CHUTES AND LADDERS

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Abstract. This paper discusses and then uses the theory of Markov chains to analyze and develop a theory of the board game *Chutes and Ladders*. Further, a method of using computer programming to simulate Chutes and Ladders is discussed, as are various 'experiments' with the rules and layout of the standard game.

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1. Introduction

The game *Chutes and Ladders*, also referred to as Snakes and Ladders, is a simple game that a child can play. Despite this the mathematics of this game - specifically with regards to Markov chains - are quite interesting. This paper is dedicated to exploring these mathematics.

First, we shall explain the rules of Chutes and Ladders. Then, we shall briefly discuss the theory of **Markov chains** in general, and then focus on **absorbing Markov chains**. Then, we shall build a theory of *Chutes and Ladders*, using, in part, the theory of Markov chains. Then, we will discuss how to use computer programming to model *Chutes and Ladders*. Next, we shall perform a few experiments on the ordinary rules and layout of *Chutes and Ladders* and derive counterintuitive results.

2. The Rules of Chutes and Ladders

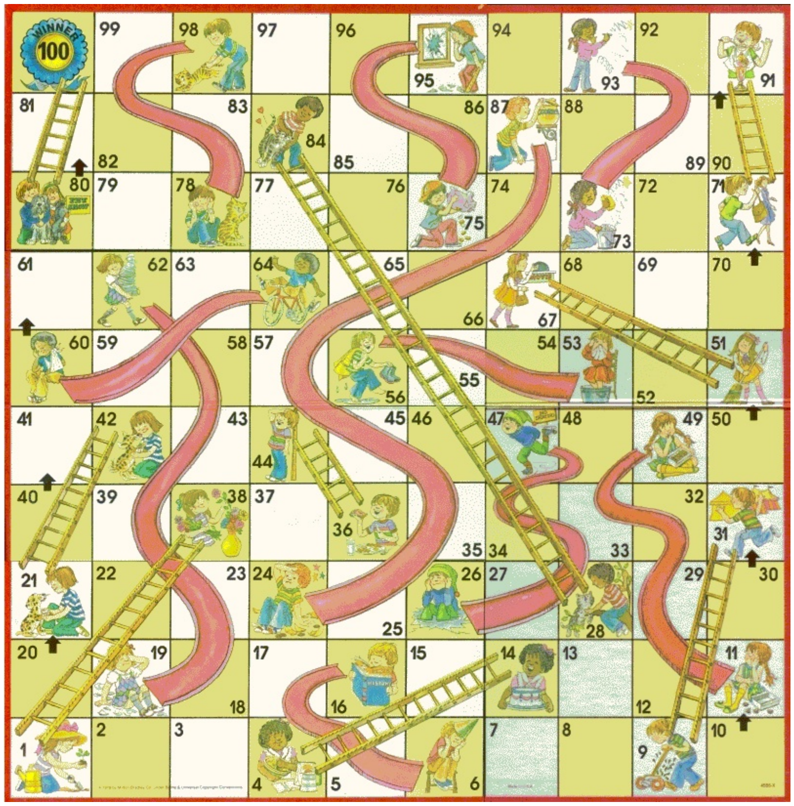

Figure 1. The Standard Board

Chutes and Ladders is played on a 100 square board game. There can be as many players as desired; however, since the actions of the players are independent from the actions of the other players, we shall only consider one player. The player begins off of the board, at a figurative square zero. The player then rolls a six-sided die, and advances the number of spaces shown on the die. For example, if the player is at position 8, and rolls a 5, they would advance to position 13. The game is finished when the player lands on square 100. There are two exceptions to the rule of movement.

The first is that if the player, after advancing, lands on a chutes (slide) or a ladder, they slide down or climb up them, respectively. For example, if the player, on their first turn, rolls a 4, the player advances to square 4, and then "climbs" to square 14, on the same turn. Thus, if the players is on square 77, and rolls a 3, the player is finished, as he advances to 80, and then climb to 100.

The second is that the player must land exactly on square 100 to win. If the player rolls a die that would advance them beyond square 100, they stay at the same place. For example, a player at square 96 must roll exactly a 4 to win. A roll of 5 would make the player stay put at position 96.

