Characteristic Classes, Chern Classes and Applications to Intersection Theory

HUANG, Yifeng

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1 Introduction

Characteristic classes provide a type of invariant for vector bundles over smooth manifolds (or more generally, Hausdorff paracompact spaces). A characteristic class associates to each isomorphism class of vector bundles over a fixed manifold M an element in $H^*(M;G)$, the cohomology ring of M with coefficients in G, in a functorial and natural way. In this article, we assume the reader knows the basics of cohomology, cup product and Poincaré duality. After some preliminaries, we will study the cohomology ring of Grassmannian manifolds, as it will later be shown that the ring of characteristic classes is isomorphic to the cohomology ring of a particular Grassmannian manifold. In the case of complex vector bundles, this cohomology ring has a rather simple structure, namely a polynomial ring in several variables. The last section will be devoted to the study of these generators of the polynomial ring, called the Chern classes.

The properties of cohomology rings can be translated to facts in intersection theory via Poincaré duality. This article will present two such applications, enumerative geometry and Bézout's theorem, both dealing with counting the number of common intersection points of subvarieties.

2 Cohomology

2.1 Preliminaries

For basic definitions related to chain complexes, we refer the reader to [2, p. 106].

Definition 2.1. (Singular homology) Given a topological space X, we define the singular chain complex $C(X)$ as [2, p. 108]. The *singular homology groups* of X are the homology groups of the chain complex $C(X)$.

Remark. Elements in homology groups are represented by *cycles*, i.e. chains whose boundaries vanish. The representatives of the zero element in homology groups are *boundaries*, i.e. the boundaries of some chains.

One can define the singular cohomology by taking the dual of the singular chain complex.

Definition 2.2. (Cohomologies of chain complexes) Given a chain complex C and an abelian group G, the *dual chain complex of C with coefficients in G* is the chain complex C^* with $C^*_n := Hom_{\mathbb{Z}}(C_n, G)$ and the coboundary map $\delta : C^n \to C^{n+1}$ defined as the pullback by boundary map ∂ on C. The *n-th cohomology group of C with coefficients in G* is the homology group of C^* at C^n, denoted $H^n(C;G)$.

Remark. Elements in cohomology groups are represented by *cocycles*, i.e. $\phi : C_n \to G$ s.t. $\delta\phi = 0$, i.e. $\phi\partial = 0$, or ϕ vanishes on boundaries. The zero element in a cohomology can be represented by any *coboundary*, i.e. $\psi = \phi\partial$ for some $\phi : C_{n-1} \to G$. A coboundary always vanishes on cycles, but the converse is not true in general.

Definition 2.3. Given a topological space X and an abelian group G, the *singular cohomology* $H^n(X; G)$ is the cohomology of the singular chain complex of X with coefficients in G.

Remark. One can also define the homology of a space X with coefficients in G by tensoring the chain complex with G over $\mathbb{Z}$.

One important fact is that the homology and cohomology groups with any coefficients are determined by the homology groups with integral coefficients. Moreover, if a chain map induces an isomorphism on each integral homology group, then it induces an isomorphism on every (co)homology group with coefficients in any group G. These are consequences of the Universal Coefficient Theorem, whose cohomology version is stated in [2, p. 195-196], and homology version in [2, p. 264].

2.2 Cellular Cohomology

To calculate the singular (co)homology for a CW-complex, we introduce the concept of cellular (co)homology, which is isomorphic to the singular (co)homology. We will define the cellular chain complex after some preparation, and the (co)homology groups of this chain complex will be isomorphic to the singular (co)homology groups.

Definition 2.4. Given a (continuous) map $f : S^n \to S^n, n > 0$, as $H_n(S^n) \cong \mathbb{Z}$, the induced map $f_* : H_n(S^n) \to H_n(S^n)$ is multiplication by a number d. Define the *degree* of f by $\deg f = d$.

Proposition 2.5. *Basic properties of degree of maps:*

1. *Functoriality:* $\deg(\mathrm{id}) = 1, \deg(fg) = \deg(f)\deg(g)$
2. *Homotopy invariance: If* $f \simeq g$ *then* $\deg f = \deg g$
3. *If* f *is not onto, then* f_* *can be factored as* $H_n(S^n) \to H_n(S^n - \mathrm{pt}) \cong 0 \hookrightarrow H_n(S^n)$, *hence* $\deg f = 0$.
4. *The degree of reflection map about a hyperplane (i.e. negating one coordinate) is* -1. *Hence the degree of the antipodal map is* $(-1)^{n+1}$.

In complex variables, a nonconstant holomorphic map $f : \mathbb{C}P^1 \to \mathbb{C}P^1$ has the following property: for each $w \in \mathbb{C}P^1$, the map $f(z)$ takes value w exactly k times (counting multiplicity) for some integer k independent of w. Viewing $\mathbb{C}P^1$ as S^2, the integer k is the degree of f. The following generalizes the concept of multiplicity.

