AN INTRODUCTION TO KNOT THEORY AND THE KNOT GROUP

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Abstract. This paper for the University of Chicago Math REU is an expository introduction to knot theory. In the first section, definitions are given for knots and for fundamental concepts and examples in knot theory, and motivation is given for the second section. The second section applies the fundamental group from algebraic topology to knots as a means to approach the basic problem of knot theory, and several important examples are given as well as a general method of computation for knot diagrams. This paper assumes knowledge in basic algebraic and general topology as well as group theory.

Contents

1. Knots and Links

We open with a definition:

Definition 1.1. A **knot** is an embedding of the circle S^1 in $\mathbb{R}^3$.

The intuitive meaning behind a knot can be directly discerned from its name, as can the motivation for the concept. A mathematical knot is just like a knot of string in the real world, except that it has no thickness, is fixed in space, and most importantly forms a closed loop, without any loose ends. For mathematical convenience, $\mathbb{R}^3$ in the definition is often replaced with its one-point compactification S^3.

Of course, knots in the real world are not fixed in space, and there is no interesting difference between, say, two knots that differ only by a translation. It is also of interest to us, when presented with a real-world knot, whether it can be "untied," since the defining property of a knot in the real world is that it can be moved around

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and warped without losing its knottedness, so long as it isn't broken. We therefore want to define an equivalence relation on knots that indicates when two knots can be smoothly transformed into each other without ever breaking or self-intersecting (de-knotting) during the process. Knots, taken as equivalence classes, could then be considered only in terms of their topologically important qualities.

Definition 1.2. Two knots $a : S^1 \to S^3$ and b are **equivalent** if there is a continuous function $F : S^3 \times [0,1] \to S^3$ for which:

(1) F_0 is the identity map,
(2) F_t is a homeomorphism $S^3 \to S^3$ for all $t \in [0,1]$, and
(3) $F_1 \circ a = b$.

It is a simple check that the following is true:

Proposition 1.3. *The above relation is an equivalence relation.*

Generally when we speak of "a knot" we are referring to an equivalence class of knots, rather than just a specific one.

Although many examples of knots exist, we will only consider a certain subclass consisting of those that are well-behaved. A **tame** knot, the type we will use, is any knot equivalent to a **polygonal** knot, which is a knot whose image is the union of finitely many line segments. Any tame knot can be represented efficiently by a **knot diagram**, which is essentially just a picture of the knot in two-space. It is obtained by projecting the knot onto a plane in such a way that only finitely many disjoint pairs of points on the knot map to the same point on the plane. For these points the diagram indicates which segment crosses "above" the other. The diagram may also indicate the orientation of the knot.

1.1. **Examples of Knots.** The simplest example of a knot is the **unknot**, which is just any knot equivalent to a simple circle in S^3, that is, any knot which is "not knotted" and thus can be "untied." Any knot diagram without any crossings is an unknot. Some unknots are represented below.

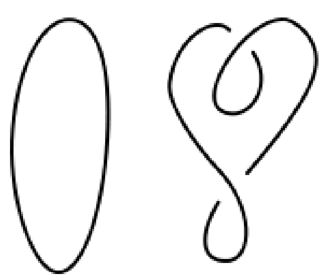

Figure 1. Examples of unknots, represented by unoriented knot diagrams.

One of the more important types of knot is that of the **torus knot**, which is any knot that is embedded onto a standard torus (one which, when solidified, deformation retracts onto an unknot) in S^3. Typical torus knots can be expressed as follows:

Definition 1.4. For any ordered pair of coprime integers (a, b), the standard torus knot $K_{a,b} : S^1 \to S^3$ corresponding to (a, b) is, defined in Euclidean coordinates,

$$\theta \mapsto \begin{pmatrix} (2+\cos b\theta)\cos a\theta \\ (2+\cos b\theta)\sin a\theta \\ -\sin b\theta \end{pmatrix}$$