Do note that these rules could easily apply to a board of any size, with any size die, and with chutes and ladders in any position.

3. Overview of Markov Chains

Imagine a frog, sitting on a lilypad. Surrounding the frog are other lilypads. At given intervals, the frog either moves from the lilypad to another, or simply stays put, with fixed probabilities. Furthermore, the frog has no memory: the frog's set of probabilities from moving from a given lilypad to another stay the same, regardless of the frog's previous movements. That is, if the frog has a probability p of moving from lilypad i to j at one instance, the probability of moving from lilypad i to lilypad j is the same regardless of the frog's previous jumps. The frog's behavior can be modeled by a **Markov Chain**.

Formally, a Markov Chain can be described as a set of states, $s_1, s_2, \dots, s_n$, and a set of probabilities p_{ij} of transitioning from one state to another. In terms of the frog analogy, the states are the lilypads. p_{ij} are the probabilities of transitioning to state j given you are at state i. For example, p_{24} is the probability of moving to state 4 given you are at state 2. These probabilities form an $n \times n$ matrix called a **transition matrix**. Note that for all $1 \leq i, j \leq n$, p_{ij} always exists - although it may be zero.

We will discuss a few examples of Markov chains and transitions matrices.

*I go out for dinner each night. However, I am quite picky. I only eat hamburgers, pizza, cereal, and foie gras. Normally, I pick one of the four at random. There are three exceptions, however: (1) I never eat pizza the night after I eat hamburgers. In this case, I pick from the other three at random. (2) The night after eating foie gras, I have a particular craving for cereal, and a distaste for hamburgers: I thus choose cereal 50 percent of the time, hamburgers 10 percent of the time, and pizza and foie gras each 20 percent of the time. (3) The night after eating cereal, I always eat cereal. The transition matrix can be represented as:

$$\begin{array}{c} \\ H \\ P \\ C \\ F \end{array} \begin{array}{c} \begin{array}{cccc} H & P & C & F \end{array} \\ \begin{pmatrix} \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 1 & 0 \\ \frac{1}{10} & \frac{1}{5} & \frac{1}{2} & \frac{1}{5} \end{pmatrix} \end{array}$$

*I am on a monorail system. There are three stations that form a loop, and each train stops at each station. I flip a coin and decide to either go one stop clockwise or one stop counterclockwise. The transition matrix is:

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \end{array}$$

*Going back to the frog example, imagine that there are 4 lilypads, yet the frog only stays on the lilypad which it is currently on. The transition matrix is:

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Notice that this is the identity matrix.

Furthermore, note that all transition matrices are square. This is because each state always has a transition probability to every other state (including itself), even if this probability is 0.

Chutes and Ladders can be thought of as being modeled by a Markov chain. Each square on the board, 1 through 100, is a state, as is the starting position, the figurative square zero. We will consider the transition matrix of *Chutes and Ladders* in the next section.

There are many different types of Markov chains. This paper shall focus on one specific type, absorbing Markov chains. First, we define an absorption state:

Definition 3.1. An **absorption state** is one in which, when entered, it is impossible to leave. That is, $p_{ii} = 1$. A state that is not an absorption state is called a **transient state**.

Definition 3.2. An **absorbing Markov chain** is a Markov chain with absorption states and with the property that it is possible to transition from any state to an absorbing state in a finite number of transitions.

In the above examples, the first and third examples were absorbing Markov chains. In the first example, 'cereal' was the absorbing state, while in the third example, all of the states were absorbing states.

Furthermore, for nearly any configuration, *Chutes and Ladders* is an absorbing Markov Chain. The final square is the absorbing state, and it is possible to reach the final square from any given square (including figurative square zero). Some exceptions to this rule are that if there are d or more chutes directly before the final square, where d is the number of sides on the dice, or if there is some loop of chutes and ladders. Denoting $\mathbf{P}$ as the transition matrix, we derive the probability of going from state i to state j in precisely n steps.