Definition 2.6. (Local degree) Let $f : S^n \to S^n$ be a continuous map. For $y \in S^n$, if $f^{-1}(y)$ is discrete, we call y a *regular point*. For $x \in S^n$ s.t. $y = f(x)$ is regular, we define the *local degree* of f at x as the following:

Choose an open neighborhood U of x such that $U \cap f^{-1}(y) = \{x\}$. We get a pair map $f : (U, U - x) \to (S^n, S^n - y)$. By the excision theorem, the inclusion map $H_n(U, U - x) \to H_n(S^n, S^n - x)$ is an isomorphism. Since $S^n - x$ is contractible, the map $H_n(S^n) \to H_n(S^n, S^n - x)$ arising from the quotient chain map is an isomorphism. We get the following diagram:

$$\begin{tikzcd}
H_n(U,U-x) \arrow[r, "f_*"] \arrow[d, "\cong"'] & H_n(S^n,S^n-y) \\
H_n(S^n,S^n-x) & \\
H_n(S^n) \arrow[u, "\cong"] \arrow[r, dashed, "d"] & H_n(S^n) \arrow[uu, "\cong"']
\end{tikzcd}$$

where d is the composition of all maps, with some isomorphisms reversed if necessary. The number d is the local degree of f at x, denoted $\deg f|_x$. The local degree is well-defined, as the maps do not change if we shrink U.

Example 2.7. By identifying $S^2 \cong \mathbb{C}P^1$, the local degree of a holomorphic map is exactly its multiplicity at that point. It suffices to show $\deg z^k|_{z=0} = k$.

Theorem 2.8. *Let $y \in S^n$ be regular under $f : S^n \to S^n$. Let $f^{-1}(y) = \{x_1, ..., x_m\}$, then we have $\deg f = \sum_i \deg f|_{x_i}$. In other words, f takes the value y $\deg f$ times.*

Proof. See [2, p. 136]. ☐

Definition 2.9. (Cellular chain complexes) Given a CW-complex X, the cellular chain complex $C^{CW}(X)$ is defined as

$$C_n^{CW}(X) = \langle e_\alpha^n : D^n \to X \mid e_\alpha^n \text{ is an n-cell of } X \rangle$$
$$\partial(e_\alpha^n) = \sum_\beta d_{\alpha\beta} e_\beta^{n-1},$$

where $d_{\alpha\beta}$ is defined by the following:

If $n = 1$, then

$$d_{\alpha\beta} = \begin{cases} 1, & \text{if } e_\beta^0 = e_\alpha^1(1), \text{ i.e. the end point of } e_\alpha \\ -1, & \text{if } e_\beta = e_\alpha(-1), \text{ i.e. the starting point of } e_\alpha \\ 0, & \text{if } e_\beta \text{ is not a vertex of } e_\alpha. \end{cases}$$

In other words, $\partial(e_\alpha^1) = e_\alpha(1) - e_\alpha(-1)$.

If $n > 1$, then $d_{\alpha\beta}$ is the degree of the attaching map formally defined below: Let B^{n-1} be the interior of D^{n-1}. We have

$$\begin{tikzcd}
 & D^n \arrow[r, "e_\alpha^n"] & X \arrow[r] & X/(X - e_\beta^{n-1}(B^{n-1})) & \\
S^{n-1} \arrow[r, "\cong"] & \partial D^n \arrow[u, hook] & D^{n-1} \arrow[u, "e_\beta^{n-1}"'] \arrow[r] & D^{n-1}/\partial D^{n-1} \arrow[u, dashed, "\cong"'] \arrow[r, "\cong"] & S^{n-1}
\end{tikzcd}$$

The dashed map that factors e_β^{n-1} is a homeomorphism as can be shown using the definitions for CW-complexes. The canonical identification $D^{n-1}/\partial D^{n-1}$ glues ∂D^{n-1} to the south pole $(\vec{0}, -1) \in S^{n-1}$. The composition map from the leftmost S^{n-1} to the rightmost S^{n-1} is the attaching map, whose degree is $d_{\alpha\beta}$.

Remark. When e^n_α does not touch e^{n-1}_β, $d_{\alpha\beta}$ is automatically zero since the map above is a constant map, which has degree 0. Hence the sum can be taken over finitely many β's.

Theorem 2.10. *Definition* 2.9 *makes $C^{CW}(X)$ a chain complex, and its homology $H^{CW}(X)$ is isomorphic to the singular homology of X.*

Proof. See [2, p. 139-140]. □

Corollary 2.11. *The cohomology of the cellular chain complex is isomorphic to singular cohomology, no matter which coefficient group we choose.*

Example 2.12. Let $X = \mathbb{C}P^n$. Then X is a CW-complex consisting of cells $\{e^0, e^2, ..., e^{2n}\}$, where e^{2i} is defined by $e^{2i}(\vec{z}) = [\vec{z} : \sqrt{1 - |\vec{z}|^2} : \vec{0}_{n-i}]$ for $\vec{z} \in D^{2i} \subset \mathbb{C}^i$. Hence $C^{CW}_k(X) = 0$ for odd k, thus boundary maps are all zero. Hence $H_k(X) \cong H^k(X) \cong C^{CW}_k(X)$, which is isomorphic to $\mathbb{Z}$ for even $k \leq 2n$ and is trivial for other k.

