HODGE THEORY AND ELLIPTIC REGULARITY

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Abstract. The central goal of this paper is a proof of the Hodge decomposition of the deRham complex for compact Riemannian manifolds. Along the way, we develop parts of the theories of distributions and pseudodifferential operators on vector bundles over manifolds. This is then used to prove classical regularity results about elliptic operators in the vector bundle context.

Contents

1. Introduction

When using the methods of algebraic topology to study spaces, we obtain algebraic invariants such as homotopy, homology, and cohomology groups by forming quotients of large spaces under some equivalence relation. In the study of a smooth manifold M of dimension m, we begin with the deRham cochain complex

$$0 \xrightarrow{0} \Omega^0(M) \xrightarrow{d_0} \Omega^1(M) \xrightarrow{d_1} \dots \xrightarrow{d_{m-1}} \Omega^m(M) \xrightarrow{0} 0$$

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of real-valued differential forms. The quotient objects of interest are the deRham cohomology groups $H^k_{\mathrm{dR}}(M) = \ker d_k / \operatorname{im} d_{k-1}$. The goal of the Hodge decomposition is to find distinguished representatives of the equivalence classes. We thus replace the study of a quotient object with the study of a nice subspace of differential forms. The Hodge decomposition accomplishes this in the case M is compact, oriented and given a Riemannian metric. Using the metric and exterior derivative we construct the Laplace-Beltrami operator, a differential operator $\Delta : \Omega^k(M) \to \Omega^k(M)$, analogous to the classical Laplacian. The solution space of the equation

$$\Delta\omega = 0 \tag{1.1}$$

is then a collection of unique representatives of the cohomology classes in $H^k_{\mathrm{dR}}(M)$.

We begin in 2.1 by motivating the Hodge decomposition as a minimization problem over deRham cohomology classes with respect to an inner product on differential forms. Applying Hilbert space theory, in 2.2 we infer that minimizers are precisely forms in the kernel of the Laplace-Beltrami operator. The major work of the paper, in sections 3 and 4 focuses on the regularity of elliptic operators between vector bundles equipped with fiber-wise inner products on compact, oriented Riemannian manifolds. We introduce notions of distributions in $3.1-2$, differential operators in 3.3, Sobolev spaces in 3.4 and a bit of the machinery of pseudo-differential operators in the general vector bundle setting in 4.4. The conclusion, stated in 4.2, is that for an elliptic differential operator L of order l we may construct a pseudo-differential operator S which increases regularity and inverts L up to smooth data. It follows that the kernel is a finite-dimensional space of smooth sections. In 4.4 we prove the Laplace-Beltrami operator is elliptic, and in 4.5 we apply the elliptic theory to prove the Hodge Decomposition in full. Section 5 focuses on a couple applications to Poincáre duality and the cohomology of compact Lie groups.

This paper assumes the reader has a certain familiarity with manifolds, vector bundles, deRham cohomology, and some basic Riemannian geometry. The pre-requisite material can be found in Warner [3]. In addition, the material on elliptic regularity and the definition of the local structure of a pseudo-differential operator is more readable if the reader has some familiarity with the basics of Sobolev spaces and Fourier analysis on $\mathbb{R}^n$. In particular the reader should be comfortable with the idea that application of a constant-coefficient differential operator to a function corresponds to multiplication by a polynomial in the Fourier transform. The derived result that growth conditions on the Fourier transform correspond to differentiability conditions on the function inform the definition of ellipticity. For a detailed background in the machinery of distributions and pseudodifferential operators a good reference is Hörmander [1][2].