Proposition 3.3. *The probability of going from state i to state j in precisely n steps is $p_{ij}^{(n)}$, the i,j-th entry of $\mathbf{P}^n$.*

Proof. The probability of going from state i to state j in two steps is the sum of the probability of going from step i to step 1, then from step 1 to step j, the probability of going from step i to step 2, then from step 2 to step j, and so on. Thus, letting $\mathbf{P}$ be a $w \times w$ matrix,

$$p_{ij}^{(2)} = p_{i1}p_{1j} + p_{i2}p_{2j} + \ldots + p_{iw}p_{wj} = \sum_{r=1}^{w} p_{ir}p_{rj}$$

This parallels the definition of matrix multiplication. That is, it is evident that $p_{ij}^{(2)}$ is the i,j-th entry of $\mathbf{P}^2 = \mathbf{P} \times \mathbf{P}$.

This proves the proposition for the $n = 2$ case; the proofs for greater n follow similarly. □

Proposition 3.4. *The probability of being in an absorbing state approaches 1, and the probability of being in a non-absorbing state approaches 0, correspondingly, as the number of steps is increased.*

Proof. Define m_j to be the minimum number of steps required to reach an absorbing state from state j, and t_j to be the probability of *not* reaching an absorbing state from state j in m_j steps. $t_j < 1$. Define m^* to be the greatest of the m_j, and t^* to be the greatest of the t_j. Thus, the probability of not being absorbed in m^* is less than or equal to $t^* < 1$, the probability of not being absorbed in $2m^*$ is less than of equal to t^{*2}. Since t^* is less than zero, as the number of steps increases, the probability of not being in an absorbing state approaches 0. □

For an absorbing Markov chain, we can define a submatrix $\mathbf{Q}$ of $\mathbf{P}$ as the transition matrix between non-absorbing states. That is, $\mathbf{Q}$ is $\mathbf{P}$, but with the rows and columns corresponding to absorbing states removed. For example, in the dinner scenario described above

$$\mathbf{Q} = \begin{array}{c} \\ H \\ P \\ F \end{array} \begin{array}{c} \begin{array}{ccc} H & P & F \end{array} \\ \begin{pmatrix} \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{3} \\ \frac{1}{10} & \frac{1}{5} & \frac{1}{5} \end{pmatrix} \end{array}$$

Remark 3.5. By Proposition 3.3 and Proposition 3.4, we see that as n approaches infinity, $\mathbf{Q}^n$ approaches $\mathbf{0}$.

Proposition 3.6. $(\mathbf{I} - \mathbf{Q})^{-1}$ *exists.*

Proof. Take $\mathbf{x}$ such that $(\mathbf{I} - \mathbf{Q})\mathbf{x} = 0$. Note that this is possible, as the trivial solution $\mathbf{x} = \mathbf{0}$ works. Distributing and subtracting yields $\mathbf{x} = \mathbf{Q}\mathbf{x}$. This implies that $\mathbf{x} = \mathbf{Q}^n\mathbf{x}$ by iteration. By Remark 3.5, we have that $\mathbf{Q}^n$ approaches $\mathbf{0}$ as n approaches infinity, which, due to the way $\mathbf{Q}^n\mathbf{x}$ is constructed by iteration, implies that $\mathbf{x} = \mathbf{0}$. Thus, the only solution is the trivial solution, which means that $(\mathbf{I}-\mathbf{Q})$ is invertible. □

Proposition 3.7. $(\mathbf{I} - \mathbf{Q})^{-1} = \mathbf{I} + \mathbf{Q} + \mathbf{Q}^2 + \mathbf{Q}^3 + \dots$.