Example 2.13. Let $X = \mathbb{R}P^n$. Define $e^k : D^k \to X$ by $e^k(\vec{x}) = [\vec{x} : \sqrt{1 - |\vec{x}|^2} : \vec{0}_{n-k}]$ for $k \leq n$. Note that $\mathrm{im}(e^k) = \mathrm{im}(e^{k+1}|_{\partial D^{k+1}}) = \mathbb{R}P^k$. Hence $\mathbb{R}P^k$ is the k-skeleton of X and $e^{k+1} : \partial D^{k+1} \to \mathbb{R}P^k$ is the attaching map, which gives X a CW-complex structure.

Hence the cellular chain complex is $0 \to \mathbb{Z} \xrightarrow{d_n} \mathbb{Z} \xrightarrow{d_{n-1}} ... \xrightarrow{d_1} \mathbb{Z} \to 0$.

First of all, $d_1(e^1) = e^0 - e^0 = 0$. Thus $d_1 = 0$.

For $1 < k \leq n$, the attaching diagram is

$$S^{k-1} \hookrightarrow D^k \xrightarrow{e^k} X \xleftarrow{e^{k-1}} D^{k-1} \to D^{k-1}/\partial D^{k-1} \cong S^{k-1}$$

Pick $(\vec{y}, z) \in S^{k-1}$, let $f : S^{k-1} \to S^{k-1}$ be the attaching map, and denote $x = f(\vec{y}, z)$.

If $z \neq 0$, then $(y, z) \mapsto [y, z, 0]$. Since $|y| < 1$, there is $x \in B^{k-1}$ such that $e^{k-1}(x) = [y, z, 0]$. Hence $x \in B^{k-1}$ and $x = y$ if $z > 0$, $x = -y$ if $z < 0$.

If $z = 0$, then $(y, z) \mapsto [y, 0, 0] \in im(e^{k-1}|_{\partial D^{k-1}})$. It follows that $f(y, z)$ is the south pole.

Now to compute the degree of f, let $N = (\vec{0}, 1)$ be the north pole. Then $f^{-1}(N) = \pm N$. Near N, f is homotopic to identity map, so $\deg f|_N = 1$. Near $-N$, f is homotopic to antipodal map, so $\deg f|_{-N} = (-1)^{k-1+1} = (-1)^k$. Hence

$$d_k = \deg f = 1 + (-1)^k = \begin{cases} 2, k \text{ is even} \\ 0, k \text{ is odd.} \end{cases}$$

Thus the chain complex is

$$... \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0.$$

As a result, if $k < n$, then

$$H_k(\mathbb{R}P^n) = \begin{cases} \mathbb{Z}_2, & k \text{ is odd} \\ 0, & k \text{ is even.} \end{cases}$$

On the other hand, if $k = n$,

$$H_n(\mathbb{R}P^n) = \begin{cases} \mathbb{Z}, & n \text{ is odd} \\ 0, & n \text{ is even.} \end{cases}$$

The cohomology can be computed similarly using this chain complex, by reversing the arrows.

$$\begin{bmatrix} 0 & 1 & * & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

Figure 1: a typical RREF with $k = 4, n = 5, J = \{2, 4, 5, 8\}$.

2.3 Cohomology of Complex Grassmannians

Fix a field $K = \mathbb{R}$ or $\mathbb{C}$ and integers $k > 0, n \geq 0$. Define the Grassmannian $Gr_k(K^{n+k})$ to be the set of k-dimensional subspaces of K^{n+k}. We shall give a cell structure to $Gr_k(K^{n+k})$ as follows.

Represent a k-subspace of vector space K^{n+k} by the row space (span of rows) of a full rank $k \times (n + k)$ matrix. As the row spaces are in one to one correspondence to reduced row echelon forms (RREFs), and full rank matrices have k pivots in RREF, the Grassmannian $\mathrm{Gr}_k(K^{n+k})$ is in bijection with the set of RREFs with a pivot in every row.

Classify RREFs according to the positions of pivots, i.e. the sequence $J = (1 \leq j_1 < j_2 < ... < j_k \leq n + k)$, such that the i-th pivot is at the j_i-th column. For each J, the set of RREFs with pivots on J is diffeomorphic (even biholomorphic if $K = \mathbb{C}$) to a K-vector space by extracting the free entries (i.e. stars in Figure 1). Denoting the number of free entries as $|J|$, we have embeddings $X_J : K^{|J|} \to \mathrm{Gr}_k(K^{n+k})$.

It turns out that X_J can be extended to a continuous map $D^m \to \mathrm{Gr}_k(K^{n+k})$ for $m = |J|$ or $2|J|$ according to whether $K = \mathbb{R}$ or $\mathbb{C}$ (here $K^{|J|}$ is viewed as the interior of D^m via a homeomorphism). They are called *Schubert cells*, as such an extension makes $\{X_J\}$ a cell structure for $\mathrm{Gr}_k(K^{n+k})$. Hence Grassmannian manifolds are CW-complexes.