This function is an embedding of the circle on the torus T, where T is the set of points of distance 1 from the circle of radius 2 around the origin in the xy-plane. It wraps around a times the long way and b times the short way. If a or b is ± 1, then the resulting knot is trivial (equivalent to the unknot). For the torus knot $K_{a,b}$, the $K_{-a,b}$ and $K_{a,-b}$ knots are mirror images of the first, and $K_{-a,-b}$ is the same as $K_{a,b}$ but with reversed orientation, and is equivalent by a rotation around the x-axis. Furthermore, $K_{a,b}$ is always equivalent to $K_{b,a}$, since the torus can be turned in S^3 in such a way that it has the same image as before, but the orientation is reversed, as are the long and short directions. The simplest nontrivial torus knots (also the simplest nontrivial knots in general, by minimal number of crossings in the knot diagram) are the right- and left-handed **trefoil knots** $K_{2,3}$ and $K_{2,-3}$.

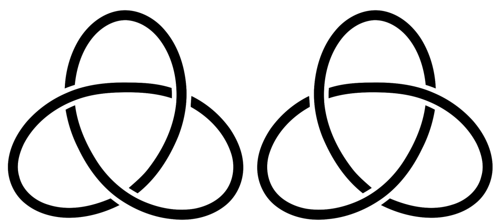

FIGURE 2. The right- and left-handed trefoil knots.

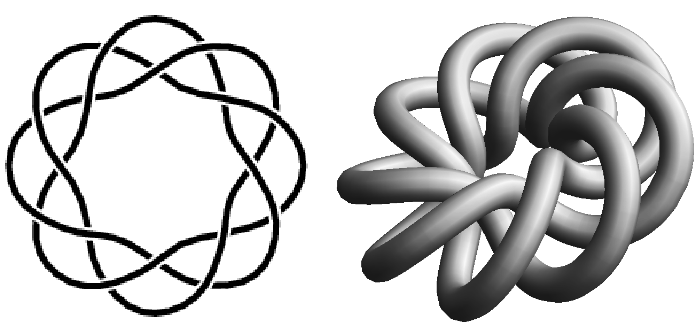

FIGURE 3. The torus knot $K_{3,-8}$ represented as a knot diagram and in space with volume.

1.2. **Links.** When working with knots it is most often convenient to instead consider the more general class of objects called links:

Definition 1.5. A **link** is an embedding of a disjoint union of finitely many circles in S^3.

In essence a link is just a bunch of knots that (possibly) link together. It can be considered to be a knot that is allowed to have multiple components. A link with only one component is just a knot, and a link that is not a knot can be called a "strict link."

Nearly every concept that applies to knots also applies to links. The definitions for equivalence and tameness are the same, and knot diagrams ("link diagrams") can still be drawn. Though it is an abuse of terminology, in this paper we will usually use the word "knot" to refer to links, and "true knot" to refer to knots.

The simplest nontrivial link that is not a true knot is the **Hopf link**, which is just two linked circles. A slightly paradoxical link is the **Borromean rings**, a nontrivial link with three components in which no pair of the components is linked.

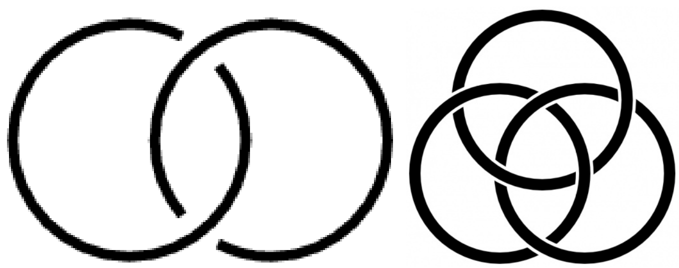

FIGURE 4. The Hopf link (left) and Borromean rings (right).

Torus knots can also be generalized to the family of **torus links**, which also lie on T. For any pair of nonzero integers (a, b), there is a corresponding torus link, which is a knot if a and b are coprime. Otherwise, the number of components is just the GCD d of a and b, and each component is a copy of $K_{a/d,b/d}$, rotated around the z axis. The whole link, analogously to knots, also wraps around the torus in total a times the long way and b times the short way in the sense that any circle going the long way around the torus intersects it a times, and any going the short way around intersects b times.

