DISCUSSION OF THE HEAT EQUATION

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Abstract. This paper discusses the heat equation from multiple perspectives. It begins with the derivation of the heat equation. Then it shows how to find solutions and analyzes their properties, including uniqueness and regularity. The uniqueness is proved in two ways- energy method and maximum principle. The former gives physical interpretation of the heat equation while the latter has its own meaning beyond proving uniqueness. Regularity shows that the solutions to the heat equation are automatically smooth.

Contents

1. Introduction

A partial differential equation (PDE) is a differential equation that contains an unknown function of two or more independent variables and certain of its partial derivatives.

Definition 1.1. Fix an $k \geq 1$ and let U denote an open set of $\mathbb{R}^n$. Function F is a continuous function from some high-dimensional Euclidean space to $\mathbb{R}$. A k^{th}-order partial differential equation is expressed as

$$F(D^k u(x), D^{k-1}u(x), ..., Du(x), u(x), x) = 0 \quad (x \in U) \tag{1.2}$$

Intuitively, there is more than one solution to a specific k^{th}-order PDE, but we do generally want a function that is at least k times continuously differentiable. To further narrow down the solution, we prescribe boundary and initial conditions.
We should also notice that there are three very important criteria to evaluate

whether a PDE is worthy of studying, **existance, uniqueness, and regularity**. We are going to figure out what they mean and show that the heat equation satisfies all of them in the rest of the paper.

Notations.

Geometric Notations.

(1) $\mathbb{R}^n$= n-dimensional real Euclidean space
(2) U is an open subset of R^n.
(3) ∂U= boundary of U, $\bar{U} = U \cup \partial U$= closure of U
(4) $U_T = U \times (0, T]$
(5) $\Gamma_T = \bar{U}_T - U_T$= parabolic boundary of U_T

Functions.

(1) $u_{x_i} = \frac{\partial u}{\partial x_i}$
(2) Given a multiindex α, $D^\alpha u(x) = \frac{\partial^{|\alpha|} u(x)}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$
(3) If k is a nonnegative integer, $D^k u(x) = \{D^\alpha u(x) | |\alpha| = k\}$

Function Spaces.

(1) $C(U) = \{u : U \to \mathbb{R} | u \text{ continuous}\}$
(2) $C(\bar{U}) = \{u \in C(U) | u \text{ uniformly continuous}\}$.
(3) $C^k(U) = \{u : U \to \mathbb{R} | u \text{ is } k\text{-times continuously differentiable}\}$
(4) $C^k(\bar{U}) = \{u \in C^k(U) | D^\alpha u \text{ is uniformly continuous for all } |\alpha| \leq k\}$
(5) $C^\infty(U) = \{u : U \to \mathbb{R} | u \text{ is infinitely differentiable}\}$
(6) $C_1^2(U_T) = \{u : \mathbb{R}^n \times [0, \infty) \to \mathbb{R} | u, D_x u, D_x^2 u, u_t \in C(\mathbb{R}^n \times [0, \infty))\}$

2. Definition of the Heat Equation and Linearity

A heat equation is a PDE that has the form:

$$u_t - k\Delta u = f \tag{2.1}$$

When $f = 0$, it is homogeneous. When $f \neq 0$, it is inhomogeneous.
Linearity is an important property of the heat equation. It greatly reduces the degree of difficulty of finding solutions.
We define that a PDE is linear by following the steps:
First, an operator $\mathcal{L}$ is said to be linear if, for any real-valued functions u and v and scalars a and b,

$$\mathcal{L}(au + bv) = a\mathcal{L}u + b\mathcal{L}v \tag{2.2}$$

The equation

$$\mathcal{L} \sum_{i=1}^{n} \alpha_i u_i = f(x) \tag{2.3}$$

is called linear if $\mathcal{L}$ is a linear operator. $f(x)$ is a function of independent variables. If $f = 0$, the equation is called a **homogeneous linear equation**. Otherwise, it is called an **inhomogeneous linear equation**.

Definition 2.4. Accordingly, the partial differential equation is called linear if it has the form,

$$\sum_{|\alpha|\leq k} a_\alpha(x) D^\alpha u = f(x) \tag{2.5}$$

Functions $a_\alpha(x)$ and $f(x)$ are given. This PDE is homogeneous if $f(x) = 0$.

Thus, by studying the structure of the heat equation, we could draw the conclusion that it is linear.

Remark 2.6. If a PDE is linear,

(1) Any linear combinations of solutions to its homogeneous problem is again a solution to it.
(2) All solutions of the inhomogeneous problem can be expressed as linear combinations of a particular solution to the inhomogeneous problem and some solutions to the homogeneous problem.