Now we count the cells of dimension the same as K^d. This amounts to finding the number of J's such that $|J| = d$. It is clear from Figure 1 that this number equals the number of solutions of

$$0 \leq y_1 \leq ... \leq y_k \leq n; y_1 + ... + y_k = d$$

In particular if $K = \mathbb{C}$, the $2d$-th (co)homology group of $\mathrm{Gr}_k(\mathbb{C}^{n+k})$ is a free abelian group with the number above as its rank. The next section will focus on the cup product, or the ring structure, of its cohomology.

3 The Cohomology Ring of Complex Grassmannians

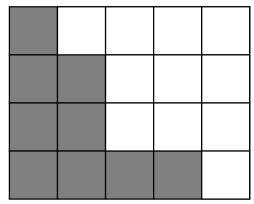
3.1 Schubert Calculus

Given a Schubert cell, say

$$\begin{bmatrix} 0 & 1 & * & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix},$$

recall J is a k-tuple such that J_k is the position of the pivot at the k-th row.

Define $\lambda_k = J_k - k$, i.e. the cumulative number of columns that are skipped in the first k rows. Draw a Young diagram such that k-th row has λ_k squares. In this example, we get



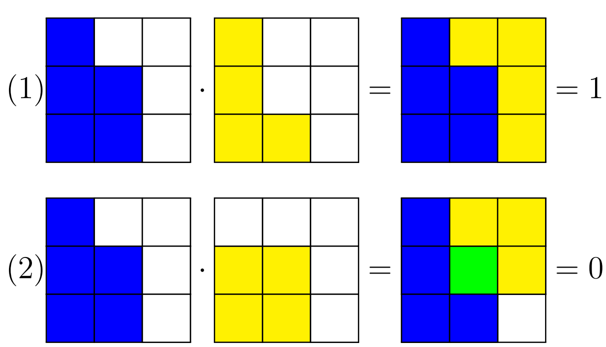
For the definition of a Young diagram, see [1, p. 1]. However, we adopt a different convention: the number of boxes is weakly increasing (instead of decreasing) from above to below.

Hence in a given flag F, the set of Schubert cells of $M = Gr_k(\mathbb{C}^{n+k})$ corresponds bijectively to the set of Young diagrams λ contained in a $k \times n$ rectangle. Denote the closed Schubert cell associated to λ with respect to the flag F as $\Omega_\lambda(F)$. Moreover, the number of squares in λ (denoted $|\lambda|$) is exactly the codimension of the Schubert cell. Note that $\Omega_\lambda \subset \Omega_{\lambda'}$ if and only if $\lambda_k \geq \lambda'_k$ for all k, thus the inclusion of Schubert cells corresponds to reversed inclusion of Young diagrams. We say the (co)homology class represented by the Schubert cell λ is the *Schubert cycle* σ_λ of λ. This is independent of the flag chosen for homotopy reason. The $2l$-th cohomology group $H^{2l}(M)$ is exactly the free abelian group generated by σ_λ for all $|\lambda| = 2l$.

To compute the cup product, we apply the Poincaré duality so that the problem is reduced to the computation of the intersection of Schubert cycles. First we consider the case involving two Schubert cells of complementary dimensions.

Lemma 3.1. *(Duality Lemma) Suppose we are working in $Gr_k(\mathbb{C}^{n+k})$ and given Young diagrams λ, λ' such that $|\lambda| + |\lambda'| = nk$. Then $\sigma_\lambda \cdot \sigma_{\lambda'}$ is 1 (more precisely, the Schubert cycle of the full rectangle as a Young diagram) if λ' is the complement of λ, i.e. λ fits with the 180 degree rotation of λ' (to be made precise in the next examples). Otherwise the product is 0.*

Example 3.2. The notion of fitting is illustrated in the following examples:



Proof. We will only illustrate the idea using the examples above. Note that the flags taken for the first and the second terms are complimentary to each other. In example (1), we have

$$\begin{bmatrix} * & 1 & & & & \\ * & 0 & 1 & & & \\ * & 0 & 0 & * & 1 & 0 \end{bmatrix} \cap \begin{bmatrix} 0 & 1 & 0 & * & 0 & * \\ & & 1 & * & 0 & * \\ & & & & 1 & * \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} = 1$$

since if two such matrices are related by left multiplication by an invertible 3×3 matrix, the matrix must be the identity.

For example (2), we have

$$\begin{bmatrix} * & 1 & & & & \\ * & 0 & 1 & & & \\ * & 0 & 0 & * & 1 & 0 \end{bmatrix} \cap \begin{bmatrix} 1 & * & * & 0 & 0 & * \\ & & & 1 & 0 & * \\ & & & & 1 & * \end{bmatrix}$$
$$= \varnothing = 0$$

The reason why the intersection is empty is that if there were a matrix X such that $XA = B$, where A is the bottom left matrix and B is the bottom right matrix, then looking at the second and third columns, we see X maps $(1, 0, 0)$ (shorthand for the column vector $[1\ 0\ 0]^T$) and $(0, 1, 0)$ to multiples of $(1, 0, 0)$, hence X is not invertible. $\square$

Remark. The same method can also be used to prove that if two Young diagrams overlap, i.e. one is not contained in the complement of the other, then their product is 0.

