# BROUWER'S FIXED POINT THEOREM AND PRICE EQUILIBRIUM

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Abstract. This is an expository paper based on Border [1]. Starting from basic convexity results, I present a proof of the so-called Equilibrium Theorem, which states the existence of a free disposal equilibrium price vector in an Arrow-Debreu economy with a continuous excess demand function.

Contents



## 1. Preliminaries

Call a **commodity** a good produced and sold for some nonnegative price. Let $\mathbb{R}^m$ denote the **commodity space**, where $m$ is the finite number of commodities. Each vector $x \in \mathbb{R}^m$ can be written as $x = \sum_{i=0}^{m-1} x_i e^i$, where $e^0, \ldots, e^{m-1}$ denote the "standard" Cartesian basis. Thus $x$ represents a commodity vector, for it denotes a particular quantity $x_i$ for the $i$-th commodity. Similarly, a price vector lists the prices of each commodity, so that the value of a commodity vector $x \in \mathbb{R}^m$ going at the price vector $p \in \mathbb{R}^m$ is simply $p \cdot x = \sum_i p_i x_i$. (I follow the convention of [1] in denoting the components of a vector by subscripts and separate vectors by superscripts.) Further, $\mathbb{R}_+$ refers to the nonnegative real numbers.

**Definition 1.1.** A set $C \subset \mathbb{R}^m$ is **convex** if for all $x, y \in C$ and $\lambda \in [0, 1]$, $\lambda x + (1 - \lambda)y \in C$.

**Definition 1.2.** For $x_1, \ldots, x_n \in \mathbb{R}^m$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}_+$ such that $\sum_i \lambda_i = 1$, we call $\sum_i \lambda_i x_i$ a **convex combination** of $x_1, \ldots, x_n$.

**Definition 1.3.** The convex hull of a set $E \in \mathbb{R}^m$, denoted by $coE$, is the set of all convex combinations of all finite subsets of the set.

Caratheodory's Theorem is a basic result about convex sets that says that a vector in the convex hull of a set in $\mathbb{R}^m$ can be written as the convex combination of at most $m+1$ vectors of the set.

**Theorem 1.4.** *(Caratheodory) Let $E \subset \mathbb{R}^m$. If $x \in coE$, then there exist $z^0, \ldots, z^m \in E$ and $\lambda_0, \ldots, \lambda_m \in \mathbb{R}_+$ with $\sum_{i=0}^{m} \lambda_i = 1$ such that*

$$x = \sum_{i=0}^{m} \lambda_i z^i$$

*Proof.* Since $x \in coE$, we have that $x$ is a convex combination of some vectors in $E$. That is, $x = \sum_{i=0}^{n} \lambda_i z^i$ for some finite $n$, where $z^0, \ldots, z^n \in E$ and $\sum_{i=0}^{n} \lambda_i = 1$. Then this is the case for $n = m + k$ for some $k > 0$—for if $k \leq 0$, we are done, as this implies $x$ is already the convex combination of at most $m+1$ vectors. We wish to show that if $x = \sum_{i=0}^{m+k} \lambda_i z^i$, $k > 0$, then $x = \sum_{i=0}^{m} \tilde{\lambda}_i \tilde{z}^i$ for some coefficients with $\sum_{i=0}^{m} \tilde{\lambda}_i = 1$.

We are given that for some $k > 0$, $x = \sum_{i=0}^{m+k} \lambda_i z^i$ for some $z^0, \ldots, z^{m+k} \in \mathbb{R}^m$. Thus, both $x$ and the $z^i$ are linear combinations of the basis vectors $e^0, \ldots, e^{m-1}$.

For $i = 0, \ldots, m+k$, let

$$\tilde{z}^i = z^i + e^m \in \mathbb{R}^{m+1}$$

Trivially, $\sum_{i=0}^{m+k} \lambda_i = 1$ implies $\sum_{i=0}^{m+k} \lambda_i e^m = e^m$. Then we have

$$x + e^m = \sum_{i=0}^{m+k} \lambda_i \tilde{z}^i$$

Note that the $\tilde{z}^i \in E$ must be linearly dependent in $\mathbb{R}^{m+1}$, as there are $m+k+1$ of them. By definition, there exist $\gamma_0, \ldots, \gamma_{m+k} \in \mathbb{R}$ satisfying

$$\sum_{i=0}^{m+k} \gamma_i \tilde{z}^i = 0 \tag{1}$$

with at least one of the $\gamma_i > 0$. Upon rewriting (1) as

$$\sum_{i=0}^{m+k} \gamma_i (z^i + e^m) = \sum_{i=0}^{m+k} \gamma_i z^i + \sum_{i=0}^{m+k} \gamma_i e^m = 0 \tag{2}$$

it becomes clear that $\sum_{i=0}^{m+k} \gamma_i = 0$.