3. Derivation of the Heat Equation

There are several physical interpretations of the heat equation. The following is a classical one:
Let D be a region in $\mathbb{R}^n$, let $u(x,t)$ be the temperature at point x, time t, and let $H(t)$ be the total amount of heat contained in D. Let c be the specific heat of the material and ρ be its density (mass per unit volume). Then the total heat in D is expressed as,

$$H(t) = \int_D c\rho u(x,t)dx \tag{3.1}$$

By taking derivatives we could find the rate of heat as

$$\frac{dH}{dt} = \int_D c\rho u_t(x,t)dx \tag{3.2}$$

According to Fourier's Law, heat flows from hot to cold regions at a rate $\kappa > 0$ proportional to the temperature gradient. We also know that the heat leaves D through the boundary, which gives the equation,

$$\frac{dH}{dt} = \int_{\partial D} \kappa \nabla u \cdot \hat{n} ds \tag{3.3}$$

where ∂D is the boundary of D, $\hat{n}$ is the outward unit normal vector to ∂D and ds is the surface measure over D. Therefore, we have

$$\int_D c\rho u_t(x,t)dx = \int_{\partial D} \kappa \nabla u \cdot \hat{n} ds \tag{3.4}$$

According to the divergence theorem,

$$\int_{\partial D} \kappa \nabla u \cdot \hat{n} ds = \int_D \nabla \cdot (\kappa \nabla u) dx \tag{3.5}$$

Thus,

$$\int_D c\rho u_t(x,t)dx = \int_D \nabla \cdot (\kappa \nabla u) dx \tag{3.6}$$

Since the domain D is arbitrary, the integrands must be the same.

$$c\rho u_t = \nabla \cdot (\kappa \nabla u) \tag{3.7}$$

As c, ρ and κ are constant, we have derived the general form of the heat equation

$$u_t = k\Delta u \tag{3.8}$$

If there is a heat source in D, the rate of heat is the sum of the rate of heat that leaves D and the rate of heat added by the heat source. Then the equation becomes

$$\frac{dH}{dt} = \int_D \nabla \cdot (\kappa \nabla u) dx + f = \int_D c\rho u_t(x,t) dx \tag{3.9}$$

where f denotes the rate of heat source. From this, we derive the inhomogeneous form of the heat equation

$$u_t = k\Delta u + f \tag{3.10}$$

4. Existence-Solution of the Heat Equation

A solution of PDE is a function $u(x)$ that satisfies the equation identically.

4.1. **Fundamental Solution.**

To makes things simpler, we have set $k = 1$. So we are looking for some specific solutions for

$$u_t = \Delta u \tag{4.1}$$

We seek a solution u with specific structure

$$u(x,t) = \frac{1}{t^\alpha} v(\frac{x}{t^\beta}) \quad (x \in \mathbb{R}^n, t > 0), \tag{4.2}$$

where α, β are constants and v is a function $\mathbb{R}^n \to \mathbb{R}$.
Insert(4.2) to (4.1),

$$\alpha t^{-(\alpha+1)} v(y) + \beta t^{-(\alpha+1)} y \cdot Dv(y) + t^{-(\alpha+2\beta)} \partial v(y) = 0 \tag{4.3}$$

for $y = t^{-\beta x}$. We want the equation to have variable y alone, so we set $-(\alpha+1) = -(\alpha+2\beta)$, $\beta = \frac{1}{2}$. Now (4.3) reduces to

$$\alpha v + \frac{1}{2} y \cdot Dv + \partial v = 0 \tag{4.4}$$

Intuitively, v is radial if we have a single source at the origin, at which all energy is concentrated at $t = 0$. This means $v(y) = w(|y|)$ for some function $w : \mathbb{R} \to \mathbb{R}$. The equation becomes

$$\alpha w + \frac{1}{2} r w' + w'' + \frac{n-1}{r} w' = 0, \tag{4.5}$$

for $r = |y|$. Now we set $\alpha = \frac{n}{2}$,

$$(r^{n-1} w')' + \frac{1}{2}(r^n w)' = 0 \tag{4.6}$$

This means,

$$r^{n-1} w' + \frac{1}{2} r^n w = a \tag{4.7}$$

for some constant a. Assuming $\lim_{r \to +\infty} w, w' = 0$, we conclude $a = 0$. Thus,

$$w' = -\frac{1}{2} r w \tag{4.8}$$

$$w = b e^{-\frac{r^2}{4}} \tag{4.9}$$

for some constant b. Therefore, $\frac{b}{t^{n/2}} e^{-\frac{|x|^2}{4t}}$ solves the equation.

Definition 4.10. The fundamental solution to the heat equation is,

$$\Phi(x,t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}} & (x \in \mathbb{R}, t > 0) \\ 0 & (x \in \mathbb{R}, t < 0) \end{cases} \tag{4.11}$$

We choose the normalizing constant, b, to be $(4\pi)^{-n/2}$ because we want $\int_{\mathbb{R}^n} \Phi(x,t)dx = 1$. As $t \to 0$, $\Phi(x,t)$ approximates the delta function on $\mathbb{R}^n$. It describes the temperature on $\mathbb{R}^n$ in the situation where a point heat source is placed at the origin at time $t = 0$.