2. The Laplace-Beltrami Operator

2.1. **Motivation by Minimization.** Given an inner product space (V, g), there is a natural extension of g to the tensor powers of V. We give $V^{\otimes p}$ the inner product $g \otimes \ldots \otimes g$, the pth tensor power of g. This is natural in the sense that it is invariant under the action of S_p on $V^{\otimes p}$. The decomposition into isotypic components is orthogonal, so either as a subrepresentation or quotient $\Lambda^p V^*$ has an induced metric which we will also denote by g. These functorial constructions generalize to the case of a vector bundle $V \to E \to M$ with a fiberwise inner product. In particular, for a Riemannian manifold (M, g), they allow us to give a natural inner product on the fibers of $\Lambda^p(TM^*)$. This yields a bilinear map:

$$\Omega^p(M) \times \Omega^p(M) \to C^\infty(M)$$

$$(\omega, \eta) \mapsto g(\omega, \eta).$$

It proves useful to introduce the Hodge star operator $* : \Omega^p \to \Omega^{n-p}$. A full description of the algebraic properties of this operator can be found in exercise 2.13 of Warner [3]. The key fact for us is that it uses orientation and orthonormal bases to map forms in Ω^p to forms in Ω^{m-p}. $g(\omega, \eta) = *(\omega \wedge *\eta)$. It is also worth noting that $*(1)$ is the canonical volume form on a Riemannian manifold which takes on the value ± 1 on orthonormal bases depending on orientation.

Now we define an inner product on $\Omega^p(M)$ by the rule:

$$\langle \omega, \eta \rangle = \int_M g(\omega, \eta) d\mu = \int_M \omega \wedge *\eta. \tag{2.1}$$

We complete $\Omega^p(M)$ into a Hilbert space $L^2(M; \Lambda^p TM^*)$. For the case of $p = 0$, this is indeed the completion of $C^\infty(M)$ to the Hilbert space $L^2(M, \mu)$. Later we will define the L^p sections of a vector bundle on a Riemannian manifold. This completion is precisely the L^2 sections in that sense. Now, given a closed p-form ω on M, we can consider the closure of the cohomology class

$$\overline{[\omega]} = \{\omega + \eta | \eta \in \overline{d(\Omega^{p-1}(M))}\}.$$

This is a closed affine subspace of a Hilbert space, so it has a unique minimal element. We denote this by ω_0. We will show later that this minimizer is in $\Omega^p(M)$. It is a standard result in Hilbert space theory that the minimizer for an affine space is the unique element of that space perpendicular to it, which follows from a variational argument. It suffices to check that ω_0 is perpendicular to all exact forms.

Suppose ω_0 is a smooth differential form. For any $d\eta$ exact, by Stokes' theorem:

$$\int_M d\eta \wedge *\omega_0 = (-1)^{p-1} \int_M \eta \wedge d(*\omega_0) = (-1)^{(p-1)+n(n-p)} \int_M \eta \wedge *(*d*)(\omega_0)).$$

The final step above uses the identity, $** = (-1)^{n(n-p)}$. This shows that the operator defined by

$$\delta = (-1)^{n(p+1)+1} * d* : \Omega^p \to \Omega^{p-1}$$

is a formal adjoint to the exterior derivative. It should be noted that this is not an adjoint in the full L^2 sense. Thus ω_0 is a minimizer in the sense of our problem if and only if $0 = \langle \omega_0, d\eta \rangle = \langle \delta\omega_0, \eta \rangle = 0$ for all $\eta \in \Omega^{p-1}$. In other words, we have the simultaneous differential equations $d\omega_0 = 0, \delta\omega_0 = 0$.

2.2. **The Hodge Decomposition.** The formal adjunction of d and δ implies that $\operatorname{im} \delta$ is perpendicular to $\ker d$ and vice versa. The cochain condition $d^2 = 0$ on d implies that $\operatorname{im} d \subset \ker d$, and the same holds for δ. Based on the preceding we might hope for an orthogonal decomposition of $\Omega^p(M)$ with respect to our inner product:

$$\Omega^p(M) = \delta(\Omega^{p+1}(M)) \oplus d(\Omega^{p-1}(M)) \oplus (\ker d \cap \ker \delta) \tag{2.2}$$

