BROUWER'S FIXED POINT THEOREM: THE WALRASIAN AUCTIONEER

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Abstract. The focus of this paper is proving Brouwer's fixed point theorem, which primarily relies on the fixed point property of the closed unit ball in $\mathbb{R}^n$. We then present an economic application of Brouwer's fixed point theorem: the Walrasian auctioneer.

Contents

1. Introduction

Fixed point theorems refer to a variety of theorems that all state, in one way or another, that a transformation from a set to itself has at least one point that remains unchanged. One primary application of fixed point theorems is confirming the existence of a solution to a set of equations. In economics, the Arrow-Debreu general equilibrium model is the standard for modeling a de-centralized economy. In this model, each individual has an endowment (a certain quantity of goods that belong to him) and a specific set of preferences (represented by a utility function). Individuals are completely autonomous, and the goal of each is to maximize their own utility [3]. Yet with only a few basic assumptions about preferences, there exists a point, the competitive equilibrium, that maximizes the utility of all individuals within the constraints of the economy (the sum of all endowments). This claim is remarkable; although the individuals of this model only seek to propagate their own interests with no regard for others, the end result is the best scenario for all members of society. The proof of this claim, the existence of competitive equilibrium, will be the primary focus of this paper.

This existence proof will first require us to establish Brouwer's fixed point theorem, one of the most important fixed point theorems. However, before we can prove Brouwer's fixed point theorem, we will need to establish a more basic result: the fixed point property of the closed unit ball in $\mathbb{R}^n$. With this result, as well as three other theorems, we will prove Brouwer's fixed point theorem [2].

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We will then begin the existence proof by formally introducing the Arrow-Debreu model for an economy with L individuals and N different goods. After making a few basic assumptions, the competitive market's price adjustment mechanism can then be modeled by a function. We will prove this function satisfies the conditions of Brouwer's fixed point theorem, and thus has a set of prices that constitutes a fixed point. Finally, we will prove that with this set of prices, competitive equilibrium is achieved.

2. Fixed Point Property of the Closed Unit Ball

We first introduce Lipschitz functions. These are useful because they are continuous, and in the case of contractions, have a fixed point in a Banach space. The latter takes the form of Theorem 2.4, which we will prove after providing a formal definition of Lipschitz functions.

Definition 2.1. Let (X, d) be a metric space. A map $F : X \to X$ is *Lipschitz* if it satisfies the following condition:

$$\forall\, x, y \in X \quad \exists\, \alpha \geq 0 \quad \text{s.t.} \quad d(F(x), F(y)) \leq \alpha\, d(x, y). \tag{2.2}$$

The smallest α for which (2.2) holds is called the *Lipschitz constant*, denoted by L. If $L < 1$, then F is defined as a *contraction*. If $L = 1$, then F is defined as *non-expansive*. Note that a Lipschitz map is always continuous.

Notation 2.3. $F^{n+1}(x) = F(F^n(x))$, where $n \in \mathbb{N}$.

Theorem 2.4 (Banach's Contraction Principle)**.** *Let (X, d) be a complete metric space. Let $F : X \to X$ be a contraction ($L < 1$). Then F has a unique fixed point $u \in X$ with $u = F(u)$.*

Proof. First we show u is unique. Suppose there exists $u, v \in X$ with $u = F(u)$ and $v = F(v)$. Then we have

$$d(u, v) = d(F(u), F(v)) \leq L\, d(u, v), \tag{2.5}$$

where the second inequality follows from the fact that F is a contraction. But (2.5) only holds if $d(u, v) = 0$. By properties of distance d, $u = v$.

Next we show that for all $x \in X$, the sequence $\{F^n(x)\}$ is Cauchy, that is

$$\forall\, \varepsilon > 0 \quad \exists\, N \in \mathbb{N} \quad \text{s.t.} \quad \forall\, m, n \geq N \quad d(F^n(x), F^m(x)) < \varepsilon. \tag{2.6}$$

Without loss of generality, assume $m > n$. Since F is a contraction, we have

$$d(F^n(x), F^{n+1}(x)) = d(F(F^{n-1}), F(F^n(x))) \leq L\, d(F^{n-1}(x), F^n(x)).$$

Repeating this process inductively, we obtain

$$d(F^n(x), F^{n+1}(x)) \leq L^n\, d(x, F(x)). \tag{2.7}$$

By the triangle inequality and (2.7), we have

$$\begin{aligned} d(F^n(x), F^m(x)) &\leq d(F^n(x), F^{n+1}(x)) + d(F^{n+1}(x), F^{n+2}(x)) \\ &\quad + \ldots + d(F^{m-1}(x), F^m(x)) \\ &\leq L^n\, d(x, F(x)) + L^{n+1}\, d(x, F(x)) + \ldots + L^{m-1}\, d(x, F(x)) \\ &\leq L^n\, d(x, F(x))\, [1 + L + L^2 + \ldots] \\ &= \frac{L^n}{1 - L}\, d(x, F(x)). \end{aligned}$$

Since $L < 1$, by the Archimedean principle we can choose N large enough so that

$$\forall\, n > N \quad \frac{L^n}{1-L}\, d(x, F(x)) < \varepsilon.$$

This satisfies (2.6), thus $\{F^n(x)\}$ is Cauchy. Because X is complete,

$$\exists\, u \in X \quad \text{s.t} \quad \lim_{n\to\infty} F^n(x) = u.$$

F is continuous since it is Lipschitz, therefore

$$u = \lim_{n\to\infty} F^{n+1}(x) = \lim_{n\to\infty} F(F^n(x)) = F(u).$$

Thus u is a fixed point as required. □

Definition 2.8. A topological space X has the *fixed point property* if every continuous map $f : X \to X$ has a fixed point.

Notation 2.9. B^n will denote the closed unit ball and S^{n-1} will denote the unit sphere, where

$$B^n := \{x \in \mathbb{R}^n : \|x\| \leq 1\} \quad \text{and} \quad S^{n-1} := \{x \in \mathbb{R}^n : \|x\| = 1\}.$$

The remainder of this section will be devoted to proving Theorem 2.10, which will be integral for the proof of Brouwer's fixed point theorem in the following section.

Theorem 2.10. *The closed unit ball B^n in $\mathbb{R}^n$ has the fixed point property.*

For the remainder of this paper, we assume $\mathbb{R}^n$ is endowed with the standard inner product and norm:

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i \quad \text{and} \quad \|x\| = \langle x, x\rangle^{\frac{1}{2}}.$$

In order to prove Theorem 2.10, we will need all six theorems introduced below. The first five theorems are not used directly in the proof of Theorem 2.10, but are required in proof of subsequent theorems. For the sake of brevity, the proofs of Theorem 2.15, 2.24, and 2.27 are omitted. Before we introduce the theorems, we provide the following definitions.

Definition 2.11. Let $A \subseteq \mathbb{R}^n$. A continuous map $f : A \to \mathbb{R}^n$ is of *class* C^1 if f has a continuous extension to an open neighbourhood of A on which it is continuously differentiable.

Definition 2.12. Let $A \subseteq \mathbb{R}^n$. A map $f : A \to \mathbb{R}^n$ is *non-vanishing* if it satisfies

$$\forall\, x \in A \quad f(x) \neq 0.$$

Definition 2.13. Let $A \subseteq \mathbb{R}^n$. A map $f : A \to \mathbb{R}^n$ is *normed* if it satisfies

$$\forall\, x \in A \quad \|f(x)\| = 1.$$

Definition 2.14. Let $A \subseteq \mathbb{R}^n$. A map $f : A \to \mathbb{R}^n$ is *tangent* to S^{n-1} if it satisfies

$$\forall\, x \in S^{n-1} \quad \langle x, f(x)\rangle = 0.$$

The following theorem is stated without proof. Consult [5] for a formal proof.