Lemma 4.12 (Integral of fundamental solution)**.** *For each time $t > 0$,*

$$\int_{\mathbb{R}^n} \Phi(x,t)dx = 1 \tag{4.13}$$

Proof.

$$\begin{aligned} \int_{\mathbb{R}^n} \Phi(x,t)dx &= \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4t}} dx \\ &= \frac{1}{\pi^{n/2}} \int_{\mathbb{R}^n} e^{-|z|^2} dz \\ &= \frac{1}{\pi^{n/2}} \cdot \pi^{n/2} \\ &= 1 \end{aligned} \tag{4.14}$$

□

The proof uses the result of **Gaussian integral**, which gives $\int_{\mathbb{R}^n} e^{-x^2} dx = \pi^{n/2}$. The proof is as following,

Proof. First, we want to show that $\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}$

$$\begin{aligned} \int_{-\infty}^{+\infty} e^{-x^2} dx &= \sqrt{(\int_{-\infty}^{+\infty} e^{-x^2} dx)(\int_{-\infty}^{+\infty} e^{-x^2} dx)} \\ &= \sqrt{(\int_{-\infty}^{+\infty} e^{-x^2} dx)(\int_{-\infty}^{+\infty} e^{-y^2} dy)} \\ &= \sqrt{(\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-(x^2+y^2)} dxdy)} \end{aligned} \tag{4.15}$$

In polar coordinates,

$$\begin{aligned} \int_{-\infty}^{+\infty} e^{-x^2} dx &= \sqrt{(\int_0^{2\pi} \int_{-\infty}^{+\infty} e^{-r^2} r dr d\theta)} \\ &= \sqrt{2\pi[-\frac{1}{2} e^{-r^2}]_0^{\infty}} \\ &= \sqrt{\pi} \end{aligned} \tag{4.16}$$

Then,

$$\int_{\mathbb{R}^n} e^{-x^2} dx = \prod_{i=1}^{n} \int_{-\infty}^{+\infty} e^{-x_i^2} dx_i = \pi^{n/2} \tag{4.17}$$

□

The physical interpretation of this lemma is the conservation of energy.

4.2. **Initial-Value Problem.** We now look for a solution when the function has a specific initial-value that is not 0.

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases} \tag{4.18}$$

If $\Phi(x,t)$ solves the heat equation away from the singularity at $(0,0)$, so does $\Phi(x-y,t)$ for each fixed $y \in \mathbb{R}^n$. Thus,

$$\begin{aligned} u(x,t) &= \int_{\mathbb{R}^n} \Phi(x-y,t)g(y)dy \\ &= \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)dy \quad (x \in \mathbb{R}^n, t>0) \end{aligned} \tag{4.19}$$

should be a solution.

Theorem 4.20 (Solution of initial-value problem)**.** *Assume g is continuous and bounded on $\mathbb{R}^n$. Define u by (4.19). Then,*

(1) $u \in C^\infty(\mathbb{R}^n \times (0,\infty))$
(2) $u_t - \Delta u = 0 \quad (x \in \mathbb{R}^n, t>0)$
(3) $\lim_{(x,t)\to(x^0,0), x\in\mathbb{R}^n, t>0} u(x,t) = g(x^0)$ *for each point* $x^0 \in \mathbb{R}^n$

Proof.

(1) The function $\frac{1}{t^{n/2}} e^{-\frac{|x|^2}{4t}}$ is infinitely differentiable and has uniformly bounded derivatives of all orders on $\mathbb{R}^n \times [\partial,\infty)$. So $u \in C^\infty(\mathbb{R} \times (0,\infty))$.

(2) According to **Leibniz's Rule**, for x in (x_0,x_1),

$$\frac{d}{dx}\int_{y_0}^{y_1} f(x,y)dy = \int_{y_0}^{y_1} f_x(x,y)dy \tag{4.21}$$

if f and its partial derivative f_x are both continuous over the region in the form $[x_0,x_1]\times[y_0,y_1]$. Then, provided both u and u_t are continuous over $\mathbb{R}^n$

$$u_t(x,t) = \int_{\mathbb{R}} \Phi_t(x-y,t)g(y)dy \tag{4.22}$$

Then,

$$u_t(x,t) - \Delta u(x,t) = \int_{\mathbb{R}} [(\Phi_t - \partial_x \Phi)(x-y,t)]g(y)dy = 0 \quad (x \in \mathbb{R}^n, t>0) \tag{4.23}$$

(3) Since g is continuous, for any $x^0 \in \mathbb{R}^n$ and $\epsilon > 0$, there exists $\delta > 0$ such that

$$|g(y) - g(x^0)| < \epsilon \quad \text{if } |y - x^0| < \delta,\ y \in \mathbb{R}^n \tag{4.24}$$