As a corollary if $l + l' = nk$, the cup product is a perfect pairing between $H^{2l}(M)$ and $H^{2l'}(M)$, where the bases of Young diagrams are dual to each other by matching a diagram with its complement. Using this, one can identify an element of $H^{2l}(M)$ by its product with Schubert cycles of codimension l'.

Example 3.3. We determine the ring structure of $H^*(Gr_2\mathbb{C}^4)$. Note that here $k = 2, n = 2$.

The cohomology ring has its additive group free with generators σ_λ where λ ranges over $(0), (1), (1, 1), (2), (2, 1), (2, 2)$. Here and hereafter we list λ from λ_k to λ_1 and omit superfluous zeros.

Clearly (0) is the multiplicative identity in this ring because the closed cell represented by (0) is the whole manifold. Most products except a few vanish for dimension reasons. By the duality lemma, we have $(1)(2, 1) = (1, 1)^2 = (2)^2 = (2, 2)$ and $(1, 1)(2) = 0$. The only three left are $(1)(1, 1), (1)(2), (1)^2$.

We first compute $(1)^2(1, 1)$ and $(1)^2(2)$. Given three flags E, E' and F in general position, we have $\Omega_{(1)}(E) = \{W : \dim W \cap E_2 \geq 1\}$, $\Omega_{(1,1)}(F) = \{W : W \subset F_3\}$, and $\Omega_{(2)}(F) = \{W : W \supset F_1\}$. As E_2, E'_2 and F_3 are in general position, we have $\dim E_2 \cap E'_2 = 0$ and $\dim E_2 \cap F_3 = 1$. Hence $\Omega_{(1)}(E) \cap \Omega_{(1)}(E') \cap \Omega_{(1,1)}(F) = \{(E_2 \cap F_3) \oplus (E'_2 \cap F_3)\} = 1$ and $\Omega_{(1)}(E) \cap \Omega_{(1)}(E') \cap \Omega_{(2)}(F) = \{\mathrm{proj}_{E_2} F_1 \oplus \mathrm{proj}_{E'_2} F_1\} = 1$.

By duality, we get $(1)^2 = (1, 1) + (2)$ and $(1)(1, 1) = (1)(2) = (2, 1)$.

Generalizing this idea, we arrive at the following formula. For a proof see [1, p. 150].

Proposition 3.4. *(Pieri's formula) Given $0 \leq l \leq n$, recall that (l) denotes the Young diagram with a row of l boxes. For a Young diagram μ, we have $\sigma_{(l)} \cdot \sigma_\mu = \sum \sigma_\nu$, where ν ranges over all Young diagrams contained in a $k \times n$ rectangle obtained from adding l boxes to μ, but no two of the l boxes are in the same column.*

Similarly if $l \leq k$, let $(1^l) := (1, ..., 1)(l$ 1's), the diagram with a column of l boxes, then $\sigma_{(1^l)} \cdot \sigma_\mu = \sum \sigma_\nu$, where ν ranges over all Young diagrams obtained from adding l boxes to μ, but no two of the l boxes are in the same row.

In fact these are all we need to perform computation for arbitrary products of Schubert cycles. The trick is to decompose a Young diagram into a polynomial of single-row/column diagrams.

Example 3.5. Let $k = 2, n = 4$, we write $\ydiagram{1,2}$ for $\ydiagram[*(gray)]{1,2}*[*(white)]{4,4}$ in short. Then Pieri's formula gives $\ydiagram{1} \cdot \ydiagram{1,1} = \ydiagram{1,2}$. Thus

$$\begin{aligned} \ydiagram{1,2}^{\,2} &= \ydiagram{1} \cdot \ydiagram{1,1} \cdot \ydiagram{1,2} \\ &= \ydiagram{1} \cdot \ydiagram{2,3} \\ &= \ydiagram{3,3} + \ydiagram{2,4} \end{aligned}$$

For more details, see [3].

In the infinite Grassmannian $Gr_k(\mathbb{C}^\infty)$, although we do not have Poincaré duality, we can still represent its cohomology classes by Young diagrams contained in the $k \times \infty$ rectangle. Then Pieri's formula still works, except we do not need to worry about the length limit of Young diagrams.

3.2 Enumerative Geometry

As an interesting application of Schubert calculus, one can take several Schubert cells such that their product has full codimension, and reinterpret the product equality in terms of an equality about intersection. Note that in the complex world, everything holomorphic has positive orientation, so we do not need to worry about the signs of intersection.

Example 3.6. Let $k = 2, n = 3$. Then given any flag F, the Schubert cell of $\ydiagram{1}$ for F is the set of 2-planes in $\mathbb{C}^5$ that intersect F_3 nontrivially. Consider the problem: if we are given six 3-planes in $\mathbb{C}^5$ in general position, how many 2-planes intersect all of them nontrivially?