Theorem 2.15. *Let A be a compact subset of $\mathbb{R}^n$. Let $f : A \to \mathbb{R}^n$ be of class C^1 on A. Then we have the following:*

$$\exists\, L \geq 0 \quad \text{s.t.} \quad \forall\, x, y \in A \quad \|f(x) - f(y)\| \leq L\, \|x - y\|$$

With the result above, we are able to prove Theorem 2.16.

Theorem 2.16. *Suppose $F : S^{n-1} \to \mathbb{R}^n$ is a normed vector field of class C^1 that is tangent to S^{n-1}. Then we have the following:*

$$\exists\, t > 0 \quad \text{s.t.} \quad f_t(S^{n-1}) = (1 + t^2)^{\frac{1}{2}} S^{n-1}, \quad \text{where} \quad f_t : x \mapsto x + tF(x).$$

Proof. Define $F^* : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}^n$ and $A \subseteq \mathbb{R}^n$ by

$$F^*(x) := \|x\|\, F\left(\frac{x}{\|x\|}\right) \quad \text{and} \quad A := \left\{x \in \mathbb{R}^n : \frac{1}{2} \leq \|x\| \leq \frac{3}{2}\right\}.$$

Note that F^* is well defined. Furthermore, we know that F^* is of class C^1 in S^{n-1} since $\|x\|$, F, and $\frac{x}{\|x\|}$ are all of class C^1 in A. A is also compact since it is bounded and closed. By Theorem 2.15 applied to F^* on A,

$$\exists\, L \geq 0 \quad \text{s.t.} \quad \forall\, x, y \in A \quad \|F^*(x) - F^*(y)\| \leq L\, \|x - y\|. \tag{2.17}$$

Put $t \leq \min\left(\frac{1}{3}, \frac{1}{L}\right)$. Fix $z \in S^{n-1}$ and define $G : A \to \mathbb{R}^n$ by

$$G(x) := z - tF^*(x).$$

We aim to apply Theorem 2.4 to G. To do so, we must show $G : A \to A$ and that G is a contraction. By the triangle inequality, our choice of $t \leq \frac{1}{3}$, and the fact that F is normed, we have

$$\begin{aligned} G(x) = \|z - tf^*(x)\| &\leq \|z\| + t\, \|F^*(x)\| \\ &\leq \|z\| + \frac{1}{3}\left[\|x\| \left\|F\left(\frac{x}{\|x\|}\right)\right\|\right] \\ &\leq 1 + \frac{1}{3}\left[\frac{3}{2}\right] = \frac{3}{2}. \end{aligned}$$

Similarly, we also have

$$G(x) = \|z - tf^*(x)\| \geq \|z\| - t\, \|F^*(x)\| \geq \frac{1}{2}.$$

Thus $G(x) \in A$, that is $G : A \to A$. By our choice of $t \leq \frac{1}{L}$ and (2.17),

$$\begin{aligned} \|G(x) - G(y)\| &= \|(z - tF^*(x) - (z - tF^*(y))\| \\ &= t\|F^*(y) - F^*(x)\| \\ &\leq tL\, \|x - y\|, \quad \text{where} \quad tL \leq 1. \end{aligned}$$

Thus G is a contraction by definition. By Theorem 2.4, G has a fixed point, that is

$$\begin{aligned} &\exists\, x \in A \quad \text{s.t.} \quad G(x) = x \\ &\text{i.e} \quad z = x + tF^*(x). \end{aligned} \tag{2.18}$$

Therefore, since $z \in S^{n-1}$

$$\langle x + tF^*(x), x + tF^*(x)\rangle = \langle z, z\rangle = \|z\|^2 = 1. \tag{2.19}$$

Since F is tangent to S^{n-1} and normed by assumption, we also have

$$\begin{aligned}
\langle x+tF^*(x), x+tF^*(x)\rangle &= \langle x,x\rangle + 2\,\langle x, tF^*(x)\rangle + \langle tF^*(x), tF^*(x)\rangle \\
&= \|x\|^2 + 2\left\langle x, t\|x\|\,F\left(\frac{x}{\|x\|}\right)\right\rangle + \left\|t\|x\|F\left(\frac{x}{\|x\|}\right)\right\|^2 \\
&= \|x\|^2 + t^2\|x\|^2\left\|F\left(\frac{x}{\|x\|}\right)\right\|^2 \\
&= \|x\|^2 + t^2\|x\|^2 \\
&= \left(1+t^2\right)\|x\|^2. \qquad (2.20)
\end{aligned}$$

By (2.19) and (2.20) we have $\left(1+t^2\right)\|x\|^2 = 1$, therefore

$$\|x\| = \left(1+t^2\right)^{-\frac{1}{2}}. \qquad (2.21)$$

Let $y = \frac{x}{\|x\|} \in S^{n-1}$. Substituting in (2.21), $y = x\left(1+t^2\right)^{\frac{1}{2}}$. By (2.21) and (2.18)

$$\begin{aligned}
f_t(y) = y + tF(y) &= x\left(1+t^2\right)^{\frac{1}{2}} + tF\left(x\left(1+t^2\right)^{\frac{1}{2}}\right) \\
&= \left(1+t^2\right)^{\frac{1}{2}}\left(x + \left(1+t^2\right)^{-\frac{1}{2}}\left[tF\left(x\left(1+t^2\right)^{\frac{1}{2}}\right)\right]\right) \\
&= \left(1+t^2\right)^{\frac{1}{2}}\left(x + \|x\|\left[tF\left(\frac{x}{\|x\|}\right)\right]\right) \\
&= \left(1+t^2\right)^{\frac{1}{2}}\left(x + tF^*(x)\right) \\
&= \left(1+t^2\right)^{\frac{1}{2}} z, \qquad (2.22)
\end{aligned}$$

where $z \in S^{n-1}$ is arbitrary. We have shown that for any $z \in S^{n-1}$, there exists $y \in S^{n-1}$ with $\left(1+t^2\right)^{\frac{1}{2}} z = f_t(y)$. Thus $\left(1+t^2\right)^{\frac{1}{2}} S^{n-1} \subseteq f_t(S^{n-1})$.

To complete the proof, all that remains is to show $f_t(S^{n-1}) \subseteq \left(1+t^2\right)^{\frac{1}{2}} S^{n-1}$. Fix $x \in S^{n-1}$. Because F is tangent to S^{n-1} (shown above) and normed by assumption, we have

$$\begin{aligned}
\|x+tF(x)\|^2 &= \langle x+tF(x), x+tF(x)\rangle \\
&= \langle x,x\rangle + 2\langle x, tF(x)\rangle + \langle tF(x), tF(x)\rangle \\
&= \|x\|^2 + t^2\|F(x)\|^2 \\
&= 1+t^2. \qquad (2.23)
\end{aligned}$$

Put $y = \left(1+t^2\right)^{-\frac{1}{2}}(x+tF(x))$. By (2.23), we have

$$\begin{aligned}
\|y\|^2 &= \left\|\left(1+t^2\right)^{-\frac{1}{2}}(x+tF(x))\right\|^2 \\
&= \left(1+t^2\right)^{-1}\|x+tF(x)\|^2 = 1.
\end{aligned}$$

Since $\|y\|^2 = 1$, we have $\|y\| = 1$, that is $y \in S^{n-1}$. Furthermore, by the definition of f_t and our choice of y, we know

$$\left(1+t^2\right)^{-\frac{1}{2}}(x+tF(x)) = y \Leftrightarrow x+tF(x) = \left(1+t^2\right)^{\frac{1}{2}} y \Leftrightarrow f_t(x) = \left(1+t^2\right)^{\frac{1}{2}} y.$$

We have shown that for any $x \in S^{n-1}$, we have $y \in S^{n-1}$ with $f_t(x) = \left(1+t^2\right)^{\frac{1}{2}} y$. Thus $f_t(S^{n-1}) \subseteq \left(1+t^2\right)^{\frac{1}{2}} S^{n-1}$. Hence $f_t(S^{n-1}) = \left(1+t^2\right)^{\frac{1}{2}} S^{n-1}$ as required. $\square$

Although we omit the proof (consult [5]), the following theorem is needed in order to induce a contradiction in the proof of Theorem 2.25.