Let $I$ denote the indices $i \in \{0, \ldots, m+k\}$ such that $\gamma_i > 0$. Since $I$ is nonempty, there exists an index in $I$ that minimizes the ratio $\lambda_i/\gamma_i$. Call this index $j$. Possibly up to reordering, $\lambda_j$ is such that $\lambda_j \leq \lambda_i$ for all $i$. Then (2) implies

$$z^j = -\sum_{i \neq j}^{m+k} \frac{\gamma_i}{\gamma_j} z^i$$

Define $\tilde{\lambda}_i = \lambda_i - (\gamma_i/\gamma_j)\,\lambda_j$. Note that $\tilde{\lambda}_i \geq 0$ for all $i$: for $i \in I$, we have $\lambda_i \geq (\gamma_i/\gamma_j)\lambda_j$ by choice of $j$; for $i \notin I$, we have $(\gamma_i/\gamma_j)\lambda_j \geq 0$.

Further, $\sum_{i=0}^{m+k} \tilde{\lambda}_i = \sum_{i=0}^{m+k} \lambda_i - \frac{\lambda_j}{\gamma_j} \sum_{i=0}^{m+k} \gamma_i = 1 - 0 = 1$. Since $\tilde{\lambda}_j = 0$, $x$ can then be written as a convex combination of only $(m+k+1) - 1 = m+k$ of the original vectors, as follows:

$$\begin{aligned} x = \sum_{i=0}^{m+k} \lambda_i z^i &= \sum_{i=0, i \neq j}^{m+k} \lambda_i z^i - \lambda_j \sum_{i=0, i \neq j}^{m+k} \frac{\gamma_i}{\gamma_j} z^i \\ &= \sum_{i=0, i \neq j}^{m+k} \tilde{\lambda}_i z^i \end{aligned}$$

The above shows that for arbitrary $k > 0$, we have that for $x \in coE$ such that $x$ is the sum of $m+k+1$ vectors in $E$, $x$ is necessarily the sum of $m+k$ vectors in $E$. This then applies recursively $k$ times until we have that $x$ can be written as a convex combination of $m+1$ vectors in $E$, which is what we wanted. □

## 2. Simplices

**Definition 2.1.** We define a partial order on $\mathbb{R}^m$ as follows. For $x, y \in \mathbb{R}^m$, $x > y$ if $x_i > y_i$ for all $i$. Similarly, $x \geq y$ if $x_i \geq y_i$ for all $i$.

**Definition 2.2.** $x^0, \ldots, x^n \in \mathbb{R}^m$ are called **affinely independent** if $\sum_{i=0}^{n} \lambda_i x^i = 0$ and $\sum_{i=0}^{n} \lambda_i = 0$ imply $\lambda_0 = \cdots = \lambda_m = 0$.

**Definition 2.3.** An $n$ **simplex** is the set of all strictly positive convex combinations of an affinely independent set of $n+1$ elements. A closed $n$ simplex is the closure of an $n$ simplex. For a set of vectors $x^0, \ldots, x^n$, we denote the simplex $x^0 \cdots x^n$ by

$$x^0 \cdots x^n = \left\{ \sum_{i=0}^{n} \alpha_i x^i \mid \alpha_i > 0, i = 0, \ldots, n; \sum_{i=0}^{n} \alpha_i = 1 \right\}$$

Each $x^i$ is called a **vertex**, and each $k$ simplex $x^{i_0} \cdots x^{i_k}$ is a **face**. Further, if $y = \sum_{i=0}^n \alpha_i x^i$ and $\chi(y) = \{i \mid \alpha_i > 0\}$, then the face $x^{j_0} \cdots x^{j_k}$ (such that $j_0, \dots, j_k \in \chi(y)$) is called the **carrier** of $y$.