The set of such 2-planes is exactly the intersection of six Schubert cells $\Omega_{\ydiagram{1}}$ in general position. So it reduces to the problem calculating $\ydiagram{1}^{\,6}$ in terms of a multiple of $\ydiagram{3,3}$. We have

$$\begin{aligned}\ydiagram{1}^6 &= \ydiagram{1}^4 \cdot (\ydiagram{1,1} + \ydiagram{2}) = \ydiagram{1}^3 \cdot (2 \cdot \ydiagram{1,2} + \ydiagram{3}) \\ &= \ydiagram{1}^2 \cdot (2 \cdot \ydiagram{2,2} + \ydiagram{1,3}) + \ydiagram{1,3}) = \ydiagram{1}^2 \cdot (2 \cdot \ydiagram{2,2} + 3 \cdot \ydiagram{1,3}) \\ &= \ydiagram{1} \cdot (2 \cdot \ydiagram{2,3} + 3 \cdot \ydiagram{2,3}) = 5 \cdot \ydiagram{3,3}\end{aligned}$$

Hence there are five 2-planes that intersect all of the six 3-planes.

4 Characteristic Classes

Characteristic classes are an important type of invariant for (real or complex) vector bundles. Notable examples like Chern classes, the Euler class and Stiefel-Whitney classes play an important role in various subjects including algebraic geometry and differential geometry. Here we present the definition and show that characteristic classes can be classified by the cohomology ring of an infinite Grassmannian.

4.1 The Classification Theorem for Characteristic Classes

Definition 4.1. (Pullback of a vector bundle) Given a (real or complex) vector bundle $E \xrightarrow{\pi} X$ over a topological space X, and a continuous map $f : Y \to X$, we define $f^*E := E \times_X Y := \{(e, y) : \pi(e) = f(y)\}$, the fiber product of E and Y over X, and the projection from f^*E to Y is the standard projection to the second coordinate. Let the induced map $\tilde{f} : f^*E \to E$ be the projection to the first coordinate. Categorically, what we have defined is exactly the pullback of the diagram $E, Y \to X$.

Proposition 4.2. *$f^*E \to Y$ has the structure of a vector bundle over Y of the same rank as E.*

Proof. Suppose $\{U_\alpha\}$ is a local trivialization with transition functions $g_{\alpha\beta}$. Then $\{f^{-1}(U_\alpha)\}$ is a local trivialization for f^*E with transition functions given by $g_{\alpha\beta} \circ f$. $\square$

Now suppose we have a smooth K-vector bundle E of rank k over a smooth manifold M^n. Our goal is to show that E can always be realized as the pullback of a canonical vector bundle over a Grassmannian.

Definition 4.3. (Tautological bundle over $Gr_k^n := Gr_k(K^n), K = \mathbb{R}$ or $\mathbb{C}$) Let

$$\gamma_k^n = \{(v, W) : W \in Gr_k^n, v \in W\}$$

be a submanifold of $Gr_k^n \times K^n$. Then projection onto the first coordinate makes γ_k^n a rank k vector bundle over Gr_k^n. Call γ_k^n the tautological bundle over Gr_k^n.

Proposition 4.4. *(Classifying map) Given a smooth vector bundle E of rank k over a smooth manifold M^n, there is a smooth map $f : M \to Gr_k^N$ for sufficient large N such that $f^*\gamma_k^N \cong E$. Such maps are called classfying maps.*

Proof. First we claim that there is a map $F : E \to K^N$ that restricts to a linear injection on every fiber $E|_p$. The idea is to first prove the compact case using partition of unity to patch up the local trivializations, and then use the idea of exhausting functions to prove the noncompact case. See [5, p. 6]. However, if $K = \mathbb{R}$ we have a direct proof using the Whitney Immersion Theorem. Let $i : E \to \mathbb{R}^N$ be an immersion for some N. We define linear injections $F|_p : E|p \to \mathbb{R}^N$ at each point $p \in M$ by the composition

$$E|_p \hookrightarrow T_{0_p}E \overset{i_*}{\hookrightarrow} T_{i(0_p)}\mathbb{R}^N \cong \mathbb{R}^N, \text{ where } 0_p \text{ is the zero section at } p.$$

It is easy to check that $F|_p$ glues to a smooth map $F : E \to \mathbb{R}^N$.

Next, define $f : M \to Gr_k^N$ by

$$f(p) = \mathrm{im}(F|_p),$$

and $\tilde{f} : E \to \gamma_k^N$ by

$$\tilde{f}(e, p) = (F(e), F(E|_p)).$$

One can check E together with $\tilde{f}$ and π satisfies the universal property for $f^*\gamma_k^N$. Hence E is a pullback of tautological bundle by f. □

For convenience, we regard f as a map into the infinite Grassmannian Gr_k^∞. As a general fact, the isomorphism class of the pullback bundle is invariant under homotopy. In other words, if E is a vector bundle over a topological space X, and $f \simeq g : Y \to X$ are homotopic continuous maps, then $f^*E \cong g^*E$. In fact more is true for the above-defined classifying map.