Proof.

$$\begin{aligned} (\mathbf{I} - \mathbf{Q})(\mathbf{I} + \mathbf{Q} + \mathbf{Q}^2 + \mathbf{Q}^3 + \dots + \mathbf{Q}^n) &= \mathbf{I} - \mathbf{Q}^{n+1} \\ (\mathbf{I} - \mathbf{Q})^{-1}(\mathbf{I} - \mathbf{Q})(\mathbf{I} + \mathbf{Q} + \mathbf{Q}^2 + \mathbf{Q}^3 + \dots + \mathbf{Q}^n) &= (\mathbf{I} - \mathbf{Q})^{-1}(\mathbf{I} - \mathbf{Q}^{n+1}) \\ \mathbf{I} + \mathbf{Q} + \mathbf{Q}^2 + \mathbf{Q}^3 + \dots + \mathbf{Q}^n &= (\mathbf{I} - \mathbf{Q})^{-1}(\mathbf{I} - \mathbf{Q}^{n+1}) \\ \mathbf{I} + \mathbf{Q} + \mathbf{Q}^2 + \mathbf{Q}^3 + \dots + &= (\mathbf{I} - \mathbf{Q})^{-1} \end{aligned}$$

The final step is true as n approaches infinity per Remark 3.5. □

Definition 3.8. Now, we shall let $\mathbf{N} = (\mathbf{I} - \mathbf{Q})^{-1}$. n_{ij} refers to the i,jth entry of $\mathbf{N}$. We call $\mathbf{N}$ the **fundamental matrix** for $\mathbf{P}$.

Proposition 3.9. *Given that we start in state i, n_{ij} is the expected number of times that state j is reached.*

Remark 3.10. This includes the starting state, so $n_{ii} \geq 1$, as if we start in a given state, we must be in that state at least once.

Proof. We define $X_{ij}^{(l)}$ as a random variable that is equal to 1 if we are in state j after l steps starting from position i, and 0 otherwise. From our previous work, we see that $P(X_{ij}^{(l)} = 1) = q_{ij}^{(l)}$. Thus, $P(X_{ij} = 0) = 1 - q_{ij}^{(l)}$. Note that $q_{ij}^{(l)}$ is the i,jth entry of $\mathbf{Q}^l$, and n_{ij} is the i,jth entry of $\mathbf{N}$. Note that this holds for $l = 0$, as we can state $\mathbf{Q}^0 = \mathbf{I}$. Further, $E(X_{ij}^{(l)}) = P(X_{ij}^{(l)} = 1) = q_{ij}^{(l)}$, as $0 \leq X_{ij}^{(l)} \leq 1$. Therefore,

$$E(X_{ij}^{(0)} + X_{ij}^{(1)} + X_{ij}^{(2)} + \dots + X_{ij}^{(l)}) = q_{ij}^{(0)} + q_{ij}^{(1)} + q_{ij}^{(2)} + q_{ij}^{(l)}$$

Letting l approach infinity,

$$E(\sum_{l=0}^{\infty} X_{ij}^{(l)}) = \sum_{l=0}^{\infty} q_{ij}^{(l)} = n_{ij}$$

□

Remark 3.11. A natural question is, given we start in state i, how much time will it take until an absorbing state is reached. By our definition of $\mathbf{N}$, the solution is to sum the entries of row i in $\mathbf{N}$. This is because n_{ij} is the number of times a particular non-absorptive state will be reached, so to get the number of times until absorption, we simply need to sum these values, $\sum_j n_{ij}$.

4. Theory of Chutes and Ladders

Now that we have discussed the terminology of Markov chains, it is time to apply them to *Chutes and Ladders*. As mentioned above, *Chutes and Ladders* can be represented as an absorbing Markov chain, with the final square as the only absorbing state.

Now, the job is to determine the transition matrix.

Remark 4.1. We shall denote a game of Chutes and Ladders by $C(k, n, M)$, where k is the number of squares on the board (not including the figurative square zero), n is the number of sides on the dice, and M is the set of chutes and ladders. A chute is denoted as, for example, (15,2), which means there is a chute from square 15 to square 2. Ladders are denoted similarly.

Remark 4.2. The standard board, described in Section 2, shall be denoted as $\mathbf{T}$. Specifically, $\mathbf{T} = C(100, 6, \{(1, 38), (4, 14), (9, 31), (21, 42), (28, 84), (36, 44), (51, 67), (71, 91), (80, 100), (16, 6), (47, 26), (49, 11), (56, 53), (62, 19), (64, 60), (87, 24), (93, 73), (95, 75), (98, 78)\})$.

For the time being, we shall consider $C(k, n, \emptyset)$, given $n \geq 2$ and $n < k$. What would this transition matrix be?