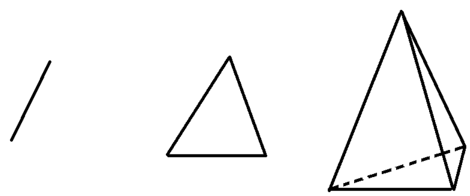

Fig. 1: A 1-simplex, 2-simplex, and 3-simplex

**Theorem 2.4.** *If the $n+1$ vectors $x^0, \dots, x^n \in \mathbb{R}^m$ are affinely independent, then $m \geq n$.*

*Proof.* For the sake of contradiction, suppose $m < n$. For each $i$, $x^i = \sum_{i=0}^{m-1} \alpha_i e^i$ for some $\alpha_i$. Put $y^i = x^i + e^m \in \mathbb{R}^{m+1}$. Suppose $\sum_{i=0}^{n} \lambda_i y^i = 0$ and $\sum_{i=0}^{n} \lambda_i = 0$. We then have

$$\sum_{i=0}^{n} \lambda_i y^i = \sum_{i=0}^{n} \lambda_i x^i + \sum_{i=0}^{n} \lambda_i e^m = 0 \implies \sum_{i=0}^{n} \lambda_i x^i = 0 \tag{3}$$

and the $x_i$ are affinely independent, so it follows that $\lambda_i = 0$ for all $i$. Then the $y^i \in \mathbb{R}^{m+1}$ are affinely independent. However, these make $n+1$ vectors in $\mathbb{R}^{m+1}$, meaning that they must be linearly dependent. That is, there exist $\gamma_0, \dots, \gamma_n \in \mathbb{R}$ such that $\sum_{i=0}^{n} \gamma_i y^i = 0$ and there exists $j$ such that $\gamma_j \neq 0$. But $\sum_{i=0}^{n} \gamma_i y^i = 0 \implies \sum_{i=0}^{n} \gamma_i e^m = 0 \implies \sum_{i=0}^{n} \gamma_i = 0$. By (3), this implies $\gamma_i = 0$ for all $i$, which is a contradiction. □

**Definition 2.5.** The **standard $n$ simplex** is $e^0 \cdots e^n$. The **standard closed $n$ simplex** $\Delta_n$ is the closure of the standard $n$ simplex.

**Lemma 2.6.** *Let $x, y \in \Delta_n$ such that $x \leq y$. Then $x = y$.*

*Proof.* $x, y \in \Delta_n$ means $x = \sum_{i=0}^{n} \alpha_i e^i$ and $y = \sum_{i=0}^{n} \beta_i e^i$ for some $\alpha_i, \beta_i \in \mathbb{R}_+$ such that $\sum_{i=0}^{n} \alpha_i = \sum_{i=0}^{n} \beta_i = 1$. By the definition of the partial ordering, we have $\alpha_i \leq \beta_i$ for all $i$. This implies $\sum_{i \in I} \alpha_i \leq \sum_{i \in I} \beta_i$ for any $I \subset \{0, \ldots, n\}$.

Suppose that there existed some $j$ such that $\alpha_j < \beta_j$. Then we have $\sum_{i \neq j} \alpha_i \leq \sum_{i \neq j} \beta_i$. Adding $\alpha_j < \beta_j$ would imply

$$\sum_{i=0}^{n} \alpha_i < \sum_{i=0}^{n} \beta_i$$

which contradicts $\sum_{i=0}^{n} \alpha_i = \sum_{i=0}^{n} \beta_i = 1$. We then have $\alpha_i = \beta_i$ for all $i$, so $x = y$. □

**Definition 2.7.** A **simplicial subdivision** of a closed simplex $\bar{T}$ is a finite set of simplices $\{T_i \mid i \in \{0, \ldots, p\}\}$ such that $\cup_i T_i = \bar{T}$ and that for any $i, j \in \{0, \ldots, p\}$, $\bar{T}_i \cap \bar{T}_j$ is either empty of equal to the closure of a common face.