1.3. **Knot Invariants.** One of the fundamental problems in knot theory is determining when two knots are equivalent. In general, it is much simpler to show that two knots are equivalent than to show that they are not. All one needs to show equivalence is to provide an ambient isotopy (the type of function in Definition 1.2). In the case of two knots given explicitly by diagrams, this can be done easily (though indirectly), through what are called "Reidemeister moves," which is essentially just manually transitioning between the two step by step. To show that no such function exists takes more work.

The most common method of distinguishing knots is by finding "knot invariants," which are properties that are the same for any two knots that are equivalent. Showing that two knots have different values of a knot invariant then proves that they are not equivalent. It follows directly from the definition of equivalence that for any two equivalent knots, the complements of the images of the knots in S^3 (their **knot complements**) are homeomorphic. Many knot invariants, including the one we will focus on, the **knot group**, work by using this fact, distinguishing nonequivalent knots by distinguishing their knot complements. Even the knot complement itself could be considered a knot invariant, albeit a very useless one on its own.

2. Knot Groups and the Wirtinger Presentation

Definition 2.1. The **knot group** of a knot a with base point $b \in S^3 - \mathrm{Im}(a)$ is the fundamental group of the knot complement of a, with b as the base point.

Unlike other knot invariants, it takes no work to show that the knot groups are isomorphic for any pair of equivalent knots and base points, since equivalent knots have homeomorphic complements and homeomorphic spaces have isomorphic fundamental groups. Just like how we use "a knot" to refer to an equivalence class of knots, we can also use "group" to refer to what are actually equvalence classes of isomorphic groups. This allows us to talk about *the* knot group of a knot, without reference to a base point.

Unfortunately, the knot group is not always enough to show nonequivalence. For example, the right- and left-handed trefoil knots, as mirror images of each other, have the same knot group, but are not equivalent. This takes more work to show. However, for almost all practical cases the knot group can be used to show two knots are distinct.

2.1. **Preliminary Examples.**

The simplest knot group to calculate is, of course, that of the unknot. However, its knot group is not trivial.

Proposition 2.2. *The knot group of the unknot is the infinite cyclic group C.*

Proof. We first construct a deformation retract of the knot complement onto a more manageable subspace. It is easiest to do this in S^3. In this space, which is just $\mathbb{R}^3 \cup \{\infty\}$, the z-axis together with infinity is a circle, and the image of an unknot. The complement of this space in S^3 can be expressed with cylindrical coordinates with θ, $r > 0$, and z. The following function then defines a deformation retract of the space onto the unit circle in the xy-plane around the origin:

$$f_t(\theta, r, z) = (\theta, r^{(1-t)}, (1-t)z)$$

It follows that the knot group of the unknot is the fundamental group of the circle, which is the infinite cyclic group. □

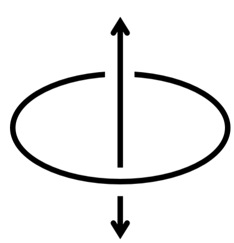

Figure 5. A Hopf link shown so that one component includes the point at infinity. The complement of each component in S^3 deformation retracts to the other.

A similar example is that of the Hopf link.

Proposition 2.3. *The knot group of the Hopf link is the free abelian group with two generators, $C \times C$.*

Proof. As the image of the Hopf link we use the vertical line with infinity as before, together with the unit circle around the origin in the xy-plane. We can again use cylindrical coordinates for the knot complement, except this time the points with both $r = 1$ and $z = 0$ are excluded. This time the space can be retracted onto a torus. A family of functions that does this is:

$$f_t(\theta, r, z) = (\theta, (1-t)r + t(r-1)/2\rho + t, (1-t)z + tz/2\rho)$$

Here $\rho = \sqrt{(r-1)^2 + z^2}$. The function deformation retracts onto the torus with tube radius $\frac{1}{2}$ and whose central circle is the unit circle around the origin of the xy-plane. This shows that the knot group is the same as the fundamental group of the torus, which is $C \times C$. □

2.2. **The Wirtinger Presentation.** A general method for finding the knot group of any tame knot was given by Wilhelm Wirtinger around the beginning of the 20th century. It has the advantage of being intuitively simple and easy to compute.