Assume for the moment that this decomposition holds. This implies an orthogonal decomposition of closed forms as $d(\Omega^{p-1}(M)) \oplus (\ker d \cap \ker \delta)$. The first summand is the exact forms, so $(\ker d \cap \ker \delta)$ would contain a unique representative of each cohomology class. Consider the composition $d\delta : \Omega^p(M) \to \Omega^p(M)$. Given the cochain condition, the only non-trivial portions of this composition are:

$$d(\Omega^{p-1}(M)) \xrightarrow{\delta} \delta(\Omega^{p-1}(M)) \xrightarrow{d} d(\Omega^{p-1}(M)).$$

The factor maps above are both injective on the given domains by adjunction. If the terms are finite-dimensional, we have isomorphisms. Likewise δd has image $\delta(\Omega^{p-1}(M))$ with image depending only on the coexact component of omega. Thus we are led to consider the operator:

$$\Delta = d\delta + \delta d : \Omega^p(M) \to \Omega^p(M). \tag{2.3}$$

This is the Laplace-Beltrami operator. As an explicit example, if f is a smooth function on M, and locally $X_1, ..., X_m$ are a collection vector fields forming an oriented orthonormal basis at each point, then we have:

$$\Delta f = -\sum_{i=1}^{m} X_i(X_i(f)) + \sum_{i=1}^{m} \sum_{j \neq i} X_i(f) g([X_j, X_i], X_j). \tag{2.4}$$

If M is some subset of Euclidean space and X_i are the vector fields given by translating the standard basis vectors, then $[X_i, X_j] = 0$. We thus reduce to the usual Laplacian

$$\Delta f = -\sum \frac{\partial^2 f}{\partial x_i^2}.$$

The claim of the Hodge Theorem is then as follows:

Theorem 2.5. *(Hodge Decomposition If M is a compact, oriented, Riemannian m-manifold there is an othogonal decomposition of the deRham complex, functorial in local isometries, as*

$$\Omega^\bullet(M) = \mathcal{H}^\bullet(M) \oplus \Delta\Omega^\bullet(M)$$

. Here $\mathcal{H}^\bullet$ is stable under the wedge product and the inclusion of complexes $\mathcal{H}^\bullet(M)$ in $\Omega^\bullet(M)$ induces an isomoprhism of graded-commutative algebras $\mathcal{H}^\bullet(M) \cong H^\bullet_{dR}(M)$. In addition, the complex $\Delta\Omega^\bullet(M)$ is exact so in particular $\Delta^2 = 0$. Finally, each $\mathcal{H}^p(M)$ is finite dimensional.

3. Distributions

3.1. **Distributions on $\mathbb{R}^n$.** Given an open set $\Omega \subset \mathbb{R}^n$, we define the space of distributions $\mathscr{D}'(\Omega)$ to be the topological dual of $C_0^\infty(\Omega)$, given the topology characterized by the uniform convergence of all partial derivatives in compact sets, with supports contained in a fixed compact set. It is standard to denote by $\langle T, \phi \rangle$ the value $T(\phi)$ for $T \in \mathscr{D}'(\Omega)$, and $\phi \in \mathscr{D}(\omega)$. For such open sets, there are natural inclusions of L^p_{loc} in $\mathscr{D}'(\Omega)$ for $1 \leq p \leq \infty$ given by

$$\langle f, \phi \rangle = \int_\Omega f \phi d\mu$$

Here μ is Lebesgue measure, $f \in L^p_{\text{loc}}$, and ϕ is a test function. We also have formal derivatives of all orders in the space of distributions by defining

$$\langle \partial^\alpha T, \phi \rangle = (-1)^{|\alpha|} \langle T, \partial^\alpha \phi \rangle$$

Here α is a multi-index, and $|\alpha|$ is the sum of the indices. This definition coincides with the result of integration by parts when $T = f$ is a smooth function.

