and partitions of unity result in the same norm, and that $H_0(M)$ is naturally isomorphic to $L_2(M)$. For a differential k-form $\omega$, we define

$$||\omega||_s = ||\langle\omega,\omega\rangle^{\frac{1}{2}}||_s, \tag{6.3}$$

where the inner product on differential forms is that defined in (2.2). We denote the completion of the space of (mixed) differential forms relative to this norm by $H_s\Lambda(M)$. As in the case of real functions, $H_s\Lambda(M)$ is independent of the chosen cover and partition of unity, and $H_0\Lambda(M) \cong L_2\Lambda(M)$.

The Rellich lemma is seen to hold on $H_s(M)$ and $H_s\Lambda(M)$ by applying the normal Rellich lemma to take Cauchy subsequences in each coordinate patch.

**Definition 6.4.** An operator $T$ that takes sections of a vector bundle $E$ over $M$ to sections of $E$ is a partial differential operator of order $t$ if it is a partial differential operator in coordinates. That is, if $\varphi$ and $\phi$ are coordinates for $E$,

$$\varphi^* T(\varphi^{-1})^*$$

is a partial differential operator of order $t$. This is well-defined, since for different coordinates $\phi$ we have

$$\phi^* T(\phi^{-1})^* = (\phi^*(\varphi^{-1})^*)(\varphi^* T(\varphi^{-1})^*)(\varphi^*(\phi^{-1})^*),$$

and by the chain rule this is also a partial differential operator of order $t$. A partial differential operator $T$ of order $t$ is elliptic if for each point $x$ in $M$, each non-zero section $u$ of $E$, and each smooth real function $\varphi$ on $M$ with $\varphi(x) = 0$ but $D\varphi(x) \neq 0$,

$$T(\varphi^t u)(x) \neq 0.$$

This is equivalent to $T$ being elliptic in coordinates. Gårding's inequality, like Rellich's lemma, holds for elliptic operators on a compact, oriented manifold $M$.

**Proposition 6.5.** *The Laplacian is elliptic.*

For the proof, we refer the reader to [4]. We are now in a position to prove the Hodge Decomposition Theorem.

**Theorem 6.6** (Hodge)**.** *For each compact, oriented, Riemannian manifold $M$ of dimension $n$, and each integer $p$ with $0 \leq p \leq n$, the space $H^p$ of Harmonic $p$-forms is finite dimensional, and we have the following orthogonal direct sum decompositions of the space $E^p(M)$ of smooth $p$-forms on $M$:*

$$E^p(M) = \Delta(E^p) \oplus H^p = d(E^{p-1}) \oplus \delta(E^{p+1}) \oplus H^p. \tag{6.7}$$

By the Laplacian's ellipticity and Gårding's inequality, we have that for $\omega$ in $H_s$,

$$||\omega||_s \leq \text{const}(||\omega||_{s-2} + ||\Delta\omega||_{s-2}).$$

The operator $D = d + \delta$ satisfies $D^2 = \Delta$. Hence,

$$||D\omega||_{s-1} \geq \text{const}||D^2\omega||_{s-2} = \text{const}||\Delta\omega||_{s-2},$$

so, because $||\omega||_{s-1} \geq ||\omega||_{s-2}$,

$$||\omega||_s \leq \text{const}(||\omega||_{s-1} + ||D\omega||_{s-1}),$$

and Gårding's inequality holds for the square root of the Laplacian as well. Thus, given the form of the decomposition in (2.13) and the fact that $\ker D = \ker \Delta$, it makes just as much sense to work with $D$ as $\Delta$.

**Lemma 6.8** (regularity for the operator D)**.** *If $\omega$ is in $H_s(M)$ for some $s$, $\eta$ is $C^\infty$, and $D\omega = \eta$, then $\omega$ is $C^\infty$. In particular, $D\omega = 0$ implies $\omega$ is smooth.*

*Proof.* By Gårding's Inequality,

$$||\omega||_{s+1} \leq \text{const}(||\eta||_s + ||\omega||_s),$$

so $\omega$ is in $H_{s+1}$. Going on in this way we see that $\omega$ is in every $H_t$, and therefore $\omega$ is smooth. □

We now prove Theorem 6.6.

*Proof.* Suppose $\ker \Delta$ is infinite dimensional in $L_2\Lambda(M) = H_0\Lambda(M)$, and let $\{\omega_i\}$ be an orthonormal basis for $\ker D$. By Gårding's inequality,

$$||\omega_i||_1 \leq \text{const}(||\omega_i||_0),$$

and $\{\omega_i\}$ is bounded in $H_1$, so by Rellich's lemma, $\{\omega_i\}$ has a convergent subsequence, a contradiction. This shows $\ker \Delta$ has finite dimension.

To prove the decomposition of the space of $C^\infty$ forms in (6.7), we will first find a decomposition for $L_2\Lambda(M)$. In order to do so, we extend $D$ in the following way. For $\alpha$ and $\omega$ in $L_2\Lambda(M)$, we say $\overline{D}\omega = \alpha$ if there is a sequence $\{\omega_n\}$ in $L_2\Lambda(M)$ such that $\omega_n \to \omega$ and $D\omega_n \to \alpha$. For an operator $T$, denote the range of $T$ by $\text{Ran}(T)$. We claim that

$$L_2\Lambda(M) = \ker(D) \oplus \text{Ran}(\overline{D}). \tag{6.9}$$

To see this, first note the following.

- Suppose $\overline{D}\omega = \alpha$. Then for $\eta$ in $\ker D$,

  $$\langle \alpha, \eta \rangle_0 = \lim \langle D\omega_n, \eta \rangle_0 = \lim \langle \omega_n, D\eta \rangle_0 = 0,$$

  and $\text{Ran}(\overline{D}) \subset \ker(D)^\perp \cap L_2\Lambda(M)$.