Constructing the Wirtinger presentation starts by considering the (oriented) knot diagram of a knot k. It is viewed as being entirely in the xy-plane in $\mathbb{R}^3$, except for the lower part of each crossing, which dips down below to avoid intersection with the above segment. Remember that a knot diagram of a tame knot consists of finitely many arcs in the plane, with finitely many crossings at the ends where one arc bridges under another. At each crossing, we consider the arc that passes over to be unbroken, so each side is part of the same arc. Meanwhile, the piece that passes under is broken, so the two sides are ends of two different arcs (or in some cases, the two ends of the same arc). With this disambiguated, we can let n be the number of arcs in the knot diagram, and we can number the arcs a_0, a_1, ..., a_{n-1}. If k is a true knot, then we can assign the numbers such that a_{i+1} is the arc that comes after a_i with the given orientation, with addition in $\mathbb{Z}/n\mathbb{Z}$. Since the Wirtinger presentation can be used for strict links as well as true knots (provided of course that they are tame), we will in general use $a_i + 1$ to refer to the arc that follows a_i. Here "+" is no longer an operation; "+1" is a function mapping the set of arcs to itself.

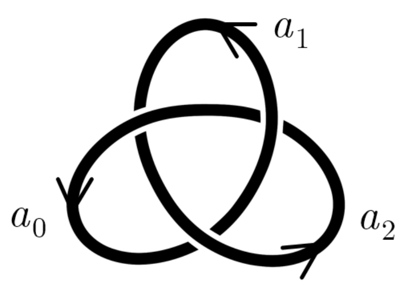

Figure 6. A trefoil knot with labelled arcs and indicated orientations.

In order to construct the Wirtinger presentation, we also need a way to talk about the crossings in the knot diagram. To each crossing b there are three (not necessarily distinct) associated arcs: the "over" arc $o(b)$, which is the one that is unbroken by the crossing, and two "under" arcs $u(b)$ and $u(b)+1$. The first, $u(b)$, is the one that is oriented toward the crossing, and $u(b)$ is oriented away. The crossings can be divided into two categories by "handedness" based on orientation, and they are treated differently. Right-handed crossings are those in which the "over" arc points to the right when facing in the direction of the other two, and left-handed crossings are those in which it points to the left. This gets used in the formulation of the Wirtinger presentation below.

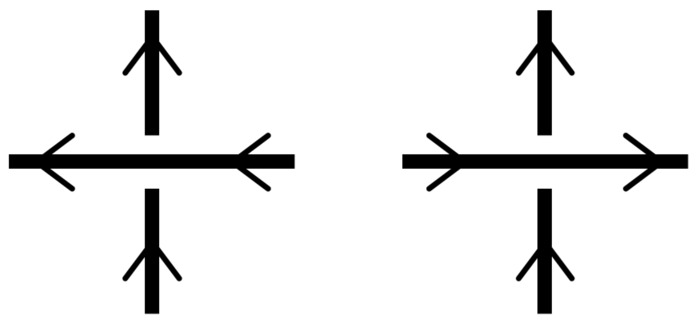

FIGURE 7. A left-handed crossing (left) and a right-handed crossing (right).

Proposition 2.4. *Let K be a tame knot expressed by a knot diagram, and let A be the set of arcs and B the set of crossings. Let W be the free group with generating set A, and let N be the subgroup of A generated by the elements $r(b)$ for each $b \in B$, with r refined as follows:*

$$r(b) = \begin{cases} (u(b)+1)o(b)u(b)^{-1}o(b)^{-1} & \text{if } b \text{ is right-handed} \\ o(b)(u(b)+1)o(b)^{-1}u(b)^{-1} & \text{if } b \text{ is left-handed} \end{cases}$$

Then W/N is the knot group of K.