Theorem 2.24. *Let $A \in \mathbb{R}^n$ be a closed and bounded domain. Let $F : A \to \mathbb{R}^n$ be of class C^1 on A. Then we have the following:*

$$\exists\, (-\varepsilon, \varepsilon) \quad \text{s.t.} \quad \phi : t \mapsto \mathrm{Vol}(f_t(A)) \quad \text{is a polynomial of degree at most n.}$$

Here Vol *is the n-dimensional Lebesgue measure volume in $\mathbb{R}^n$.*

With Theorem 2.16 and 2.24, we can prove the following result.

Theorem 2.25. *Let $k \in \mathbb{N}$. Then there are no normed vector fields of class C^1 that are tangent to S^{2k}.*

Proof. We aim to prove the above statement by contradiction. Suppose a vector field F satisfying the above conditions does exist. Define $A \subseteq \mathbb{R}^n$ and $F^* : A \to \mathbb{R}^n$ as in the proof of Theorem 2.16. F is tangent to S^{2k} by assumption, therefore F^* is tangent to all concentric spheres of S^{2k} in A. Define $f_t : A \to \mathbb{R}^n$ as follows:

$$f_t(x) := x + tF^*(x).$$

A is composed of concentric spheres with radii ranging from $\frac{1}{2}$ to $\frac{3}{2}$. Restricting F^* to each sphere in turn, we apply Theorem 2.16. Recombining the results,

$$\exists\, t > 0 \quad \text{s.t.} \quad f_t(A) = (1 + t^2)^{\frac{1}{2}} A. \tag{2.26}$$

(Note that the same t holds for all spheres since we only need $t < \min\left(\frac{1}{3}, \frac{1}{L}\right)$ and L is a constant inherent to F^*.)
By (2.26) and the definition of the Lebesgue measure volume, we have

$$\mathrm{Vol}(f_t(A)) = \mathrm{Vol}\left(\left(1 + t^2\right)^{\frac{1}{2}} A\right) = \left(1 + t^2\right)^{\frac{2k+1}{2}} \mathrm{Vol}(A).$$

By Theorem 2.24 applied to F^* on A

$$\exists\, (-\varepsilon, \varepsilon) \quad \text{s.t.} \quad \phi : t \mapsto \mathrm{Vol}(f_t(A)) \quad \text{is a polynomial of degree at most n.}$$

This is a contradiction; regardless of ε, $\left(1 + t^2\right)^{\frac{1}{2}}$ is not a polynomial of any degree. Our initial assumption was false, and such a F does not exist as required. □

In Theorem 2.28 that follows, we will extend Theorem 2.25 by dropping the restrictive C^1 and normed conditions for the vector fields. They will be replaced with (more general) non-vanishing and continuity conditions. The proof of this result will require the Weierstrass approximation theorem stated below. This theorem will then be used directly in the proof of Theorem 2.10 to provide a contradiction.

Theorem 2.27 (Weierstrass Approximation Theorem)**.** *If f is continuous and real-valued on the interval $[a, b]$, then the following holds:*

$$\forall\, \varepsilon > 0 \quad \exists \text{ polynomial } p(x) \quad \text{s.t.} \quad \forall\, x \in [a, b] \quad \|f(x) - p(x)\| < \varepsilon.$$

Theorem 2.28. *Fix $k \in \mathbb{N}$. There are no non-vanishing, continuous vector fields that are tangent to S^{2k}.*

Proof. We aim to prove the above statement by contradiction. Suppose such a vector field $F : S^{2k} \to \mathbb{R}^{2k+1}$ does exist. Since F is continuous and maps to $\mathbb{R}^{2k+1}$ by assumption, we can apply Theorem 2.27 to each of the $2k + 1$ coordinate components of F. Recombining the resulting polynomial components, we obtain:

$$\forall\, \varepsilon > 0 \quad \exists\, P(x) : S^{2k} \to \mathbb{R}^{2k+1} \quad \text{s.t.} \quad \forall\, x \in S^{2k} \quad \|F(x) - P(x)\| < \varepsilon.$$

Let $m = \min\{\|F(x)\| : x \in S^{2k}\}$. Note that $m > 0$ since F is non-vanishing. Thus $\frac{m}{2}$ is a valid choice for ε;

$$\exists\, P(x) : S^{2k} \to \mathbb{R}^{2k+1} \quad \text{s.t.} \quad \forall\, x \in S^{2k} \quad \|F(x) - P(x)\| < \frac{m}{2}. \tag{2.29}$$

Now P is of class C^∞ since polynomials are infinitely differentiable. By the triangle inequality and the fact that m is a minimum, we have

$$\begin{aligned} \|P(x)\| &= \|F(x) - (F(x) - P(x))\| \\ &\geq \|F(x)\| - \|P(x) - F(x)\| \geq m - \frac{m}{2} > 0. \end{aligned} \tag{2.30}$$

Thus P is non-vanishing by definition. Define $Q : S^{2k} \to \mathbb{R}^n$ as follows:

$$Q(x) := P(x) - \langle P(x), x\rangle x.$$

Because Q is also a polynomial, it is also of class C^∞. Furthermore, for all $x \in S^{2k}$

$$\begin{aligned} \langle x, Q(x)\rangle &= \langle x, P(x) - \langle P(x), x\rangle x\rangle \\ &= \langle x, P(x)\rangle - \langle x, P(x)\rangle\langle x, x\rangle \\ &= \langle x, P(x)\rangle - \langle x, P(x)\rangle = 0. \end{aligned}$$

Thus Q is tangent to S^{2k}. By the triangle inequality, (2.30), the tangency of F (by assumption), the Cauchy-Schwarz inequality, and (2.29), for all $x \in S^{2k}$ we have

$$\begin{aligned} \|Q(x)\| &= \|P(x) - (P(x) - Q(x))\| \\ &\geq \|P(x)\| - \|Q(x) - P(x)\| \\ &\geq \frac{m}{2} - \|\langle P(x), x\rangle\| \\ &= \frac{m}{2} - \|\langle P(x) - F(x), x\rangle\| \\ &\geq \frac{m}{2} - \|P(x) - F(x)\| > 0. \end{aligned}$$

We can therefore consider $Q' = \frac{Q}{\|Q\|}$. Q' is normed by definition. It is also of class C^1 and tangent to S^{2k}, since Q was of class C^∞ and tangent to S^{2k}. This is a contradiction; by Theorem 2.25 such a thing does not exist. Our initial assumption was false, and such a field F does not exist as required. □

With the results above, we are now able to provide a proof of Theorem 2.10. Before we begin, first note that any vector $(x_1, \ldots, x_n, x_{n+1}) \in \mathbb{R}^{n+1}$ can be rewritten as (x, x_{n+1}), where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. The unit sphere S^n can thus be divided into an upper hemisphere S^n_+ and lower hemisphere S^n_- defined by

$$S^n_+ := \{(x, x_{n+1}) \in S^n : x_{n+1} \geq 0\} \quad \text{and} \quad S^n_- := \{(x, x_{n+1}) \in S^n : x_{n+1} \leq 0\}.$$

The two hemispheres intersect at the equator, given by $S^{n-1} = S^n_+ \bigcap S^n_-$. The vector $e_{n+1} = (0, \ldots, 0, 1) \in S^n$ is the *north pole*, while the vector $-e_{n+1} = (0, \ldots, 0, -1) \in S^n$ is the *south pole*.