Theorem 4.5. *(Classification Theorem for vector bundles, [5, p. 17]) Given a smooth manifold M and integer $k \geq 0$, the isomorphism classes of rank-k vector bundles over M are in bijection with the homotopy classes of maps $M \to Gr_k^\infty$.*

This classifies all characteristic classes, a concept to be defined as the following.

Definition 4.6. (Characteristic classes) A characteristic class c with coefficient group G (where G is an abelian group) is a family of 'natural' maps c_X for each space X from the set of isomorphism classes of vector bundles over X to the cohomology ring $H^*(X; G)$, in the sense that if $f : X \to Y$ is a continuous map and E is a bundle over Y, then $c_X(f^*(E)) = f^*(c_Y(E))$. Similar notions apply to the smooth case.

Note that one can take linear combination and cup product of characteristic classes in a natural way to make the set of characteristic classes a graded commutative ring, because pullback map of cohomology preserves linear combination and cup product.

Example 4.7. Fix an element α of $H^*(Gr_k^\infty; G)$. Define the characteristic class c by

$$c(E \to X) = f^*(\alpha),$$

where f is a classifying map of the bundle $E \to X$. It is well-defined because the homotopy class of f is determined by the bundle. Moreover c is natural. If $g : X \to Y$ is a smooth map and E is a bundle over Y, then for a classifying map f of E over Y, we have $f^*(\alpha) \cong E$, so $(f \circ g)^*(\alpha) = g^* f^*(\alpha) \cong g^* E$, i.e. $f \circ g$ is a classifying map of $g^* E$. Hence $c_X(g^*(E)) = (f \circ g)^*(\alpha) = g^*(f^*(\alpha)) = g^*(c_Y(E))$.

Moreover, all characteristic classes arise from this construction. Take a characteristic class c, and let $\alpha = c(\gamma_k^\infty)$. By the naturality condition, for a vector bundle E of rank k over X and a classifying map $f : X \to Gr_k^\infty$, we have $c(E) = c(f^*\gamma_k^\infty) = f^* c(\gamma_k^\infty) = f^*(\alpha)$. This construction establishes an isomorphism between the graded commutative ring of characteristic classes and the graded commutative ring $H^*(Gr_k^\infty; G)$. To summarize,

Theorem 4.8. *(Classification Theorem for characteristic classes) The ring of characteristic classes is isomorphic to the cohomology ring of an infinite Grassmannian, with the isomorphism given by Example* 4.7*.*

4.2 Chern Classes

In Example 3.5, we saw an example of decomposing a Schubert cycle into a polynomial of Schubert cycles of the form $\sigma_{(1^l)}$. One might expect that such a decomposition always exists and is unique. This can be proven true by studying the characteristic classes associated to such Schubert cycles, as follows.

Definition 4.9. (Chern classes) The n-th Chern class c_n for k-dimensional complex vector bundles is defined such that $c_n(\gamma_k^\infty) = \sigma_{(1^n)}$, by means of Example 4.7. By convention, $c_0 = 1$ and $c_n = 0$ for $n > k$.

To state the properties more conveniently, we define the total Chern classes

$$c = c_0 + c_1 + ... = 1 + c_1 + c_2 + ... + c_k.$$

Proposition 4.10. *([4, p. 42]) Chern classes satisfy the following properties:*

1. For complex line bundles L_1 and L_2, we have $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$.

2. (Whitney sum formula) For complex vector bundles E and F, we have $c(E \oplus F) = c(E)c(F)$.

3. Given a complex vector bundle $E \to M$ of rank k, the top Chern class $c_k(E)$ is the Poincaré dual (in M) of the zero set of any smooth section that intersects the base manifold transversally (which means, the smooth section viewed as a submanifold of E intersects transversally with the zero section viewed as a submanifold of E diffeomorphic to M). In particular, if the bundle admits a nonvanishing section, then the top Chern class is zero.

The above properties immediately yield the following application.

Example 4.11. (Bézout's Theorem) Suppose we have n complex homogeneous polynomials $p_1, ..., p_n$ in n variables of degrees $d_1, ..., d_n$ respectively. Each of them determine a projective variety V_i in $\mathbb{C}P^n$. In the 'general' case, the number of intersection points of all of the V_i is $d_1 d_2 ... d_n$. (General case means assuming all regularity and transversality. By Sard's theorem, this happens with probability 1.) In other words, the polynomials p_i have $d_1 ... d_n$ common zeroes in general.

Proof. Consider the section s_i of the complex line bundle $\mathcal{O}(d_i) := \mathcal{O}(1)^{\otimes d_i}$ over $\mathbb{C}P^n$ defined as follows. Let $[z_0 : ... : z_n]$ be the homogeneous coordinate representation of a point in $\mathbb{C}P^n$, and U_j be the chart consisting of points where $z_j \neq 0$. Define the value of s_i in the local trivialization on U_j to be

$$s_i^j = \frac{p_i(z_0, ..., z_n)}{z_j^{d_i}}.$$

It is easy to verify the transition function is exactly that of $\mathcal{O}(d_i)$.