Remark 4.3. Since figurative square zero is a state, the first row in the transition matrix is the row representing state 0. However, for ease of notation, we shall index our matrix to begin at row 0. That is, row 0 represents square 0, row 1 represents square 1, and so on.

Take some $i \in \mathbb{N} \cup \{0\}$ such that $n + i \leq k$. i is some square on the board, including the figurative square zero. Rolling the dice will yield a probability of $\frac{1}{n}$ of advancing $1, 2, \ldots, n$ squares. Therefore, given being in square i, after the next transition, there is a $\frac{1}{n}$ chance of being in square $i+1$, a $\frac{1}{n}$ chance of being in square $i + 2$ and so on until there is a $\frac{1}{n}$ chance of being in square $i + n$. The ith row of the transition matrix thus looks like:

$$\begin{array}{c} \\ i \end{array} \begin{array}{c} \begin{array}{cccccccccccc} 0 & 1 & \ldots & i & i+1 & i+2 & \ldots & i+n & i+n+i & \ldots & k \end{array} \\ \begin{pmatrix} 0 & 0 & \ldots & 0 & \frac{1}{n} & \frac{1}{n} & \ldots & \frac{1}{n} & 0 & \ldots & 0 \end{pmatrix} \end{array}$$

However, the issue is slightly different for i within n squares of k. Recall that a player must land precisely on k. If the player rolls something that would place them beyond k, they stay put. For example, in $\mathbf{T}$, if a player is on square 97 and rolls a 4, they remain on 97. This means that our transition matrix over $C(k, n, \emptyset)$, for the final n rows (including row k) will be different from the remaining rows discussed above. Row k will look as such:

$$\begin{array}{cccccccc} & 0 & 1 & \dots & k-3 & k-2 & k-1 & k \\ k & (0 & 0 & \dots & 0 & 0 & 0 & 1) \end{array}$$

This is because it is an absorbing state.

Row $k-1$ will look as such:

$$\begin{array}{cccccccc} & 0 & 1 & \dots & k-3 & k-2 & k-1 & k \\ k-1 & (0 & 0 & \dots & 0 & 0 & \frac{n-1}{n} & \frac{1}{n}) \end{array}$$

This is because there is a $\frac{1}{n}$ probability of rolling a 1 on an n-sided dice, and a probability of $1-\frac{1}{n}=\frac{n-1}{n}$ of not rolling a one and staying put.

For similar reasons, row $k-2$ looks as such:

$$\begin{array}{ccccccccc} & 0 & 1 & \dots & k-3 & k-2 & k-1 & k & \\ k-2 & (0 & 0 & \dots & 0 & \frac{n-2}{n} & \frac{1}{n} & \frac{1}{n}) & \end{array}$$

This pattern continues until row $k-n$, which is simply an 'ordinary' row.

Now, we shall add in 1 ladder (or chute) from square f to square g. Now, anytime we roll the die that would result in going to square f, we move instead to square g. We therefore need to alter our transition matrix accordingly.

Remark 4.4. The same algorithm applies regardless if $f<g$ (a ladder) or $f>g$ (a chute).

The probability of going to square f is now 0, and the probability of going to square g is now what used to be the probability of going to square f plus the original probability of going to square g, which is in most cases 0. The transition matrix would look like:

$$\begin{array}{c} \\ f-n-1 \\ f-n \\ f-n+1 \\ \dots \\ f-2 \\ f-1 \end{array} \begin{array}{c} \begin{array}{ccccccccccc} f-n & f-n+1 & f-n+2 & \dots & f-1 & f & \dots & g & \dots & k \end{array} \\ \begin{pmatrix} \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \dots & \frac{1}{n} & 0 & \dots & 0 & \dots & 0 \\ 0 & \frac{1}{n} & \frac{1}{n} & \dots & \frac{1}{n} & 0 & \dots & \frac{1}{n} & \dots & 0 \\ 0 & 0 & \frac{1}{n} & \dots & \frac{1}{n} & 0 & \dots & \frac{1}{n} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \frac{1}{n} & 0 & \dots & \frac{1}{n} & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & \dots & \frac{1}{n} & \dots & 0 \end{pmatrix} \end{array}$$

This still leaves the question of what row f looks like.