3.2. **Distributions on a Manifold.** To deal with questions of regularity for differential operators on a manifold (in particular, on the spaces of differential forms), we would like an analogous notion of distributions as generalized functions, and further a notion of distributional sections of vector bundles. We consider the question of generalizing $C^\infty(M)$ first, and then move onto $C^\infty(M;E)$, the smooth sections of a vector bundle of M.

We need a bundle F over M with a natural pairing $C^\infty(M;E) \times C_0^\infty(M;F) \to \mathbb{R}$. In general this will be $F := E^* \otimes D = \mathrm{Hom}(E,D)$, where D is the line bundle of densities. Distributions are elements of the topological dual of the space of compactly supported smooth sections. The bundle F has a natural pairing

$$C^\infty(M;F) \times C^\infty(M;E) \to C^\infty(M;D),$$

and if the section of F is compactly supported, then the resulting density will be as well. Integration gives a map

$$C_c^\infty(M;F) \times C^\infty(M;E) \to \mathbb{R}.$$

Using this we embed $\mathbb{C}^\infty(M;E)$ into the space of distributions.

We are primarily interested in the case of an oriented Riemannian m-manifold. Orientation allows us to replace D with $\Lambda^m(M)$. In an oriented Riemannian m-manifold this is canonically trivialized. And the tensor bundles we are considering are equipped with a fiberwise inner product that allows us to identify E^* with E. Thus $F = E$.

Given an oriented Riemannian manifold M, and a vector-bundle $E \to M$ with fiberwise inner product $g(\cdot,\cdot) \in C^\infty(M;\mathrm{Sym}^2 E)$, we can topologize the space $\mathscr{D}(M;E)$ of compactly supported sections of E as follows. Choose an atlas of charts that also trivializes the bundle, and endow $\mathscr{D}(M;E)$ with the topology of uniform convergence of all partial derivatives on compact sets. The same argument regarding the chain rule shows that this topology is independent of trivializations and coordinates. We can then take the topological dual $\mathscr{D}'(M;E)$, which we define to be the space of distributions. The Riemannian structure gives a canonical Borel measure μ on the manifold associated to integration against the volume form. Thus, for any section $f : M \to E$, if we define $|f| = \sqrt{g(f,f)} : M \to \mathbb{R}$ then we have a way of defining $L^p(M;E), L^p_{\mathrm{loc}}(M;E)$ by testing if $|f|$ is in $L^p(M), L^p_{\mathrm{loc}}(M)$. A section f in any of these classes will determine a distribution by

$$\langle f,\phi\rangle = \int_M g(f,\phi)d\mu$$

for $\phi \in \mathscr{D}'(M;E)$.

To treat the Hodge Theorem from this perspective we need a few basic results. First, we will need to describe the application of a differential operator to a distribution. Then we establish the ellipticity of the Laplace-Beltrami operator. Once this is done, we prove two regularity results about elliptic operators: using distributions we will define a notion of the Sobolev spaces on a manifold, and then show that for an elliptic operator L of degree s the extended operator is Fredholm as an operator $H^s = W^{2,s} \to L_2$. Better yet, if u is a distributional solution of $Lu = f$ for f in H^l then $u \in H^{s+l}$,. In particular, if f is smooth, u is as well. The Fredholm condition implies that the kernel of the Laplace-Beltrami operator (or any elliptic operator) is a finite dimensional space of smooth sections.

3.3. **Differential Operators for Distributions on Manifolds.** For vector bundles E, F over a manifold M, the class of linear differential operators with smooth coefficients may be described as those operators which, choosing some atlas that locally trivializes both E, F, have the local structure of a linear differential operator. That is, if n_1, n_2 are the fiber dimensions of E, F, and U is a coordinate patch on M, then a differential operator P of order s restricts to a map

$$C^\infty(U) \otimes \mathbb{R}^{n_1} \to C^\infty(U) \otimes \mathbb{R}^{n_2}$$

Which may be written as a $n_2 \times n_1$ matrix with entries of the form $\sum_{|\alpha| \leq s} a_\alpha \partial^\alpha$ where α ranges over multi-indices, and each a_α is a smooth function.