Then if $|x - x^0| < \frac{\delta}{2}$, we have

(4.25)

$$\begin{aligned}|u(x,t) - g(x^0)| &= |\int_{\mathbb{R}^n} \Phi(x-y,t)[g(y) - g(x^0)]dy| \\ &\leq \int_{B(x^0,\delta)} \Phi(x-y,t)|g(y) - g(x^0)|dy + \int_{\mathbb{R}^n - B(x^0,\delta)} \Phi(x-y,t)[g(y) - g(x^0)]dy \\ &= I + J\end{aligned}$$

Then,

$$I \leq \epsilon \int_{\mathbb{R}^n} \Phi(x-y,t)dy = \epsilon \tag{4.26}$$

If $|x - x^0| \leq \frac{\delta}{2}$ and$|y - x^0| \geq \delta$,

$$|y - x^0| \leq |y - x| + \frac{\delta}{2} \leq |y - x| + \frac{1}{2}|y - x^0| \tag{4.27}$$

$$\frac{1}{2}|y - x^0| \leq |y - x| \tag{4.28}$$

Consequently,

$$\begin{aligned} J &\leq 2||g||_{L^\infty} \int_{\mathbb{R} - B(x^0,\delta)} \Phi(x-y,t)dy \\ &\leq \frac{C}{t^{n/2}} \int_{\mathbb{R} - B(x^0,\delta)} e^{-\frac{|x-y|^2}{4t}} dy \\ &\leq \frac{C}{t^{n/2}} \int_{\mathbb{R} - B(x^0,\delta)} e^{-\frac{|x-y|^2}{16t}} dy \\ &= \frac{C}{t^{n/2}} \int_{\delta}^{\infty} e^{-\frac{r^2}{16t}} r^{n-1} dr \to 0 \quad \text{as } t \to 0^+ \end{aligned} \tag{4.29}$$

Thus, for any $\epsilon > 0$, there exists some$\frac{\delta}{2}$ such that if $|x - x^0| < \frac{\delta}{2}$ and t is small enough, then $u(x,t) - g(x^0) < \epsilon$.
This proves $\lim_{(x,t)\to(x^0,0), x\in\mathbb{R}^n, t>0} u(x,t) = g(x^0)$ for each point $x^0 \in \mathbb{R}$

□

4.3. **inhomogeneous Problem.** For inhomogeneous problem, the equation has the form:

$$\begin{cases} u_t - \Delta u = f & \text{in } \mathbb{R} \times (0,\infty) \\ u = 0 & \text{on } \mathbb{R} \times \{t = 0\} \end{cases} \tag{4.30}$$

To find the solution, we notice that mapping $(x,t) \to \Phi(x-y, t-s)$ is also a solution of the heat equation (for given $y \in \mathbb{R}, 0 < s < t$). Thus, for fixed s,

$$u = u(x,t;s) = \int_{\mathbb{R}^n} \Phi(x-y, t-s) f(y,s)dy \tag{4.31}$$

solves

$$\begin{cases} u_t(x,t;s) - \Delta u(x,t;s) = 0 & \text{in } \mathbb{R} \times (s,\infty) \\ u(x,t;s) = f(x,t;s) & \text{on } \mathbb{R} \times \{t = s\} \end{cases} \tag{4.32}$$

which is just an initial value problem.
However, according to **Duhamel's principle**, we can find a solution of inhomogeneous problem by integrating the solution of initial value problem over s, which gives us

$$\begin{aligned} u(x,t) &= \int_0^t u(x,t;s)ds \\ &= \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s)f(y,s)dyds \\ &= \int_0^t \frac{1}{(4\pi(t-s))^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4(t-s)}} f(y,s)dyds \quad (x\in\mathbb{R}^n, t>0) \end{aligned} \tag{4.33}$$

Theorem 4.34 (Solution of inhomogeneous problem)**.** *For simplicity we are assuming that $f \in C_1^2(\mathbb{R}^n \times [0,\infty))$ and f has compact support.*
Then

(1) $u \in C_1^2(\mathbb{R}^n \times (0,\infty))$
(2) $u_t(x,t) - \Delta u(x,t) = f(x,t) \quad (x\in\mathbb{R}^n, t>0)$
(3) $\lim_{(x,t)\to(x^0,0), x\in\mathbb{R}^n, t>0} u(x,t) = 0$ *for each point* $x^0 \in \mathbb{R}$

Proof.

(1) By changing the variables, we write

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s)f(x-y,t-s)dyds \tag{4.35}$$

Since $f \in C_1^2(\mathbb{R}^n \times [0,\infty))$ has compact support and $\Phi(y,s)$ is smooth near $s=t>0$, we compute

$$u_t(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s)f_t(x-y,t-s)dyds + \int_{\mathbb{R}^n} \Phi(y,s)f_t(x-y,0)dy \tag{4.36}$$

and

$$\frac{\partial^2 u}{\partial x_i \partial x_j}(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s)\frac{\partial^2}{\partial x_i \partial x_j} f(x-y,t-s)dyds \quad (i,j=1,...,n) \tag{4.37}$$

Thus u belongs to $C(\mathbb{R}^n \times (0,\infty))$.