Proof. The base point used for the knot group is $(0,0,1)$, or really any point that lies "above" the knot diagram in space, or is in the same direction from which the diagram is viewed. For each arc a, the generating element a in the group presentation above corresponds to the loop that starts at the base point, travels down to the xy-plane where the diagram is and hooks underneath the arc a before returning back to the base point. Under this convention the loop descends below the arc on its left side and rises on its right side, as determined by its orientation. Of course, the reverse loop corresponds to a^{-1}.

Before we start taking any quotients, we need to check that the loops corresponding to each arc generate all possible loops, up to homotopy. Given any loop c starting and ending at the base point, there is a finite sequence of arcs underneath which c passes and the directions of the passes relative to the orientation. By moving the path back to the base point after each crossing under and straightening, we construct a homotopy from c to the composition of generator loops and their inverses corresponding to each pass-under.

There are two special cases in which a homotopy of the loop c changes the series of instances in which it crosses under an arc, and any other change is a combination of

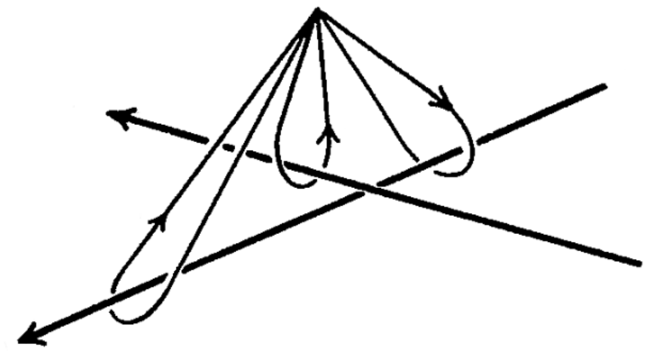

FIGURE 8. The loops associated to the three arcs in a right-handed crossing. The base point is apparent at the top of the image.

these. The first is when the loop moves so that it crosses under an arc and then back in the other direction without any other crossings, which is of course homotopic the the loop that skips these entirely. This is accounted for in our construction by the identity $aa^{-1} = e$. The second case is when a homotopy moves the loop underneath a crossing. When a loop travels around a crossing and entirely underneath it, it passes below the "over" arc twice in opposite directions and the "under" arcs once each in opposite directions, with exactly one of the later two crossings between the former pair, with the exact directions and orders determined by the starting point and the handedness of the crossing. This piece of path is homotopic to a piece with no crossings, so we add the relations in the definition of N to identify these segments with the identity. This completes the Wirtinger presentation.

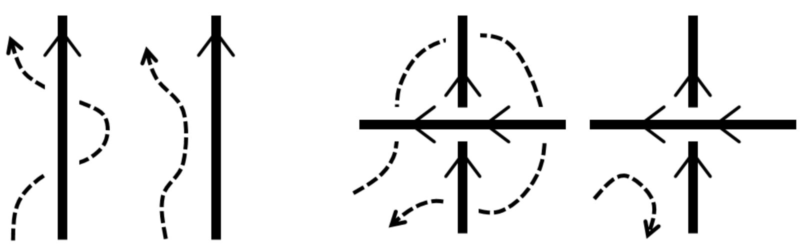

FIGURE 9. The homotopies showing that $aa^{-1} = e$ for any arc a and $o(b)(u(b)+1)o(b)^{-1}u(b)^{-1} = e$ for any left-handed knot b.

□

Example 2.5. A standard knot diagram for a trefoil knot consists of three arcs, as shown in Figure 6, which can be labelled a_0, a_1, and a_2. The Wirtinger presentation is given as:

$$W/N = \langle a_0, a_1, a_2 | a_1 a_2 a_0^{-1} a_2^{-1} = a_0 a_1 a_2^{-1} a_1^{-1} = a_2 a_0 a_1^{-1} a_0^{-1} = e \rangle$$

which can then be simplified. One identity follows from the other two, so it can be removed. Another can be used to indicate that $a_2 = a_0 a_1 a_0^{-1}$. With this knowledge,

a_2 can be removed as a generator, and substitution can be used in the third relation, yielding the following presentation for W/N:

$$W/N = \langle a_0, a_1 | a_0 a_1 a_0 = a_1 a_0 a_1 \rangle$$

2.3. **Knot Groups for Torus Knots.** While the Wirtinger presentation can be used for any reasonable knot, it often fails to provide an intuitive understanding of the knot itself. For example, the Wirtinger presentation for the knot group of the trefoil has no immediately apparent connection to the shape of the knot. In the case of the trefoil and true torus knots in general, a different approach gives a much more geometrically understandable presentation for the knot group.