Note that the zero set of s_i is exactly V_i. Hence V_i is Poincaré dual to the first Chern class $c_1(\mathcal{O}(d_i)) = d_i c_1(\mathcal{O}(1))$ by property 1 in Proposition 4.10. We shall let x denote $c_1(\mathcal{O}(1))$ for short.

Observe that $V := \bigcap_{i=1}^n V_i$ is the zero set of $(s_1, ..., s_n)$ in $\bigoplus_{i=1}^n \mathcal{O}(d_i)$. Hence V is Poincaré dual to $c_n(\bigoplus \mathcal{O}(d_i)) = \prod c_1(\mathcal{O}(d_i)) = d_1...d_n x^n$.

Now it remains to calculate the Poincaré dual of a point in $\mathbb{C}P^n$. Consider the special case $f_i = z_i$, $i = 1, ..., k$. Note that $i = 0$ is not in the list. Then

$$V = \{[z_0 : ... : z_n] : z_1 = z_2 = ... = z_n = 0\} = \{[1 : 0 : ... : 0]\}$$

is dual to $d_1...d_n x^n = x^n$. Thus the dual of a point is x^n, which concludes that in the general case V consists of $d_1...d_n$ points. □

The Chern classes of higher dimensional bundles can always be decomposed into the Chern classes of line bundles, by the following principle:

Proposition 4.12. *(Splitting principle, [4, p. 43]) Given a vector bundle $E \to M$ of rank k. Then there is a manifold M' and smooth map $f : M' \to M$ such that $f^*(E)$ is a direct sum of line bundles and the pull back map $f^* : H^*(M) \to H^*(M')$ is injective.*

Now we are ready to sketch a proof of the assertion made at the end of last section.

Theorem 4.13. $H^*(Gr_k^\infty) \cong \mathbb{C}[x_1, ..., x_k]$*, the polynomial ring of k variables, where the isomorphism is given by mapping x_i to the Chern class $c_i(\gamma_k^\infty)$.*

Proof. For $k = 1$, $Gr_1^\infty = \mathbb{C}P^\infty$, and one can prove this theorem by directly working out the cohomology ring by Pieri's formula. Here $c_1(\mathbb{C}P^\infty) = \sigma_{(1)}$ and its n-th power is just $\sigma_{(n)}$.

For general k, consider the manifold $M = \mathbb{C}P^\infty \times \cdots \times \mathbb{C}P^\infty$ (k copies). Let p_i be the projection onto the i-th coordinate and consider the bundle $E = p_1^*\gamma_1^\infty \oplus \cdots \oplus p_k^*\gamma_1^\infty$.

Let $h : M \to Gr_k^\infty$ be a classifying map for E. Using the splitting principle, one can prove $h^* : H^*(Gr_k^\infty) \to H^*(M)$ is injective. Hence $H^*(Gr_k^\infty)$ can be viewed as a subring of $H^*(M)$, which by Künneth formula, is the tensor product of k copies of $H^*(\mathbb{C}P^\infty)$. Write the first Chern class of the i-th copy of $\mathbb{C}P^\infty$ as a_k. Then $H^*(M) \cong \mathbb{C}[a_1] \otimes \cdots \otimes \mathbb{C}[a_k] \cong \mathbb{C}[a_1, ..., a_k]$. Therefore we can represent an element in $H^*(Gr_k^\infty)$ by a polynomial in a_i. The polynomial corresponds to Schubert cycle σ_λ is called the *Schur polynomial* S_λ.

From the construction of $E \to M$, it has an obvious action of the symmetric group S_k by permuting coordinates. This gives an S_k action on $H^*(M)$ by taking pullback. Under the isomorphism $H^*(M) \cong \mathbb{C}[a_1, ..., a_k]$, S_k acts on it by permuting a_i's.

Pick $g \in S_k$, then $g^*h^*(\gamma_k^\infty) = g^*(E) = E = h^*(\gamma_k^\infty)$. By the classification theorem (Theorem 4.5), we have $h \circ g \simeq h$. Hence $g^*h^* = h^*$ on cohomology classes, i.e. the image of h^* is invariant under S_k action. Hence the polynomials represented by elements of $H^*(Gr_k^\infty)$ are symmetric.

It can be shown that the Schur polynomial of $\lambda = (1^l)$ is

$$S_{(1^l)} = \sum_{1 \le i_1 < ... < i_l \le k} a_{i_1} ... a_{i_k} = e_l,$$

the l-th elementary polynomial in $a_1, ..., a_k$. By a standard algebra fact, elementary polynomials generate the ring of symmetric polynomials with no algebraic dependence. Hence the image of h^* contains all of the symmetric polynomials and $H^*(Gr_k^\infty) = \mathbb{C}[c_1, ..., c_k]$ as a polynomial ring.

□

Corollary 4.14. *Every characteristic class of complex vector bundles is a unique polynomial of Chern classes.*

As a final remark, the entire discussion can also be applied to the case where we consider real instead of complex vector bundles, if we change the coefficient group from $\mathbb{Z}$ to $\mathbb{Z}/2\mathbb{Z}$. The analogy of the Chern classes are called the Stiefel-Whitney classes (see [4, p. 19]).

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