(2)

$$\begin{aligned} u_t(x,t) - \Delta u(x,t) &= \int_0^t \int_{\mathbb{R}^n} \Phi(y,s)[(\frac{\partial}{\partial t} - \Delta_x) f(x-y,t-s)]dyds \\ &\quad + \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)dy \\ &= \int_\epsilon^t \int_{\mathbb{R}^n} \Phi(y,s)[-(\frac{\partial}{\partial s} - \Delta_y) f(x-y,t-s)]dyds \\ &\quad + \int_0^\epsilon \int_{\mathbb{R}^n} \Phi(y,s)[-(\frac{\partial}{\partial s} - \Delta_y) f(x-y,t-s)]dyds \\ &\quad + \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)dy \\ &= I_\epsilon + J_\epsilon + K \end{aligned} \tag{4.38}$$

By **integral of fundamental solution**,

$$|J_\epsilon| \leq (||f_t||_{L^\infty} + ||D^2 f||_{L^\infty}) \int_0^\epsilon \int_{\mathbb{R}^n} \Phi(y,s)dyds \leq \epsilon C \tag{4.39}$$

By integrating by parts,

$$\begin{aligned} |I_\epsilon| &= \int_\epsilon^t \int_{\mathbb{R}^n} \Phi(y,s)[-(\frac{\partial}{\partial s} - \Delta_y) f(x-y,t-s)]dyds \\ &+ \int_{\mathbb{R}^n} \Phi(y,\epsilon) f(x-y,t-\epsilon)dy \\ &- \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)dy \\ &= \int_{\mathbb{R}^n} \Phi(y,\epsilon) f(x-y,t-\epsilon)dy - K \end{aligned} \tag{4.40}$$

Combining (4.36) to (4.38), we have,

$$\begin{aligned} u_t(x,t) - \Delta u(x,t) &= \lim_{\epsilon \to 0} \int_{\mathbb{R}^n} \Phi(y,\epsilon) f(x-y,t-\epsilon)dy \\ &= f(x,t) \quad (x \in \mathbb{R}^n, t > 0) \end{aligned} \tag{4.41}$$

□

Remark 4.42. Combining the initial problem and the inhomogeneous problem, we find the solution of

$$\begin{cases} u_t - \partial u = f & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases} \tag{4.43}$$

is

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s)dyds + \int_{\mathbb{R}^n} \Phi(x-y,t) g(y)dy \tag{4.44}$$

5. Uniqueness

Uniqueness is a very important property of the heat equation. It means that any solution is determined completely by its initial and boundary conditions. There are two approaches to prove this property: **energy method** and **maximum principle**.

5.1. **Energy method and uniqueness.** If both u and $\tilde{u}$ are solutions, $w = u - \tilde{u}$ satisfies,

$$\begin{cases} w_t - \Delta w = 0 & \text{in } U_T) \\ w = 0 & \text{on } \Gamma_T \end{cases} \tag{5.1}$$

Now define the following ”energy”

$$E(t) = \int_U w^2(x,t)dx \quad (0 \leq t \leq T) \tag{5.2}$$

Then

$$E_t(t) = 2\int_U ww_t dx = 2\int_U w\partial w dx = -2\int_U |Dw|^2 dx \le 0 \tag{5.3}$$

so

$$E(t) \le E(0) = 0 \quad (0) \tag{5.4}$$

Consequently,

$$w = u = \tilde{u} = 0 \quad \text{in } U_T \tag{5.5}$$

5.2. **Maximum Principle and Uniqueness.** u is a solution of the heat equation. Maximum principle asserts that the maximum of u in a domain is to be found on the boundary of that domain. If it achieves maximum in the interior of the domain, then it is constant over the domain. To prove the maximum principle, we prove the **mean value property** first.

Definition 5.6 (Heat Ball)**.** For fixed $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, $r > 0$, we define

$$E(x,t;r) = \{(y,s) \in \mathbb{R}^{n+1} \mid s \le t, \Phi(x-y,t-s) \ge \frac{1}{r^n}\} \tag{5.7}$$

Theorem 5.8 (Mean value property for the heat equation)**.** *Let $u \in C_1^2(U_T)$ solve the heat equation. Then*

$$u(x,t) = \frac{1}{4r^n}\iint_{E(x,y;r)} u(y,s)\frac{|x-y|^2}{(t-s)^2} dyds \tag{5.9}$$

for each $E(x,y;r) \subset U_T$

Proof. Assume $x = 0, t = 0$ and set

$$\phi(r) = \frac{1}{r^n}\iint_{E(r)} u(y,s)\frac{|y|^2}{s^2} dyds = \iint_{E(1)} u(ry, r^2 s)\frac{|y|^2}{s^2} dyds \tag{5.10}$$

$$\phi'(r) = \iint_{E(1)} \sum_{i=1}^n u_{y_i}\frac{|y|^2}{s^2} + 2ru_s\frac{|y|^2}{s^2} dyds = \frac{1}{r^{n+1}}\iint_{E(r)} \sum_{i=1}^n u_{y_i}(y_i)\frac{|y|^2}{s^2} + 2u_s\frac{|y|^2}{s^2} dyds = A + B \tag{5.11}$$