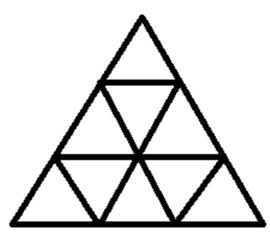

Fig. 2: A subdivision of a 2-simplex

**Definition 2.8.** Let the closed simplex $\bar{T}$, the closure of $x^0 \cdots x^n$, be simplicially subdivided and $V$ denote all the vertices of all the subsimplices. A function $\lambda : V \to \{0, \ldots, n\}$ such that $\lambda(v) \in \chi(v)$ for all $v \in V$ is called a **proper labeling** of the subdivision.

A subsimplex is **completely labeled** if $\lambda$ assumes all the values $0, \ldots, n$ on its vertices. A subsimplex is **almost completely labeled** if $\lambda$ assumes all the values $0, \ldots, n-1$.

The proofs of the remaining results in this paper follow the exposition in [1].

**Lemma 2.9.** *(Sperner) Let $\bar{T}$ be the closure of the simplex $x^0 \cdots x^n$. $\bar{T}$ is simplicially subdivided and properly labeled by the function $\lambda$. Then there are an odd number of completely labeled subsimplices in the subdivision.*

*Proof.* Call the subdivision $S$. When $n = 0$, the simplex is the single vector $x^0$. The proper labeling requires that $x^0$ be labeled 0. This is the only labeled vertex, and itself is the only completely labeled subsimplex. Since 1 is an odd number, we are done.

We proceed by induction, assuming that the lemma holds for $n = k - 1$. Note that an $n-1$ subsimplex is necessarily a face of an $n$ simplex. As such, it either lies on the boundary of $\bar{T}$ or is the shared face of two $n$ simplices. Define the following sets:

$$A = \{T_i \in S \mid T_i \text{ is an almost completely labeled } n \text{ simplex}\}$$
$$B = \{T_i \in S \mid T_i \text{ is a completely labeled } n-1 \text{ simplex on the boundary}\}$$
$$C = \{T_i \in S \mid T_i \text{ is a completely labeled } n \text{ simplex}\}$$
$$D = A \cup B \cup C$$
$$E = \{T_i \in S \mid T_i \text{ is a } n-1 \text{ simplex with all the labels } 0, 1, 2, \ldots, n-1\}$$

The point here is to recognize $D$ as a set of nodes and $E$ as a set of edges in a graph. We say that $d \in D$ and $e \in E$ are incident if either $d \in A \cup C$ and $e$ is a face of $d$, or $e = d \in B$.

It is known that for $d \in A$, $d$ has exactly one pair of vertices with the same label, being an almost-completely labeled $n$-simplex. This implies that it has two $n-1$ simplices (faces) in $E$. So $\deg d = 2$. But if $d \in B \cup C$, we have $\deg d = 1$.

Further, note that every $e \in E$ is indeed adjacent to two nodes in $D$: either between itself (but as a *node* in $B$) and a node in $A \cup C$, or between two nodes in $A \cup C$. So what we have is a simple graph.

So $\sum_{d \in D} \deg d = 2|A| + |B| + |C|$. By the Handshake Theorem in graph theory [2] this same quantity must be even, and by the inductive hypothesis, $|B|$ is odd, so $|C|$ is odd.

□

## 3. The Equilibrium Theorem

**Theorem 3.1.** *(Brouwer's Fixed Point Theorem for Simplices) Let* $f : \Delta_m \to \Delta_m$ *be continuous. Then* $f$ *has a fixed point.*

*Proof.* Take $\epsilon > 0$. Subdivide $\Delta$ simplicially into subsimplices of diameter no greater than $\epsilon$. If $V$ is the set of vertices of this subdivision, define a labeling $\lambda : V \to \{0, \ldots, n\}$ such that for $v \in x^{i_0} \cdots x^{i_k}$, we have

$$\lambda(v) \in \{i_0, \ldots, i_k\} \cap \{i : f_i(v) \leq v_i\}$$

(By $f_i(v)$ , we mean the $i$th coordinate of $f(v)$ with respect to the standard Cartesian basis.)

This intersection is known to be nonempty; if it were empty, then we would have $f_i(v) > v_i$ for all $i \in \{i_0, \ldots, i_k\}$, so that $\sum_{i=0}^{m} f_i(v) = 1 > \sum_{j=0}^{k} v_i$. But $v \in x^{i_0} \cdots x^{i_k}$, which means $\sum_{j=0}^{k} v_{i_j} = \sum_{i=0}^{m} v_i = 1$, contradicting $1 = 1$.