From a sheaf-theoretic perspective, we can define the space of differential operators $C^\infty(M; E) \to C^\infty(M; E)$ as follows. Let $\mathscr{A}$ be the sheaf of smooth sections of E. Then we have a sheaf of algebras $\mathrm{End}(\mathscr{A})$ which are the local linear operators. The subsheaf generated by multiplication by smooth functions and derivation against vector fields will be the differential operators.

A linear differential operator $P : C^\infty(M; E) \to C^\infty(M; F)$ with smooth coefficients can be extended to a map$\mathscr{D}'(M; E) \to \mathscr{D}'(M; F)$. To do this we claim there is an operator $P^t : C^\infty(M; F^* \otimes \Omega^n) \to C^\infty(E^* \otimes \Lambda^m(TM)^*)$ (the more general bundle pairing for all oriented manifolds). such that for all $f \in C^\infty(M; E), \phi \in C_0^\infty(M; F^* \otimes \Lambda^m(TM)^*)$ we have:

$$\langle Pf, \phi \rangle = \langle f, P^t \phi \rangle \tag{3.1}$$

Finding local expressions for P^t is an exercise in integration by parts. For simplicity we will deal with the case of $\mathscr{D}'(M)$ and differential operators $L : C^\infty(M) \to C^\infty(M)$, and will work on a Riemannian manifold where there is a standard volume form trivializing Ω^n. Suppose our distribution is a smooth function f. Let $\Omega \subset \mathbb{R}^m$ be identified with some coordinate chart. Let ϕ be in $C_0^\infty(\Omega)$. Finally, let $\mu \in C^\infty(\Omega)$ be such that the pullback of the canonical volume form on M onto Ω is $\mu dx_1 \wedge ... \wedge dx_n$. Then we have

$$\langle f, \phi \rangle = \int_\Omega f \phi \mu \; dx$$

As in the case of distributions on a subset of $\mathbb{R}^m$ we can multiply a distribution by a smooth function g by the expression:

$$\langle gf, \phi \rangle = \langle f, g\phi \rangle. \tag{3.2}$$

We now treat the differential component. Let ∂^α be a basic differential operator in the local coordinates for some multi-index α. Repeated integration by parts, removing boundary terms by the compact support hypothesis, and applying the Leibniz differentiation rule yields:

$$\langle \partial^\alpha f, \phi \rangle = \int_\Omega (\partial^\alpha f)(\phi \mu) \; dx = (-1)^\alpha \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \int_\Omega f \partial^{\alpha - \beta} \phi \partial^\beta \mu \; dx. \tag{3.3}$$

Thus motivated we define:

$$\langle \partial^\alpha T, \phi \rangle = \langle T, \frac{(-1)^\alpha}{\mu} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^{\alpha - \beta} \phi \partial^\beta \mu \rangle \tag{3.4}$$

These definitions describe, in local coordinates, application of a differential operator with smooth coefficients to a distribution T. Similar expressions locally define differential operators between two bundles E, F of arbitrary dimension.

3.4. **Sobolev Sections of Vector Bundles.** As in previous sections, we consider a vector bundle E over an m-manifold M. When necessary we will assume M to be compact, Riemannian, and orientable, and E to be equipped with a smoothly varying inner product on fibers.

On an open region Ω of Euclidean space $\mathbb{R}^n$, we define Sobolev space by introducing a notion of weak partial derivative. The Sobolev spaces H^s for $s \in \mathbb{N}$ are then space of L^2 functions u with all weak partial derivatives $\partial^\alpha u$ up to and including order s lying in L^2. A norm arises from the inner product:

$$\langle u, v\rangle = \int_\Omega uv + \sum_{|\alpha|\leq s} \partial^\alpha u \partial^\alpha v \;\, d\mu$$

A result of this set-up is that any linear differential operator D of order l with smooth and bounded coefficients defines a bounded linear map $D : H^l \to H^0 = L^2$, or more generally $H^{s+l} \to H^s$.