Definition 2.31. The *stereographic projection* from e_{n+1} to S^n is the map $S_+ : R^n \to S^n$ given by

$$S_+(x) := \left(\frac{2x}{1 + \|x\|^2}, \frac{\|x\|^2 - 1}{1 + \|x\|^2}\right).$$

Similarly, $S_- : R^n \to S^n$, is given by

$$S_-(x) := \left(\frac{2x}{1+\|x\|^2}, \frac{1-\|x\|^2}{1+\|x\|^2} \right).$$

Note that S_+ and S_- are both infinitely differentiable and thus of class C^∞. Furthermore, for any $x \in B^n$ we have

$$\frac{\|x\|^2 - 1}{1+\|x\|^2} \leq 0 \quad \text{and} \quad \frac{1-\|x\|^2}{1+\|x\|^2} \geq 0.$$

Thus $S_+ : B^n \to S^n_-$ and $S_- : B^n \to S^n_+$. Notice also, that for any $x \in S^{n-1}$

$$S_+(x) = \left(\frac{2x}{1+1}, \frac{1-1}{1+1} \right) = S_-(x) = (x, 0) = x. \tag{2.32}$$

We are now ready to prove Theorem 2.10.

Proof of Theorem 2.10. We consider two cases: n even and n odd, where n is the dimension of R^n. Recall that we aim to prove B^n has the fixed point property.

Case 1: n even

Let $n = 2k$ where $k \in \mathbb{N}$. We proceed by contradiction. Suppose B^n does not have the fixed point property, that is there exists a continuous map $f : B^{2k} \to B^{2k}$ with no fixed point. We claim that for all $x \in B^{2k}$ we have $1 - \langle x, f(x) \rangle > 0$, or equivalently $\langle x, f(x) \rangle < 1$. By the Cauchy-Schwarz inequality, we have

$$\langle x, f(x) \rangle \leq \|x\| \|f(x)\| \leq 1.$$

Suppose equality holds, that is $\langle x, f(x) \rangle = 1$. Then

$$1 = \langle x, f(x) \rangle \leq \|x\| \|f(x)\| \leq 1 \Rightarrow 1 = \|x\| \|f(x)\| = \langle x, f(x) \rangle. \tag{2.33}$$

Since $\|x\| \|f(x)\| = \langle x, f(x) \rangle$, we can write $f(x) = \alpha x$ for some $\alpha > 0$. Because $x, f(x) \in B^{2k}$, (2.33) implies $\|x\| = \|f(x)\| = \alpha \|x\| = 1$. Therefore $\alpha = 1$. But this means that $x = f(x)$, which contradicts the assumption that f has no fixed points. We cannot have $\langle x, f(x) \rangle = 1$, thus $\langle x, f(x) \rangle < 1$ for all $x \in B^{2k}$ as claimed. Because $1 - \langle x, f(x) \rangle > 0$, we can now define $F : B^{2k} \to \mathbb{R}^{2k}$ as follows:

$$F(x) := x - \left(\frac{1-\|x\|^2}{1 - \langle x, f(x) \rangle} \right) f(x).$$

Note that for any $x \in S^{2k-1}$, we have

$$F(x) = x - \left(\frac{1-1^2}{1 - \langle x, f(x) \rangle} \right) f(x) = x. \tag{2.34}$$

We aim to prove F is non-vanishing through contradiction. Suppose that for some $x \in B^{2k}$, $F(x) = 0$. Then we have

$$x - \left(\frac{1-\|x\|^2}{1 - \langle x, f(x) \rangle} \right) f(x) = 0 \quad \text{i.e.} \quad x = \left(\frac{1-\|x\|^2}{1 - \langle x, f(x) \rangle} \right) f(x). \tag{2.35}$$

This shows that 0, x, and $f(x)$ are collinear since by (2.35) x is some constant times $f(x)$. Therefore for some α, $f(x) = \alpha x$. As a result,

$$\langle x, f(x) \rangle x = \left(\alpha \|x\|^2 \right) x = \|x\|^2 f(x). \tag{2.36}$$

By (2.35) and (2.36) we have

$$x = \left(\tfrac{1-\|x\|^2}{1-\langle x, f(x)\rangle}\right) f(x) \Leftrightarrow x\left(1 - \langle x, f(x)\rangle\right) = \left(1 - \|x\|^2\right) f(x)$$
$$\Leftrightarrow x - \langle x, f(x)\rangle x = f(x) - \|x\|^2 f(x) \Leftrightarrow x = f(x).$$

This is a contradiction; f has no fixed points. Our initial assumption was false and F is non-vanishing.

Let $x \in B^{2k}$, and consider $A := \{x + tF(x) : t \in [0,1]\}$. Since the stereographic projection S_+ is of class C^∞ and maps to S_-^{2k}, the image of S_+ restricted to A is a differentiable arc in S_-^{2k}. We therefore define $T_- : S_-^{2k} \to \mathbb{R}^{2k+1}$ as the following:

$$\begin{aligned}
T_-(y) &:= \left\{\frac{d}{dt} S_+(x + tF(x))\right\}\Big|_{t=0}, \quad \text{where} \quad y = S_+(x) \in S_-^{2k} \\
&= \left\{\frac{d}{dt}\left(\frac{2(x+tF(x))}{1+\|x+tF(x)\|^2}, \frac{\|x+tF(x)\|^2 - 1}{1+\|x+tF(x)\|^2}\right)\right\}\Big|_{t=0} \\
&= \left\{\frac{2F(x)}{1+\|x+tF(x)\|^2} + \left[-\frac{2(x+tF(x))}{\left(1+\|x+tF(x)\|^2\right)^2}\right]\left[2\langle x+tF(x), F(x)\rangle\right],\right. \\
&\quad \left.\frac{2\langle x+tF(x), F(x)\rangle}{1+\|x+tF(x)\|^2} + \left[-\frac{\left(\|x+tF(x)\|^2 - 1\right)\left(2\langle x+tF(x), F(x)\rangle\right)}{\left(1+\|x+tF(x)\|^2\right)^2}\right]\right\}\Big|_{t=0} \\
&= \frac{2}{\left(1+\|x\|^2\right)^2}\left(\left(1+\|x\|^2\right)F(x) - 2\langle x, F(x)\rangle x, 2\langle x, F(x)\rangle\right).
\end{aligned}$$