By Sperner's lemma, there must exist a completely labeled subsimplex, i.e. a simplex $p_\epsilon^0 \cdots p_\epsilon^m$ such that $f_i(p_\epsilon^i) \leq p_\epsilon^i$ for each $i$. Since $\Delta$ is compact, we can get a convergent subsequence of simplices such that $p_\epsilon^i \to z$ as $\epsilon \to 0$ for all $i = 0, \ldots, m$. Since $f$ is continuous, for all $i = 0, \ldots, m$ we have $f_i(z) \leq z_i$. By Lemma 2.6, $f(z) = z$.

□

Let $p \in \mathbb{R}_+^m$ be a price vector. If the supply vector $s(p)$ denotes the total quantity supplied for each commodity and $x(p)$ denotes the total quantity demanded, we call $f(p) = x(p) - s(p)$ an **excess demand vector**. $f(p)$ is generally assumed to be continuous. Walras' law [1], a sort of accounting identity, says that in a closed economy all markets must clear, so that we have $p \cdot f(p) = 0$.

The weak form of Walras' law says that in a closed economy there is no net borrowing, so that $p \cdot f(p) \leq 0$. In other words, some commodities are in excess supply. This situation is referred to as a **free disposal equilibrium**, and a price vector $p$ that makes $f(p) \leq 0$ is called a **free disposal price vector**. Note that since prices are nonnegative and the weak form of Walras' law holds, when $f_i(p) < 0$ we must have $p_i = 0$. In other words, the only cases in which markets do not clear completely is when the commodity comes for free.

**Theorem 3.2.** *(Equilibrium Theorem) Let* $f : \Delta_m \to \mathbb{R}^{m+1}$ *be continuous such that for all* $p$*,* $p \cdot f(p) \leq 0$*. Then the set* $\{p \in \Delta_m \mid f(p) \leq 0\}$ *of free disposal equilibrium prices is compact and nonempty.*

*Proof.* Define $h : \Delta_m \to \Delta_m$ by:

$$h(p) = \frac{p + f(p)^+}{1 + \sum_i f_i(p)^+}$$

where $f_i(p)^+ = \max\{f_i(p), 0\}$. To see why $h$ maps to $\Delta_m$, take $p \in \Delta_m$. This means $p = \sum_{i=0}^m \lambda_i e^i$ with $\sum_i \lambda_i = 1$ and $\lambda_0, \ldots, \lambda_m \in \mathbb{R}_+$. Then

$$\begin{aligned} h(p) &= \frac{\sum_{i=0}^m (\lambda_i + f_i(p)^+)\, e^i}{\sum_{i=0}^m \lambda_i + f_i(p)^+} \\ &= \sum_{i=0}^m \gamma_i e^i \end{aligned}$$

where $\gamma_i = \dfrac{\lambda_i + f_i(p)^+}{\sum_{i=0}^m \lambda_i + f_i(p)^+}$. Since $\gamma_i \in \mathbb{R}_+$ for all $i$ and $\sum_i \gamma_i = 1$, we have $h(p) \in \Delta_m$.

By Brouwer's fixed point theorem, there exists a $\bar{p}$ such that

$$\bar{p} = \frac{\bar{p} + f(\bar{p})^+}{1 + \sum_i f_i(\bar{p})^+} \tag{4}$$

Since $\bar{p} \cdot f(\bar{p}) \leq 0$, there must exist an $i$ such that $\bar{p}_i > 0$ and $f_i(\bar{p}) \leq 0$—otherwise, if such an $i$ did not exist, then we would have $\bar{p} \cdot f(\bar{p}) > 0$. For this $i$, $f(\bar{p})^+ = 0$. By (4), we must have $\sum_i f_i(\bar{p})^+ = 0$, which implies $f(\bar{p}) \leq 0$. This shows that the set of free disposal prices is nonempty.

Compactness is proven more easily. Since the set of free disposal prices is a subset of $\Delta_m$, it is bounded. Since this set is also the preimage of the closed set $(-\infty, 0]$ via the function $f$, it is also closed. Thus the set of free disposal prices is compact.

□

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