Introducing the useful function

$$\psi = -\frac{n}{2}log(-4\pi s) + \frac{|y|^2}{4s} + nlogr \tag{5.12}$$

$\psi = 0$ on $\partial E(r)$

$$\begin{aligned} B &= \frac{1}{r^{n+1}} \iint_{E(r)} 4u_s \sum_{i=1}^{n} y_i \psi_{y_i} dyds \\ &= -\frac{1}{r^{n+1}} \iint_{E(r)} 4nu_s\psi + 4\sum_{i=1}^{n} u_{sy_i} y_i \psi dyds \\ &= \frac{1}{r^{n+1}} \iint_{E(r)} -4nu_s\psi + 4\sum_{i=1}^{n} u_{y_i} y_i \psi_s dyds \\ &= \frac{1}{r^{n+1}} \iint_{E(r)} -4nu_s\psi + 4\sum_{i=1}^{n} u_{y_i} y_i (-\frac{n}{2s} - \frac{|y|^2}{4s^2}) dyds \\ &= \frac{1}{r^{n+1}} \iint_{E(r)} -4nu_s\psi - \frac{2n}{s}\sum_{i=1}^{n} u_{y_i} y_i dyds - A \end{aligned} \tag{5.13}$$

Since u solves the heat equation,

$$\begin{aligned} \phi'(r) &= A + B \\ &= \frac{1}{r^{n+1}} \iint_{E(r)} -4n\partial u\psi - \frac{2n}{s}\sum_{i=1}^{n} u_{y_i} y_i dyds \\ &= \sum_{i=1}^{n} \frac{1}{r^{n+1}} \iint_{E(r)} -4nu_{y_i}\psi_{y_i} - \frac{2n}{s} u_{y_i} y_i dyds \\ &= 0 \end{aligned} \tag{5.14}$$

This means that ϕ is constant. Then,

$$\phi(r) = \lim_{t\to 0} \phi(t) = u(0.0)(\lim_{t\to 0} \frac{1}{t^n}) \iint_{E(t)} \frac{|y|^2}{s^2} dyds) = 4u(0,0) \tag{5.15}$$

$$\frac{1}{t^n} \iint_{E(t)} \frac{|y|^2}{s^2} dyds = \iint_{E(1)} \frac{|y|^2}{s^2} dyds = 4 \tag{5.16}$$

□

Theorem 5.17 (Maximum Principle)**.** *Assume $u \in C_1^2(U_T) \cap C(\bar{U}_T)$ solves the heat equation in U_T.*

(1) Then

$$\max_{\bar{U}_T} u = \max_{\Gamma_T} u \tag{5.18}$$

(2) Furthermore, if U is connected and there exists a point $(x_0, t_0) \in U_T$ such that

$$u(x_0, t_0) = \max_{\bar{U}_T} u \tag{5.19}$$

then u is constant in $\bar{U}_{t_0}$.

Proof. Suppose there exists $(x_0, t_0) \in U_T$, $u(x_0, t_0) = \max_{\bar{U}_T} u = M$. Pick r small enough so that $E(x_0, t_o; r) \subset U_T$. Applying the mean value property,

$$M = u(x_0, t_0) = \frac{1}{4r^n} \iint_{E(x_0,t_0;r)} u(y, s) \frac{|x_0 - y|^2}{(t_0 - s)^2} dyds \leq M \tag{5.20}$$

The equality holds only if

$$u(y, s) = M \quad \text{for all } (y, s) \in E(x_0, t_0; r) \tag{5.21}$$

Now we have proved u is constant within in a small heat ball. Then we try to extend the result to the whole space.
First, we draw any segment L in U_T connecting (x_0, t_0) with some other $(y_0, s_0) \in U_T$, with $s_0 < t_0$ and we want to prove $u \equiv M$ on L. Consider

$$r_0 = min\{s \geq s_0 | u(x, t) = M \text{ for all points } (x, t) \in L, s \leq t \leq t_0\} \tag{5.22}$$

Assume $r_0 > s_0$. Then $u(z_0, r_0) = M$ for some point (z_0, r_0) on $L \cap U_T$ and so $u = M$ for all sufficiently small $r > 0$. It contradicts that $E(z_0, r_0; r)$ contains $L \cap \{r_0 - \sigma \leq t \leq r_0\}$ for some $\sigma > 0$. Thus $r_0 = s_0$ and $u = M$ on L.
Next, we fix a point $x \in U$ and any time $0 \leq t \leq t_0$. We can connect the point to (x_0, t_0) with a finite number of line segments in U_T. According to the first step, $u = M$ on each segment so $u(x, t) = M$. Then u is constant over the whole space. □

With the maximum principle we can easily prove uniqueness.