Theorem 2.6. *The knot group of $K_{a,b}$ is given by $\langle x_1, x_2 | x_1^b = x_2^a \rangle$ for all coprime positive integers a and b.*

Proof. Recall that for coprime positive a and b, the image K of a torus knot $K_{a,b}$ as given by Definition 1.4 is homeomorphic to a circle and lies completely on a standard torus T with the z axis as its axis of rotation. A rotation of this set around the z-axis by an angle of π/b yields a similarly-shaped set that also lies on T and weaves through the original image. Call this set K'. It can also be obtained by twisting the torus in its other direction by an angle of π/a.

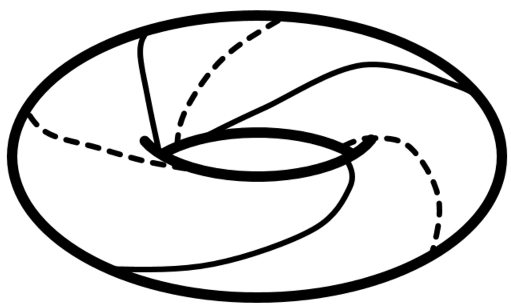

FIGURE 10. The torus T, showing a left-handed trefoil knot K (solid) and K' (dashed).

Our goal will be to apply Van Kampen's theorem to the complement of K in S^3. First we just consider the solid torus T_1 consisting of T and its interior. Since T_1 deformation retracts to the circle at its center, its fundamental group is C, and since K exists completely on the boundary, $\pi_1(T_1 - K) = \pi_1(T_1) = C$. The same is true for the solid torus T_2 consisting of T and its exterior, which is identical to T_1 (since we use S^3 instead of $\mathbb{R}^3$) except for the way that K and K' wraps around it. Remember that the union of these two pieces is the entire space S^3 and their intersection is just T.

The intersection between $T_1 - K$ and $T_2 - K$ is just $T - K$, which is path-connected and also has fundamental group C, since it deformation retracts to K'. Since K', like K, wraps a times around T the short way and b times around T the long way, It is homotopic to x_1^b in $T_1 - K$ and x_2^a in $T_2 - K$, where x_1 and x_2 are the generators for the fundamental group in $T_1 - K$ and $T_2 - K$, respectively. By Van Kampen's theorem, the fundamental group of $S^3 - K$ and knot group of $K_{a,b}$ is then given by the presentation:

$$\langle x_1, x_2 | x_1^b = x_2^a \rangle$$

□

For torus knots, this result has the clear advantages over the Wirtinger presentation of its simplicity and immediate connection to the parameters in its definition, and of the use of the geometry of the torus.

The two presentations of the trefoil knot ($\langle x_1, x_2 | x_1^3 = x_2^2 \rangle$ and $\langle a_0, a_1 | a_0 a_1 a_0 = a_1 a_0 a_1 \rangle$) can by reconciled with each other by setting $x_1 = a_0 a_1$ and $x_2 = a_0 a_1 a_0$, or $a_0 = x_1^{-1} x_2$ and $a_1 = x_2^{-1} x_1^2$. It is a simple check that the two are compatible.

Unfortunately, this trick does not apply to torus links that are strict links. The simplest example is that of the Hopf link, which can be defined as the torus link $K_{2,2}$. We have already calculated its knot group to be $C \times C$. Since $\langle x_1, x_2 | x_1^2 = x_2^2 \rangle$ is not abelian like $C \times C$ is, the two are clearly not isomorphic. The reason that it fails is that for the Hopf link embedded on the torus, $T - K$ is not path-connected so K' cannot be constructed in the same way and Van Kampen's theorem cannot be applied.

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