T_- is continuous since S_+ is infinitely differentiable. We also claim that T_- is non-vanishing. To see this, consider $\|T_-(y)\|^2$:

$$\begin{aligned}
\|T_-(y)\|^2 &= \frac{4}{\left(1+\|x\|^2\right)^4}\left[\left\langle \left(1+\|x\|^2\right)F(x) - 2\langle x, F(x)\rangle x, \left(1+\|x\|^2\right)F(x) - 2\langle x, F(x)\rangle x\right\rangle\right. \\
&\quad \left. + 4\langle x, F(x)\rangle^2\right] \\
&= \frac{4}{\left(1+\|x\|^2\right)^4}\left[\left(1+\|x\|^2\right)^2\|F(x)\|^2 - 4\left(1+\|x\|^2\right)\langle x, F(x)\rangle^2\right. \\
&\quad \left. + 4\|x\|^2\langle x, F(x)\rangle^2 + 4\langle x, F(x)\rangle^2\right] \\
&= \frac{4}{\left(1+\|x\|^2\right)^4}\left[\left(1+\|x\|^2\right)^2\|F(x)\|^2\right] \\
&= \frac{4}{\left(1+\|x\|^2\right)^2}\|F(x)\|^2.
\end{aligned}$$

We know F is non-vanishing ($\|F(x)\| \neq 0$), thus T_- is non-vanshing as claimed. Lastly, we claim that T_- is tangent to S_-^{2k}. Since S_+ maps to S_-^{2k}, we have

$$\begin{aligned} &\langle S_+(x+tF(x)), S_+(x+tF(x))\rangle = 1 \\ \text{(2.37)} \qquad &\text{i.e.} \quad \tfrac{d}{dt}\langle S_+(x+tF(x)), S_+(x+tF(x))\rangle = 0. \end{aligned}$$

But because the inner product is symmetric, we also have

$$\begin{aligned} \frac{d}{dt}\langle S_+(x+tF(x)), S_+(x+tF(x))\rangle &= 2\left\langle \frac{d}{dt}S_+(x+tF(x), S_+(x+tF(x)\right\rangle \\ \text{(2.38)} \qquad &= 2\langle T_-(y), y\rangle, \quad \text{where} \quad y = S_+(x+tF(x)) \in S_-^{2k}. \end{aligned}$$

Combining (2.37) and (2.38), we obtain $\langle T_-(y), y\rangle = 0$ for $y \in S_-^{2k}$.
Thus T_- is tangent to S_-^{2k} as claimed. We define $T_+ : S_+^{2k} \to \mathbb{R}^{2k+1}$ by

$$\begin{aligned} T_-(y) &:= \left\{\frac{d}{dt}S_-(x+tF(x))\right\}\Bigg|_{t=0}, \quad \text{where} \quad y = S_-(x) \in S_+^{2k} \\ &= \frac{2}{\left(1+\|x\|^2\right)^2}\left(2\langle x, F(x)\rangle - \left(1+\|x\|^2\right)F(x), 2\langle x, F(x)\rangle\right). \end{aligned}$$

By similar arguments as those above, T_+ is also continuous, non-vanishing, and tangent to S_+^{2k}. Now consider $T : S^{2k} \to \mathbb{R}^{2k+1}$ defined as follows:

$$T(y) := \begin{cases} T_-(y) & \text{if } y \in S_-^{2k} \\ T_+(y) & \text{if } y \in S_+^{2k}. \end{cases}$$

For all $y \in S^{2k-1}$ (the equator of S^{2k}), by (2.32) $y = x$, thus $x \in S^{2k-1}$. By (2.34),

$$\begin{aligned} \forall\, y \in S^{2k-1} \quad T_-(y) &= \frac{2}{(1+1^2)}\left(\left(1+1^2\right)F(x) - 2\langle x,x\rangle x, 2\langle x,x\rangle\right) \\ &= \frac{1}{2}\left(2F(x) - 2(1)x, 2(1)\right) \\ &= \frac{1}{2}\left(2(1)x - 2F(x), 2(1)\right) \\ &= \frac{2}{(1+1^2)}\left(2\langle x,x\rangle x - \left(1+1^2\right)F(x), 2\langle x,x\rangle\right) \\ \text{(2.39)} \qquad &= T_+(y). \end{aligned}$$

T is continuous on S^{2k} by (2.39) and the continuity of T_- and T_+. T is also non-vanishing and tangent to S^{2k} since T_- and T_+ are. This is a contradiction; by Theorem 2.28, such a vector field T should not exist. Our initial assumption was false and f has a fixed point for n even as required.

Case 2: n odd

Let $n = 2k-1$ where $k \in \mathbb{N}$. We proceed by contradiction. Suppose there exists a continuous map $f : B^{2k-1} \to B^{2k-1}$ with no fixed point. Define $g : B^{2k} \to B^{2k}$ by

$$g(x, x_{2k}) := (f(x), 0).$$

We know g is continuous since f is continuous. By our proof of the fixed point property for n even, g has a fixed point. Therefore for some $x \in B^{2k}$ we have $(x, x_{2k}) = (f(x), 0)$, that is $x = f(x)$. This is a contradiction; f has no fixed points. Our initial assumption was false and f has a fixed point for n odd as required. $\square$

3. Brouwer's Fixed Point Theorem

As the title suggests, this section of the paper focuses on proving Brouwer's fixed point theorem, stated below.

Theorem 3.1 (Brouwer's Fixed Point Theorem)**.** *Every non-empty, bounded, closed, and convex subset C of $\mathbb{R}^n$ has the fixed point property.*

The proof relies on the main result of the previous section, Theorem 2.10, as well as three additional theorems that we present below.

Definition 3.2. Two topological spaces X and Y are *homeomorphic* if there exists a bijective map $f : X \to Y$ such that both f and f^{-1} are continuous. The map f is then a *homeomorphism.*

Theorem 3.3. *Suppose X has the fixed point property and X is homeomorphic to Y. Then Y has the fixed point property.*

Proof. Let $h : X \to Y$ be a homeomorphism. Suppose $g : Y \to Y$ is continuous. To prove Y has the fixed point property, it suffices to show that g has a fixed point. Define $f : X \to X$ by

$$f(x) := h^{-1}(g(h(x))).$$

We know f is continuous since g, h, and h^{-1} are all continuous by assumption. Because X has the fixed point property, f has a fixed point, that is

$$\exists\, x_0 \in X \quad \text{s.t.} \quad h^{-1}(g(h(x_0))) = x_0.$$

Applying h to both sides, we obtain

$$h(h^{-1}(g(h(x_0)))) = h(x_0)$$
$$\text{i.e.} \quad g(h(x_0)) = h(x_0), \quad \text{where} \quad h(x_0) \in Y.$$

Hence g has a fixed point $h(x_0)$ as required. □

Definition 3.4. Let X be a topological space. A subset A of X is a *retract* if it satisfies the following condition

$$\exists \text{ continuous } r : X \to A \quad \text{s.t.} \quad \forall\, a \in A \quad r(a) = a.$$

The map r is then a *retraction.*

Theorem 3.5. *Every non-empty, closed, and convex subset C of $\mathbb{R}^n$ is a retract.*

Proof. Define $P_C : \mathbb{R}^n \to C$ by the following:

$$P_C(x) := y \in C \quad \text{s.t.} \quad \|x - y\| = \inf\{\|x - u\| : u \in C\}.$$

P_C is the map sending each point $x \in \mathbb{R}^{\ltimes}$ to the nearest point in C. We aim to show P_C is a non-expansive, that is

$$\forall\, x, y \in \mathbb{R}^n \quad \|P_C(x) - P_C(y)\| \leq \|x - y\|.$$

Let $x' = P_C(x)$ and $y' = P_C(y)$. Because $x', y' \in C$ and C is convex, we know

$$\forall\, t \in (0, 1) \quad (1 - t)x' + ty' \in C.$$

By definition $\|x' - x\|$ is the minimum distance between x and any point in C. Thus

$$\|[(1 - 0)x' + (0)y'] - x\|^2 = \|x' - x\|^2 \leq \|[(1 - t)x' + ty'] - x\|^2.$$