- Further, $\mathrm{Ran}(\overline{D})$ is closed. Let $\alpha$ be a limit point of $\mathrm{Ran}(\overline{D})$. Then there is a sequence $\{\beta_n\}$ with $D\beta_n \to \alpha$ in $L_2\Lambda(M)$. Without loss of generality, we can take $\{\beta_n\}$ to be in $\ker(D)^\perp \cap L_2\Lambda(M)$. $\{\beta_n\}$ is in $H_1\Lambda(M)$ by Gårding's inequality; suppose it is unbounded in $H_1\Lambda(M)$. Then, if
$$\gamma_n = \beta_n/||\beta_n||_1,$$
$D\gamma_n \to 0$ in $L_2\Lambda(M)$. By Gårding's inequality, $\{\gamma_n\}$ is bounded in $H_1$, and so by Rellich's some subsequence $\{\gamma_{n_k}\}$ converges to a $\gamma$ in $\ker(D)^\perp \cap L_2\Lambda(M)$. But for every smooth $\theta$,
$$\langle D\gamma, \theta\rangle_0 = \lim\langle D\gamma_{n_k}, \theta\rangle_0 = \lim\langle \gamma_{n_k}, D\theta\rangle_0 = 0,$$
and $\gamma$ is in both $\ker D$ and $\ker D^\perp$. This is impossible, though, because $||\gamma||_1 = 1$, and so $\{\beta_n\}$ is bounded in $H_1$. Hence, by Gårding's, it has a convergent subsequence in $L_2\Lambda(M)$, $\alpha$ is in $\mathrm{Ran}(\overline{D})$, and $\mathrm{Ran}(\overline{D})$ is closed.
- Another fact about $\overline{D}$ is that if $\overline{D}(\beta) = \alpha$ is smooth, then $\beta$ is smooth and $D\beta = \alpha$. To see this, let $\{\beta_n\} \to \beta$ with $D\beta_n \to \alpha$. Then since $D$ is continuous as a map from $L_2\Lambda(M)$ to $H_{-1}$ and $D\beta_n \to \alpha$ in $H_{-1}$, $D\beta = \alpha$ in $H_{-1}$. It follows from the regularity lemma for $D$ that $\beta$ is smooth.

Now since $\ker(D)^\perp$ and $\mathrm{Ran}(\overline{D})^\perp$ are closed subspaces, if the decomposition in (6.9) does not hold, then we can find $\omega \neq 0$ in both $\ker(D)^\perp$ and $\mathrm{Ran}(\overline{D})^\perp$. For all $C^\infty$ forms $\theta$, then,
$$\langle \omega, D\theta\rangle_0 = 0.$$
If $\{\omega_n\}$ is a $C^\infty$ sequence tending to $\omega$,
$$\langle D\omega_n, \theta\rangle_0 \to 0.$$
Consequently, $D\omega_n \to 0$, and $\overline{D}\omega = 0$. So by the fact that 0 is smooth, $D\omega = 0$, a contradiction because $\omega$ was nonzero and in $\ker(D)^\perp$. We thus have that
$$L_2\Lambda(M) = \ker(D) \oplus \mathrm{Ran}(\overline{D}),$$
and any smooth $\omega$ can be written as $\eta + \overline{D}\alpha$, where $\eta$ is in $\ker(D)$. $\eta$ is smooth, and so $\omega - \eta = \overline{D}\alpha$ is smooth, which implies $\alpha$ is smooth and $D\alpha = \omega - \eta$, or $D\alpha + \eta = \omega$. This proves the decomposition in (6.7).

□

## 7. Corollaries

A couple corollaries of the Hodge Theorem can now be obtained without much effort.

**Proposition 7.1.** *Every de Rham cohomology class of a compact, oriented manifold $M$ contains a unique harmonic form, and the de Rham cohomology groups are therefore finite dimensional.*

**Corollary 7.2.** *The dimension of* $\ker\Delta$ *is independent of the Riemannian structure on $M$.*

*Proof.* Every manifold can be given a Riemannian metric. Then, that each de Rham cohomology class contains a harmonic form is clear from the Hodge Theorem, while Lemma 2.9 gives us uniqueness. □

**Theorem 7.3** (Poincaré Duality)**.**

$$H^p_{dR} \cong (H^{n-p}_{dR})^*$$

*Proof.* First equip the manifold with a Riemannian structure. Then, in view of Definition 2.4, it is enough to exhibit a non-degenerate pairing of $H^p_{dR}$ and $H^{n-p}_{dR}$. We claim the function

$$I : H^p_{dR} \times H^{n-p}_{dR} \to \mathbb{R}$$

defined by

$$I([\omega], [\eta]) = \int_M \omega \wedge \eta \, dV$$

is such a pairing. For a nonzero element of $H^p_{dR}$, take a representative harmonic form $h$. Because $*$ and $\Delta$ commute, $*h$ is also harmonic, and therefore closed, so that

$$I([h], [*h]) = \int_M h \wedge *h \, dV = ||h||_0^2 > 0.$$

This shows $I$ is a non-degenerate pairing. □

**Corollary 7.4.** *For every compact, orientable, connected manifold $M$,*

$$H^n_{dR}(M) \cong \mathbb{R}.$$

*Proof.* This follows from the previous theorem and the fact that every closed 0-form on a connected manifold is constant. □

**Corollary 7.5.** *Every compact, oriented manifold $M$ of odd dimension $n$ has Euler characteristic zero.*

*Proof.*

$$\chi(M) = \sum_{p=0}^{n} (-1)^p \dim H^p_{dR}(M) = 0.$$

□

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