Theorem 5.23 (Uniqueness on bounded domains)**.** *Let $g \in C(\Gamma_T), f \in C(U_T)$. Then there exists at most one solution $u \in C_1^2(U_T) \cap C(\bar{U}_T)$ of the initial/boundary value problem*

$$\begin{cases} u_t - \partial u = f & \text{in } U_T \\ u = g & \text{on } \Gamma_T \end{cases} \tag{5.24}$$

Proof. Assume u and $\tilde{u}$ are two solutions, set $v = u - \tilde{u}$. Then v satisfies with zero boundary condition and initial condition $u(x, 0) - \tilde{u}(x, 0)$. According to the first part of the maximum principle, the maximum value of v, $\max_{\bar{U}_T} u = \max_{\Gamma_T} = 0$. According to the second part of the maximum principle, $v = 0$ on $u \in C_1^2(U_T) \cap C(\bar{U}_T)$. □

The maximum principle and uniqueness also apply to the initial-value problem (Cauchy problem).

Theorem 5.25 (Maximum principle for the initial-value problem)**.** *Suppose $u \in C_1^2(\mathbb{R}^n \times (0, T]) \cap C(\mathbb{R}^n \times [0, T])$ solves*

$$\begin{cases} u_t - \Delta u = f & \text{in } \mathbb{R} \times (0, T) \\ u = g & \text{on } \mathbb{R} \times \{t = 0\} \end{cases} \tag{5.26}$$

and satisfies the growth estimate

$$u(x, t) \leq Ae^{a|x|^2} \quad (x \in \mathbb{R}^n, 0 \leq t \leq T) \tag{5.27}$$

for constants A, $a > 0$. Then

$$\sup_{\mathbb{R}^n \times [0,T]} u = \sup_{\mathbb{R}^n} g \tag{5.28}$$

Proof. To make the proof simpler, we impose specific restriction on time T first. Assume

$$4aT < 1 \tag{5.29}$$

Then there exists $\epsilon > 0$ such that

$$4a(T+\epsilon) < 1 \tag{5.30}$$

Fix $y \in \mathbb{R}^n, u > 0$ and define

$$v(x,t) = u(x,t) - \frac{u}{(T+\epsilon-t)^{n/2}} e^{\frac{|x-y|^2}{4(T+\epsilon-t)}} \tag{5.31}$$

Then calculation shows that $v(x,t)$ satisfies

$$v_t - \Delta v = 0 \text{ in } \mathbb{R}^n \times (0,T] \tag{5.32}$$

Fix $r > 0$ and set $U = B^0(y,r)$, $U_T = B^0(y,r) \times (0,T]$. According to **the maximum principle**,

$$\max_{\bar{U}_T} u = \max_{\Gamma_T} u \tag{5.33}$$

Now if $x \in \mathbb{R}^n$,

$$\begin{aligned} v(x,0) &= u(x,0) - \frac{u}{(T+\epsilon)^{n/2}} e^{\frac{|x-y|^2}{4(T+\epsilon)}} \\ &\leq u(x,0) = g(x) \end{aligned} \tag{5.34}$$

If $|x-y| = r$, $0 \leq t \leq T$, then

$$\begin{aligned} v(x,t) &= u(x,t) - \frac{u}{(T+\epsilon-t)^{n/2}} e^{\frac{|x-y|^2}{4(T+\epsilon-t)}} \\ &\leq Ae^{a|x|^2} - \frac{u}{(T+\epsilon-t)^{n/2}} e^{\frac{|x-y|^2}{4(T+\epsilon-t)}} \\ &\leq Ae^{a(|y|+r)^2} - \frac{u}{(T+\epsilon-t)^{n/2}} e^{\frac{|x-y|^2}{4(T+\epsilon-t)}} \end{aligned} \tag{5.35}$$

Since there exists $\gamma > 0$ such that $\frac{1}{4(T+\epsilon)} = a + \gamma$, we can choose γ large enough that

$$v(x,t) \leq Ae^{a(|y|+r)^2} - \mu(4(a+\gamma))^{n/2} e^(a+\gamma)r^2 \leq \sup_{\mathbb{R}^n} g \tag{5.36}$$

Thus, let $\mu \to 0$

$$v(y,t) \leq \sup_{\mathbb{R}^n} g \tag{5.37}$$

for all $y \in \mathbb{R}^n, 0 \leq t \leq T$. Now we remove the restriction on T. If (5.29) is not true, we can simply repeatedly apply the result above on time intervals $[0,T_1], [0,T_2]$... for $T_1 = \frac{1}{8a}$ □

Theorem 5.38 (Uniqueness for initial value problem)**.** *Let $g \in C(\mathbb{R}^n), f \in C(\mathbb{R}^n \times [0,T])$. Then there exists at most one solution $u \in C_1^2(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$ of the initial value problem*

$$\begin{cases} u_t - \Delta u = f & \text{in } \mathbb{R} \times (0,T) \\ u = g & \text{on } \mathbb{R} \times \{t = 0\} \end{cases} \tag{5.39}$$

which satisfies the growth estimate

$$|u(x,t)| \le Ae^{a|x|^2} \quad (x \in \mathbb{R}^n, 0 \le t \le T) \tag{5.40}$$

for constants A, $a > 0$.