Therefore $\|[(1-t)x' + ty'] - x\|^2$ is increasing at $t = 0$, that is

$$\begin{aligned} \frac{d}{dt}\|(1-t)x' + ty' - x\|^2\Big|_{t=0} &= \frac{d}{dt}2\langle (1-t)x' + ty' - x, (1-t)x' + ty' - x\rangle\Big|_{t=0} \\ &= 2\langle (1-t)x' + ty' - x, y' - x'\rangle\Big|_{t=0} \\ &= 2\langle x' - x, y' - x'\rangle \geq 0. \end{aligned} \tag{3.6}$$

Similarly, because $\|y' - y\|^2 \leq \|[tx' + (1-t)y'] - y\|^2$ by definition, we also have

$$\frac{d}{dt}\|tx' + (1-t)y' - y\|^2\Big|_{t=0} = 2\langle y' - y, x' - y'\rangle \geq 0. \tag{3.7}$$

Consider the function $d : (0,1) \to \mathbb{R}$ defined by

$$d(t) := \|x' - y' + t[x - x' - (y - y')]\|^2.$$

It is clear that $d(t)$ is a quadratic polynomial with a non-negative coefficient for t^2. Its graph is thus an upwards-opening parabola. By (3.6) and (3.7), we have

$$\frac{d}{dt}\|x' - y' + t[x - x' - (y - y')\|^2\Big|_{t=0} = 2\langle y' - x', x' - x\rangle + 2\langle x' - y', y' - y\rangle \geq 0.$$

This means that $d(t)$ is non-decreasing at 0. Because $d(t)$ is an upwards sloping parabola, $d(t)$ must also be non-decreasing on the interval $[0, \infty)$. In particular,

$$d(0) = \|x' - y'\| \leq \|x - y\| = d(1).$$

Hence P_C is non-expansive. P_C is therefore also continuous (since it is Lipschitz). Furthermore, note that

$$\forall\, x \in C \quad P_C(x) = x.$$

Thus P_C is a retraction. Hence C is a retract as required. □

Theorem 3.8. *Suppose X has the fixed point property and $A \subseteq X$ is a retract. Then A has the fixed point property.*

Proof. Let $r : X \to A$ be a retraction. Suppose $f : A \to A$ is continuous. We need to show f has a fixed point. Define $g : X \to A$ by

$$g(x) := f(r(x)). \tag{3.9}$$

We know g is continuous since f and r are continuous by assumption. Moreover, since $A \subseteq X$, we have $g : X \to X$. By the fixed point property of X, g has a fixed point, that is

$$\exists\, x_0 \in X \quad \text{s.t.} \quad f(r(x_0)) = x_0. \tag{3.10}$$

But $f(r(x_0)) \in A$ since $g : X \to A$ by definition, therefore $x_0 \in A$. Because r is a retraction and $x_0 \in A$

$$r(x_0) = x_0. \tag{3.11}$$

Combining (3.10) and (3.11), we have $f(r(x_0)) = f(x_0) = x_0$. Hence f has a fixed point x_0 as required. □

We are now ready to prove Theorem 3.1 (Brouwer's fixed point theorem).

Proof of Theorem 3.1. C is bounded, therefore by definition

$$\exists\, M > 0 \quad \text{s.t.} \quad \forall\, x \in C \quad \|x\| < M.$$

C is therefore contained within the closed ball of radius M in $\mathbb{R}^n$, denoted by B^*. By Theorem 2.10, we know B^n has the fixed point property. By Theorem 3.3, since B^* and B^n are homeomorphic (consider the map $f : B^* \to B^n$ defined by $f(x) := \frac{1}{M}x$), B^* has the fixed point property. C is non-empty, closed, and convex by assumption, therefore by Theorem 3.5 C is a retract. Applying Theorem 3.8 (since $C \subseteq B^*$, C is a retract, and B^* has the fixed point property), C has the fixed point property as required. □

4. Application: The Walrasian Auctioneer

We now provide an application of Brouwer's fixed point theorem in economics: the existence proof of competitive equilibrium in a market with N goods and L individuals. We first establish the economic model used and any necessary definitions before introducing the proof.

In a N-good endowment economy, an individual i is endowed with a fixed quantity of each good, denoted by $\omega_1^i, \ldots, \omega_N^i$, where $\omega_j^i \geq 0$.

Definition 4.1. A *bundle* is any collection of goods $(x_1^i, \ldots, x_N^i)$, where x_j^i is the quantity of good x_j that individual i possesses.

Individuals have a set of preferences, represented mathematically by a "utility function" $U_i(x_1^i, \ldots, x_N^i)$ that assigns a numerical value in utils to a particular bundle. Preferences are assumed to be monotonic in this model: "more is better". However, individuals are restricted by a budget constraint given by

$$\sum_{k=1}^{N} p_k x_k^i = \sum_{k=1}^{N} p_k \omega_k^i. \tag{4.2}$$

In order to consume any bundle besides their endowment, individuals convert their endowment to nominal income by selling their endowed goods at the market prices $p_1, \ldots, p_N$. Based on their preferences, individuals then purchase the bundle that maximizes their utility at the market prices. As a result, individuals cannot spend more than the nominal value of their endowment. Because of monotonicity, we assume the individual spends all of their income, hence the equality in (4.2).

To determine the 'ideal' bundle given a particular endowment, we solve individual i's utility maximization problem (in essence a constrained optimization problem). Doing so yields a set of demand functions x_j^i for each good j that is dependent upon market prices and the individual's nominal income:

$$x_j^i(p_1, \ldots, p_N, \sum_{k=1}^{N} p_k \omega_k^i).$$

Since the endowment of individual i is fixed, we consider the demand functions as solely dependent on price. Given a set of market prices, $x_j^i(p_1, \ldots, p_N)$ gives the quantity of good x_j that maximizes individual i's utility.

The first condition for market equilibrium is that individual i must maximize his utility; this is exactly x_j. Although demand functions are derived from the preferences (utility function) of the individual, to simplify the model we take them as given.

Definition 4.3. A *perfectly competitive* market is one in which the individual has no influence upon the market price.

In a perfectly competitive market economy with L individuals, the supply of each good x_j is given by $\sum_{i=1}^{L} \omega_j^i$. An apt analogy is to think of a farmer; his crop yield is his endowment, and the total supply of food cannot exceed the summation of all farmers' crop yields. At equilibrium, the total demand for good x_j is therefore restricted by its supply:

$$\sum_{i=1}^{L} x_j^i \leq W_j, \quad \text{where} \quad W_j = \sum_{i=1}^{L} \omega_j^i. \tag{4.4}$$

To guarantee (4.4) holds with equality, we introduce a new slack variable s_j that represents the quantity of good x_j not consumed. Now we have

$$\sum_{i=1}^{L} x_j^i + s_j = W_j. \tag{4.5}$$

We now use vectors (bolded) to represent market prices, consumption bundles, and the slack variables. Each entry p_j (resp. x_j^i, s_j) of the price vector $\mathbf{p} \in \mathbb{R}^{\mathbb{N}}$ (resp. consumption vector $\mathbf{c_i}$, slack vector $\mathbf{s}$) gives the market price (resp. quantity consumed, excess supply) of good x_j.