The answer to this question is that there is a 1 in the gth column, and 0 elsewhere. This is because if a players is on square f, they move with probability 1 to square g. This is quite irrelevant for a computer simulation, as it is impossible to start a turn on square f, but is necessary for accurately calculating the fundamental matrix.

Remark 4.5. This algorithm still works in the edge cases, such as where f and g are close together, or g is near k.

Remark 4.6. The same algorithm can be used to add in as many chutes and ladders as we wish. For example, we could use this algorithm to generate $\mathbf{T}$.

Since only the square k is an absorbing state, the $\mathbf{Q}$ matrix is simply the transition matrix, with the row k and column k removed. Note that $\mathbf{Q}$ is a $k\times k$ matrix, as the transition matrix is a $(k+1)\times(k+1)$ matrix, due to the figurative square zero.

At this point, we can hand over the **Q** matrix to the computer, which, provided k is not very large, can calculate **N**, and give us the expected number of turns, completing the problem.

Remark 4.7. The computer package Jama for Java is capable of calculating inverses quickly and efficiently; attempting to calculate a 100×100 matrix's inverse, as is necessary for **T**, is quite challenging to do by hand. However, Jama can do it in under one second.

Remark 4.8. For **T**, the expected number of turns needed is approximately 39.2251223.

5. The Programming

Programming was used to determine the answer experimentally. The programming was done in the Java language. The program had a number of methods.

*The first method was to create a transition matrix of the form $C(k, n, \emptyset)$. The values of k and n were inputted from the user, and the transition matrix was constructed as described in section 3.

*The second method added the chutes and ladders. The user inputs the coordinates of the chutes and the ladders, and the transition matrix was altered accordingly, by the algorithm described in the section above.

Remark 5.1. For convenience, there exists a method that combined the two methods above to automatically create **T** without input from the user.

* The third method ran through one iteration of the game chutes and ladders. The player started at figurative square zero, corresponding to the 0th row in the matrix (using the convention described above). Then, a random number is generated corresponding to the dice roll, and the player moves according to the established transition matrix, including using the chutes and ladders.[1] Meanwhile, it is recorded that the die has been rolled once. The process is repeated until the player reaches k exactly. Then, the number of dice rolls needed is output.

* The fourth method ran the third method multiple times in order to reap the benefits of the law of large numbers. The user inputs the number of times they wish to third method to run. The average of these runs is then output. For example, running the fourth method, with input 1,000,000 on **T** 5 times yields:

$$39.225808, 39.251338, 39.221559, 39.295085, 39.211283$$

These are remarkably close to the value predicted by the theoretical methods described in section 4.

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[1]The actual details are a little bit technical, and thus best described in a footnote. The random number generated was *not* between, say, 1 and n. Instead, a random number was generated between 0 and 1. Each entry in the row corresponding to the space the player is currently in was assigned an "area" according to it's value. Since the sum of the values on a given row is equal to 1, the square moved to corresponded to which "area" it fell into. For example, if the player was on the ith square, and row i of the transition matrix looked like,

$$\begin{array}{cccccccccccc} & 0 & \dots & i & i+1 & i+2 & i+3 & i+4 & \dots & i+15 & \dots & k \\ i & \Big(0 & \dots & 0 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \dots & \frac{1}{4} & \dots & 0 \Big) \end{array}$$

then if the random variable was between 0 and .25, then the player would move to square $i+1$, if the value of the random variable was between .25 and .5, the player would move to square $i+2$, if the value of the random variable was between .5 and .75, the player would move to square $i+4$, and if the value of the random variable was between .75 and 1, the player would move to square $i+15$. This represents a ladder between square $i+3$ and square $i+15$.

6. Miscellany

In this section, we shall focus on three different 'experiments,' which shall yield somewhat counterintuitive results. These counterintuitive results are largely due to some of the large ladders and chutes on the board.