Proof. If there are two solutions u and $\tilde{u}$, we can apply **maximum principle for initial value problem** to $w = u - \tilde{u}$ and find contradictions. □

6. Regularity

Below we prove that the solutions of the heat equation are automatically smooth.

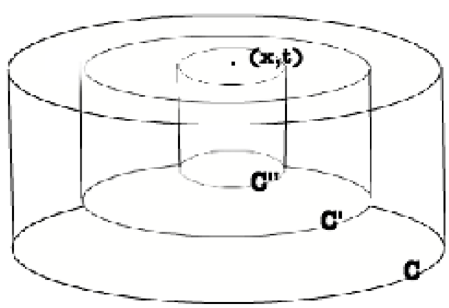
Theorem 6.1 (Smoothness)**.** *Suppose $u \in C_1^2(U_T)$ solves the heat equation in U_T. Then*

$$u \in C^\infty(U_T) \tag{6.2}$$

Proof. We denote the closed circular cylinder of radius r, height r^2 and top center point (x,t) by

$$C(x,y;r) = \{(y,s) || x-y| \le r, t-r^2 \le s \le t\} \tag{6.3}$$

Fix $(x_0,t_0) \in U_T$. We can pick a small $r > 0$ such that $C = C(x_0,y_0;r), C' = C(x_0,y_0;\frac{3}{4}r), C'' = C(x_0,y_0;\frac{1}{2}r) \subset U_T$.



Choose a smooth cutoff function $\zeta = \zeta(x,t)$ with $0 \le \zeta \le 1$:

$$\begin{cases} \zeta \equiv 1 & \text{on } C'', \\ \zeta \equiv 0 & \text{near the parabolic boundary of } C \text{ and in } (\mathbb{R} \times [0,t_0]) - C \end{cases} \tag{6.4}$$

We want to derive a formula of u, so we assume temporarily that $u \in C^\infty(U_T)$. Set

$$v(x,t) = \zeta(x,t)u(x,t) \quad (x \in \mathbb{R}^n, 0 \le t \le t_0). \tag{6.5}$$

Then

$$v_t = \zeta u_t + \zeta_t u, \quad \Delta v = \zeta \Delta u + 2D\zeta \cdot Du + u\Delta\zeta \tag{6.6}$$

Consequently

$$v = 0 \quad \text{on } \mathbb{R}^n \times \{t = 0\} \tag{6.7}$$

Define function $\tilde{f}$ by

$$\tilde{f} = v_t - \Delta v = \zeta_t u - 2D\zeta \cdot Du - u\Delta\zeta \quad \text{in } \mathbb{R}^n \times (0,t_0) \tag{6.8}$$

Now set

$$\tilde{v}(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s)\tilde{f}(y,s)dyds \tag{6.9}$$

Then $\tilde{v}$ is a solution to the inhomogeneous equation

$$\begin{cases} \tilde{v}_t - \Delta\tilde{v} = \tilde{f} & \text{in } \mathbb{R}^n \times (0, t_0) \\ \tilde{v} = 0 & \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases} \tag{6.10}$$

According to **the uniqueness for cauchy problem**, since $|v|, |\tilde{v}| \leq A$

$$\tilde{v}(x,t) = v(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s)\tilde{f}(y,s)dyds \tag{6.11}$$

Now suppose $(x,t) \in C''$. Since $\zeta \equiv 0$ in $\mathbb{R} \times [0, t_0] - C$,

(6.12)

$$u(x,t) = \iint_C \Phi(x-y, t-s)[(\zeta_s(y,s) - \Delta\zeta(y,s))u(y,s) - 2D\zeta(y,s) \cdot Du(y,s)]dyds$$

The expression in the square brackets vanishes in the region near the singularity of Φ. By integration by parts,

(6.13)

$$u(x,t) = \iint_C [\Phi(x-y, t-s)(\zeta_s(y,s) + \Delta\zeta(y,s)) + 2D_y\Phi(x-y, t-s) \cdot D\zeta(y,s)]u(y,s)dyds$$

We can derive the same formula without assuming $u \in C^\infty(U_T)$ by replacing u with $u^\epsilon = \eta_\epsilon * u$. η_ϵ is the standard mollifier in the variables x and t and let $\epsilon \to 0$. Thus (6.13) is still true if u only satisfies the hypotheses of the theorem.
Since $\zeta = 1$ on C', u has the form

$$u(x,t) = \iint_C K(x,t,y,s)u(y,s)dyds \quad ((x,t) \in C'') \tag{6.14}$$

where

$$K(x,t,y,s) = 0 \quad \text{for all points } (y,s) \in C'' \tag{6.15}$$

K is smooth on $C - C'$. From the expression of u, we see that u is infinitely differentiable within $C'' = C(x_0, y; \frac{1}{2}r) \subset U_T$. □

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