Notation 4.6. If every entry v_j is non-negative, then $\mathbf{v} \geq \mathbf{0}$. Similarly, $\mathbf{v} \leq \mathbf{0}$ indicates every entry v_j is non-positive. If $\mathbf{v} \geq \mathbf{0}$ and at least one entry v_j is strictly positive, then $\mathbf{v} > \mathbf{0}$. The equality $\mathbf{v} = \mathbf{0}$ indicates that every entry is zero.

Observe that $\mathbf{p} \geq 0$. This is because x_j is assumed to be a good (as opposed to 'bad'); x_j is desirable, thus the individual must pay to obtain it.

Note that 4.5 can be rewritten as

$$\sum_{i=1}^{L} (x_j^i - \omega_j^i) = -s_j. \tag{4.7}$$

The left-hand side of (4.7) represents the total excess demand for good x_j as a function of price (recall x_j^i is a function of price). We thus define the excess demand vector $\mathbf{z(p)} \in \mathbb{R}^N$ as the collection of excess demand functions for each good x_j. By (4.7) and (4.4), we know $\mathbf{z}\left(\mathbf{p^E}\right) = -\mathbf{s^E} \leq 0$, where $\mathbf{p^E}$ and $\mathbf{s^E}$ are the equilibrium price and slack vectors; this is the second condition for equilibrium. Rearranging (4.2) and factoring out p_k^E, we have

$$\sum_{k=1}^{N} p_k^E (x_k^i - \omega_k^i) = 0 \tag{4.8}$$

for each individual i. The summation across all L individuals yields

$$\sum_{i=1}^{L} \left(\sum_{k=1}^{N} p_k^E (x_k^i - \omega_k^i) \right) = 0. \tag{4.9}$$

But by rearranging terms, we also have

$$\begin{aligned}\sum_{i=1}^{L}\left(\sum_{k=1}^{N} p_k^E(x_k^i-\omega_k^i)\right) &= \sum_{k=1}^{N} p_k^E(x_k^1-\omega_k^1)+\ldots+\sum_{k=1}^{N} p_k^E(x_k^L-\omega_k^L)\\ &= p_1^E\left[\sum_{i=1}^{L}(x_1^i-\omega_1^i)\right]+\ldots+p_N^E\left[\sum_{i=1}^{L}(x_N^i-\omega_N^i)\right]\\ &= \mathbf{p^E}\cdot(-\mathbf{s^E}). \qquad (4.10)\end{aligned}$$

Combining (4.9) and (4.10), we have $-\mathbf{p^E}\cdot\mathbf{s^E}=0$, which is the third and final condition for equilibrium. We now provide a formal definition of equilibrium.

Definition 4.11. A *competitive equilibrium* in an L person endowment economy with N goods is a price vector $\mathbf{p^E}\geq 0$ and an allocation $\left\{\mathbf{c_1^E},\ldots,\mathbf{c_L^E}\right\}$ such that the following conditions hold:

(1) The slack vector $\mathbf{s^E}$ satisfies $\mathbf{s^E}\geq\mathbf{0}$ and $-\mathbf{p^E}\cdot\mathbf{s^E}=0$.
(2) The consumption vectors $\left\{\mathbf{c_1^E},\ldots,\mathbf{c_L^E}\right\}$ solve the utility maximization problem for each individual $i=1,\ldots,L$.
(3) Markets clear, that is $\mathbf{z}\left(\mathbf{p^E}\right)=-\mathbf{s^E}$.

We can collapse this definition into a single condition. Note that (2) holds automatically because we assumed the demand functions x_j^i for each individual i were given. Combining (1) and (3), we obtain

$$\mathbf{z}\left(\mathbf{p^E}\right)\leq\mathbf{0} \quad\text{and}\quad \mathbf{p^E}\cdot\mathbf{z}\left(\mathbf{p^E}\right)=0. \qquad (4.12)$$

We now introduce the following theorem, which is the main result of this paper.

Theorem 4.13. *Let $\boldsymbol{z}(\boldsymbol{p})$ be a function defined for all $\boldsymbol{p}\geq\boldsymbol{0}$ and $\boldsymbol{p}\neq\boldsymbol{0}$. Suppose $\boldsymbol{z}(\boldsymbol{p})$ satisfies the following conditions:*

(1) $\boldsymbol{z}(\boldsymbol{p})$ is homogeneous of degree 0 (with respect to $\boldsymbol{p}$).
(2) Individuals spend all of their income, that is $\boldsymbol{p}\cdot\boldsymbol{z}(\boldsymbol{p})=0$.
(3) $\boldsymbol{z}(\boldsymbol{p})$ is continuous.

Then there exists a price vector $\boldsymbol{p^E}\geq\boldsymbol{0}$ that constitutes a competitive equilibrium.

Before we provide a proof of the above theorem, we will need to satisfy the conditions of Brouwer's fixed point theorem before we are able to apply it. We thus introduce a useful non-empty, bounded, closed, and convex set in $\mathbb{R}^N$ below.

The excess demand function $\mathbf{z(p)}$ is homogeneous of degree 0 with respect to $\mathbf{p}$. This means that we are able to scale $\mathbf{p}$ by any constant without affecting $\mathbf{z(p)}$. Let $\mathbf{p}$ be the price vector for the N-good market. We define a new normalized price vector $\mathbf{p}'$ whose j^{th} entry is given by

$$p_j'=\frac{p_j}{\sum_{k=1}^{N}p_k}.$$

Definition 4.14. The *standard simplex* [4] in $\mathbb{R}^n$ is given by

$$\Delta^{N-1}=\left\{\mathbf{q}\in\mathbb{R}^N:\mathbf{q}\geq\mathbf{0}\text{ and }\sum_{k=1}^{N}q_k=1\right\}.$$

It is clear that $\mathbf{p}'$ lies in the simplex (there are no negative prices), thus it is non-empty. We know that the simplex is also bounded since

$$\|\mathbf{q}\| = \sum_{j=1}^{N} {q_j}^2 \leq \left(\sum_{j=1}^{N} q_j\right)^2 = 1.$$

To show the simplex is closed, consider the functions $f_i : \Delta^{N-1} \to \mathbb{R}$ given by

$$f_i(\mathbf{q}) := q_i, \quad \text{where } \mathbf{q} \in \Delta^{N-1}$$

These coordinate projection functions are continuous, thus their summation is also continuous, that is, the function $S : \Delta^{N-1} \to \mathbb{R}$ defined by

$$S(\mathbf{q}) := \sum_{i=1}^{N} q_i$$

is continuous. Now the image of S is simply $\{1\} \in \mathbb{R}$, a closed set. The simplex is therefore the pre-image of a continuous function on a closed set. Hence the simplex is closed.

Lastly, we need to show that the simplex is convex, that is

$$\forall\, \mathbf{x}, \mathbf{y} \in \Delta^{N-1},\, t \in (0,1) \quad t\,\mathbf{x} + (1-t)\,\mathbf{y} \in \Delta^{N-1}.$$

Let $\mathbf{x}, \mathbf{y} \in \Delta^{N-1}$, that is

$$\sum_{j=1}^{N} x_j = 1 \quad \text{and} \quad \sum_{j=1}^{N} y_j = 1. \tag{4.15}$$

Fix $t \in (0,1)$. By (4.15) we have

$$\sum_{j=1}^{N} (t\,x_j + (1-t)\,y_j) = t\sum_{j=1}^{N} x_j + (1-t)\sum_{j=1}^{N} y_j = 1.$$

Hence $t\,\mathbf{x} + (1-t)\,\mathbf{y} \in \Delta^{N-1}$ and the simplex is convex.