6.1. ***Sides on the Die.*** On **T**, a 6-sided die is used. Here, we shall modify **T** by changing the number of sides of the die. We would expect that, as we increase the number of sides on the die, the expected number of rolls needed to complete the game decreases, for each roll has a higher expected value. Using the theoretical methods described in Section 4[2] we have established the following table, with n representing the number of sides and e representing the expected number of rolls needed to complete the game.

n	e
2	60.7625788
3	65.9007753
4	54.4937116
5	45.5619456
6	**39.2251223**
7	34.6965984
8	31.8532909
9	30.2952849
10	28.768692
11	27.427206
12	27.017742
13	26.221553
14	25.980534
15	25.805895

This table corresponds roughly to an intuitive understanding: in general, as the number of sides of the die increases, the number of turns it takes to complete the game increases. However, there is an exception. Rolling a 2-sided die yields an e value of approximately 60.8, yet rolling a 3-sided yields an e value of 65.9. That is, it would take *longer* to play a game of Chutes and Ladders with a 3-sided die than with a 2-sided die. This is likely due to the ladder from square 1 to 38, a large ladder: There is a higher probability of landing on this ladder immediately with a two-sided rather than 3-sided die. Beyond this, however, increasing the number of die functions as expected.

6.2. ***The Effects of Adding a Ladder (or a Chute).*** This subsection shall address the idea of adding one additional ladder or chute to **T**. It would seem likely that adding a ladder would decrease the number of rolls expected to finish the game, while adding a chute would correspondingly increase the number of times. However, there are some circumstances when a non-intuitive effect would occur. Looking at the board earlier in this paper, it is clear that a ladder from square 27

[2]The empirical methods described in Section 5 would likely work as well. However, when using such methods, there is a small but existing probability of error. Because the matrix that needs to be inverted in order to calculate the precise value is small (100×100), it can be inverted quickly in Java using Jama. If we were dealing with larger matrices, the empirical method would probably be better.

to 29 would probably increase the expected number of rolls needed, for if a player is "lucky" enough to land on square 27 and advance to square 29, they would miss the opportunity to land on a much bigger ladder, that from square 28 to 84. Similarly, a chute from square 29 to 27 would probably decrease the expected number of rolls needed. In fact, these guesses is correct. The e value, calculated using the theoretical methods described in section 4, for **T** with a ladder added from square 27 to square 29 is approximately 40.20 - nearly one roll longer than **T**, and the e value for **T** with a chute added from square 29 to 27 is approximately 38.05, more than one roll shorter than **T**.

However, these are fairly trivial examples that do not, in general, counteract our notion that adding ladders decreases the expected number of rolls needed while adding chutes increases the number of expected rolls. For one, these examples were only of length two, and were situated immediately 'over' a very important ladder. Therefore, to see if our intuition is correct, we should check for long ladders.

For the remainder of this section, we shall only consider ladders of length 10 or more.[3] First, however, a definition.

Definition 6.1. A **non-ladder** square is one which is neither the starting nor finishing point of any existing chute or ladder.

We shall only permit our additional chutes and ladders to go from non-ladder squares to non-ladder squares.[4] Using the theoretical method (combined with a pair of nested for-loops to allow us to check all possible feasible ladders (or chutes) of length 10 or greater), we find that there are 5496 feasible ladders (or chutes) of length 10 or greater, and that **254** of them change the expected value against our intuition. $\frac{254}{5496} = 4.6\%$. Although a small percentage, 4.6% is not insignificant either. This means that of all ladders and chutes that can be added onto the board

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[3]10 is mostly an arbitrary, reasonably-large number

[4]If this were not the case, there would be errors. There are a few possible cases:

(1) An additional ladder (or chute) goes from square i to square k, and there already exists a ladder (or chute) from square i to square j. This makes it so that when a player lands on square i, there is no defined square to go to.
(2) An additional ladder (or chute) goes from square i to square k, and there already exists a ladder (or chute) from square j to square i. Thus, if a player were to land on square j, they would jump to square i, then immediately jump to square k. Thus, we are both adding a ladder (or chute) from square i to square k, *and*, in essence, *modifying* the existing ladder (or chute) from square j to square i to a ladder (or chute) from square j to square k. However, we solely wished to add a ladder.
(3) An additional ladder (or chute) goes from square i to square k, and there already exists a ladder (or chute) from square k to square j. For the same reasons discussed in (2), this does not work.
(4) An additional ladder (or chute) goes from square i to square k, and there already exists a ladder (or chute) from square j to square k. This would make the **Q** matrix look like:

$$\begin{array}{c} \\ i \\ \dots \\ j \end{array} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & \dots & k-1 & k & k+1 & \dots & 99 \end{array} \\ \begin{pmatrix} 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \end{pmatrix} \end{array}$$