Consider the function given by

$$\mathbf{f(q)} := \frac{\mathbf{q} + \mathbf{g(q)}}{\sum_{j=1}^{N} (q_j + g_j(\mathbf{q}))}, \tag{4.16}$$

where $g_j(\mathbf{q}) = \max(0, z_j(\mathbf{q}))$.

The Walrasian auctioneer is a (hypothetical) price adjustment process that models how market equilibrium is achieved. First, the Walrasian auctioneer announces a set of prices given by $\mathbf{q} \in \Delta^{N-1}$ where $\mathbf{q} \geq \mathbf{0}$ and $\mathbf{q} \neq \mathbf{0}$. Each individual i responds by stating the quantities $\{x_1^i, \ldots, x_N^i\}$ that they wish to purchase at those prices. The Walrasian auctioneer then checks the excess demand $z_j(\mathbf{q})$ for each good x_j. If demand exceeds supply, that is $z_j(\mathbf{q}) > 0$, then the Walrasian auctioneer responds by increasing the price of good x_j. After the price of x_j has increased, the Walrasian normalizes the increased price. This new price is then announced again, and the process repeats.

It is evident that (4.16) mimics the role of the Walrasian Auctioneer: the numerator increases the price depending on the excess demand for that good, while the denominator normalizes the increased price. The new price vector $\mathbf{f(q)}$ lies in the simplex, thus $\mathbf{f(q)} : \Delta^{N-1} \to \Delta^{N-1}$.

Lemma 4.17. *If $\boldsymbol{z}\colon \Delta^{N-1} \to \Delta^{N-1}$ is continuous, then $\boldsymbol{f}\colon \Delta^{N-1} \to \Delta^{N-1}$ is also continuous.*

Proof. Suppose $\mathbf{z} : \Delta^{N-1} \to \Delta^{N-1}$ is continuous. As a result, both the coordinate projections $z_j : \Delta^{N-1} \to \mathbb{R}$ with $1 \leq j \leq N$) and the absolute value of the coordinate projections $\|z_j(\mathbf{q})\|$ are also continuous. Now the coordinate functions $g_j : \Delta^{N-1} \to \mathbb{R}$ given in (4.16) can be written as

$$g_j(\mathbf{q}) = \frac{z_j(\mathbf{q}) + \|z_j(\mathbf{q})\|}{2}, \tag{4.18}$$

which is a sum of continuous functions and thus also continuous. Continuity of the coordinate functions g_j implies continuity of $\mathbf{g}$. Hence $\mathbf{f}$ is composed solely of continuous functions and is therefore continuous as required. □

We are now ready to prove Theorem 4.13.

Proof of Theorem 4.13. By assumption (1), $\mathbf{z(p)}$ is homogeneous of degree 0 in $\mathbf{p}$. As a result, without loss of generality we consider only the price vectors $\mathbf{q} \in \Delta^{N-1}$. By Brouwer's fixed point theorem, since Δ^{N-1} is non-empty, bounded, closed, and convex, Δ^{N-1} has the fixed point property. Now $\mathbf{f(q)} : \Delta^{N-1} \to \Delta^{N-1}$. Since $\mathbf{z(q)}$ is continuous by assumption (3), applying Lemma 4.17 yields the fact that $\mathbf{f(q)}$ is also continuous. Because Δ^{N-1} has the fixed point property, $\mathbf{f(q)}$ has a fixed point, that is

$$\exists\, \mathbf{q^E} \in \Delta^{N-1} \quad \text{s.t.} \quad \mathbf{q^E} = \mathbf{f}\left(\mathbf{q^E}\right). \tag{4.19}$$

We claim that $\mathbf{q^E}$ satisfies (4.12) and thus constitutes a competitive equilibrium. By assumption (2), we have

$$\mathbf{q^E} \cdot \mathbf{z}\left(\mathbf{q^E}\right) = 0. \tag{4.20}$$

Therefore to satisfy (4.12), all that remains is to show $\mathbf{z}\left(\mathbf{q^E}\right) \leq \mathbf{0}$. We will prove this by contradiction. Suppose $\mathbf{z}\left(\mathbf{q^E}\right) > \mathbf{0}$. This implies

$$g_j\left(\mathbf{q^E}\right) = z_j\left(\mathbf{q^E}\right) \quad \text{for} \quad j = 1, \ldots, N. \tag{4.21}$$

By (4.21) and $\mathbf{z}\left(\mathbf{q^E}\right) > \mathbf{0}$ we then have

$$\begin{aligned} \mathbf{g}\left(\mathbf{q^E}\right) \cdot \mathbf{z}\left(\mathbf{q^E}\right) &= \sum_{j=1}^{N} \left[g_j\left(\mathbf{q^E}\right) z_j\left(\mathbf{q^E}\right)\right] \\ &= \sum_{j=1}^{N} \left(z_j\left(\mathbf{q^E}\right)\right)^2 \\ &= \left\|\mathbf{z}\left(\mathbf{q^E}\right)\right\|^2 > 0. \end{aligned} \tag{4.22}$$

However, plugging (4.19) into (4.20) we have

$$\mathbf{f}\left(\mathbf{q^E}\right) \cdot \mathbf{z}\left(\mathbf{q^E}\right) = 0. \tag{4.23}$$

But by (4.16) and (4.20) we also know that

$$\begin{aligned}
\mathbf{f}\left(\mathbf{q}^{\mathbf{E}}\right)\cdot\mathbf{z}\left(\mathbf{q}^{\mathbf{E}}\right) &= \left[\frac{\mathbf{q}^{\mathbf{E}}+\mathbf{g}\left(\mathbf{q}^{\mathbf{E}}\right)}{\sum_{j=1}^{N} q_j^E + g_j(\mathbf{q}^{\mathbf{E}})}\right]\cdot\mathbf{z}\left(\mathbf{q}^{\mathbf{E}}\right) \\
&= \frac{\mathbf{q}^{\mathbf{E}}\cdot\mathbf{z}\left(\mathbf{q}^{\mathbf{E}}\right)+\mathbf{g}\left(\mathbf{q}^{\mathbf{E}}\right)\cdot\mathbf{z}\left(\mathbf{q}^{\mathbf{E}}\right)}{\sum_{j=1}^{N} q_j^E + g_j(\mathbf{q}^{\mathbf{E}})} \\
&= \frac{\mathbf{g}\left(\mathbf{q}^{\mathbf{E}}\right)\cdot\mathbf{z}\left(\mathbf{q}^{\mathbf{E}}\right)}{\sum_{j=1}^{N} q_j^E + g_j(\mathbf{q}^{\mathbf{E}})}. \qquad (4.24)
\end{aligned}$$

Combining 4.23 and 4.24 we have $\mathbf{g}\left(\mathbf{q}^{\mathbf{E}}\right)\cdot\mathbf{z}\left(\mathbf{q}^{\mathbf{E}}\right)=0$. This is a contradiction; by (4.22), we have $\mathbf{g}\left(\mathbf{q}^{\mathbf{E}}\right)\cdot\mathbf{z}\left(\mathbf{q}^{\mathbf{E}}\right)>0$. Our initial assumption was false. Hence $\mathbf{z}\left(\mathbf{q}^{\mathbf{E}}\right)\leq\mathbf{0}$ and $\mathbf{q}^{\mathbf{E}}$ constitutes a competitive equilibrium as required. □

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