Thus, the **Q** matrix is singular, so $(\mathbf{I} - \mathbf{Q})$ is singular, so $\mathbf{N} = (\mathbf{I} - \mathbf{Q})^{-1}$ does not exist, which makes our theoretical method for calculating the expected number of turns needed not work.

and are at least of length 10, nearly 1 in 20 defies our intuition about their effect upon the expected value.

Remark 6.2. Most of these are clustered near either the (28,84) or (80,100) ladder. This is because the (28,84) ladder is quite large, and the (80,100) ladder allows the player to land precisely on the finishing square.

6.3. ***Starting Somewhere Besides Square Zero.*** A rule that we have followed so far is that the player begins at the figurative square zero. What happens if we change this rule? Recalling Remark 3.11, we have a method for calculating the expected number of rolls needed to complete the game given a specified starting position.

Using this, we can see, for example, that starting at square 2 has an expected value of 39.6964061, and that starting at square 5 has an expected value of 39.2950265, both of which are larger than the expected value of starting at the figurative square zero. That is, it would be preferable to start at figurative square zero than to start at either square 2 or square 5, a counterintuitive result. What causes this? There are ladders going from square 1 to square 38, and from square 4 to square 14. If a player is on square 2, they miss the opportunity to use the 1-38 ladder, and if a player is on square 5, they miss the opportunity to use them both.

This analysis shows that it is often better to be just before a ladder rather than just after, even though there are fewer squares until the finishing square, and there is no guaranteed advancement onto the ladder.

Remark 6.3. There is not a similar analysis for chutes, because this analysis would show that it is better to be just after a chute rather than before it, which already corresponds to our intutition.

However, as we move further away from figurative square zero, this effect is not strong enough to increase the expected value above the expected value of starting at zero. For example, starting at square 29 - just after the massive (28,84) ladder - has an expected number of rolls of 36.8911770. Thus, another way to see if unintuitive results occur is to compare specific squares to the square preceding them. Because the expected value is not precisely defined from starting at the bottom of a ladder, we shall only consider cases not as such.[5] Running this algorithm from the computer, we find that it is preferable to start at the prior square (or the prior non-ladder square, if necessary) for the following squares on **T**:

$$2, 5, 10, 23, 24, 25, 26, 27, 29, 41, 43, 52, 57$$

$$66, 67, 72, 73, 75, 76, 77, 78, 79, 81, 89, 92$$

There are a number of things apparent from this list. First and foremost is the sheer quantity of such 'unintuitive' squares, 25 in total. Considering that we are not counting a number of squares, this amounts to over $\frac{1}{3}$ of the 'eligible' squares. Clearly, our intuition is off-kilter.

Further, the placement of these squares is quite interesting. These squares show how important the (28,84) and (80,100) ladders are; it is but slightly more probably to land on these latters being, say, 5 rather than 4 squares before the ladder (recalling that this is over multiple turns), and the fact that even *8* squares out,

[5]For technological reasons, we shall also exclude squares at the top of ladders and chutes, and the bottom of chutes.

for the (80,100) ladder and *4* for the (28,84) one, shows precisely how important they are.

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References

[1] Altheon, King, and Schilling. **How Long Is a Game of Snakes and Ladders?** *The Mathematical Gazette*, Vol. 77, No. 478 (Mar., 1993), pp. 71-76[6]
[2] Grinstead and Snell. **Introduction to Probability** (Chapter 11)[7]

[6]A good overview of the topic. Used in Section 4. Was the source of the experiment in section 6.2.
[7]Used extensively in Section 3; many of the proofs rely, to varying degrees, on this source.