For a vector bundle with fiberwise product over a general oriented Riemannian manifold we first define H^s_{loc}. It is the space of distributional sections $T \in \mathscr{D}'(M; E)$ such that for any linear differential operator D with smooth coefficients of order at most s, $DT \in L^2_{\text{loc}}(M; E)$. In the compact case $H^s_{\text{loc}} = H^s$ and the above definition is independent of the choice of metric. Although it is not canonical, we can make $H^s(M; E)$ a Hilbert space as follows. Let $\{U_\alpha\}_{\alpha\in A}$ be a finite cover by compact neighborhoods of M on which we have charts ψ_α and trivializations of E. If E is n dimensional then we have maps

$$H^s(M; E) \to \bigoplus_{\alpha\in A} H^s(\psi_\alpha(U_\alpha); \mathbb{R}^n) \to \bigoplus_{\alpha,\beta\in A} H^s(\psi_\alpha(U_\alpha \cap U_\beta); \mathbb{R}^n) \to ...$$

Here $H^s(\psi_\beta(U_\beta) : \mathbb{R}^n)$ is given the standard Sobolev structure, and the second map is difference of restrictions. The exactness of this composition allows $H^s(M; E)$ to be made a Hilbert space, as it is a kernel. Although the inner product depends on the choice of atlas, we have finitely many transition maps on compact domains, so they can be uniformly bounded. It follows that the norms given by two different atlases will be equivalent. Motivated by the standard case of Sobolev Space in $\mathbb{R}^n$ we define H^s for s a negative integer as the topological dual of H^{-s}.

It is worth noting at this point that lying in Sobolev space roughly corresponds to a bound on the Fourier transform of the form

$$\hat{f}(\xi) \leq C(1 + |\xi|^2)^{-s/2}$$

.

4. Elliptic Operators and Regularity

4.1. **Elliptic Operators on Vector Bundles.** Given two vector bundles E, F over M of fiber dimensions n_1, n_2, we can write any linear differential operator $L : \Gamma(E) \to \Gamma(F)$ in local trivialization coordinates as an $n_2 \times n_1$ matrix. The entries are sums of the form:

$$L_{ij} = \sum_{|\alpha|\leq l} a_{ij,\alpha}\partial^\alpha,$$

with each $a_{ij,\alpha}$ a smooth function $C^\infty(M, \mathbb{C})$.

In Fourier analysis, the action of a differential operator with constant coefficients corresponds to applying a Fourier transform, multiplying by some polynomial, and then applying an inverse Fourier transform. Motivated by this we consider the polynomials:

$$\sum_{|\alpha|\leq l} a_{ij,\alpha}\xi^{\alpha}.$$

Here $\xi = (\xi_1, ..., \xi_m) \in \mathbb{R}^m$ (viewed as the cotangent space of M) and for $\alpha = (\alpha_1, ..., \alpha_m) \in \mathbb{N}^m$ a multi-index we have $\xi^\alpha = \xi_1^{\alpha_1}...\xi_m^{\alpha_m}$. This doesn't transform well under changes of coordinate, for example all but the most simple changes of coordinates needn't maintain homogeneity of a differential operator. Instead, we restrict attention to the terms of top order. Operators of order l stay operators of order l under changes of coordinate, so we can be sure that the top level terms are independent. This is the principal symbol the matrix, for $p \in M, \xi \in \mathbb{R}^m$:

$$\sigma_l(L)(p,\xi) = \left(\sum_{|\alpha|=l} a_{ij,\alpha}(p)\xi^{\alpha}\right)_{1\leq i\leq n_2, 1\leq j\leq n_1}.$$

An operator is said to be elliptic if this matrix is non-singular for every p, ξ (obviously, this entails that $n_1 = n_2$). A coordinate free definition may be given as follows, adapted from section 6.28 of Warner [3]: We say that L of order l is elliptic if $n_1 = n_2$ and for every smooth section u of E, $u(p) \neq 0$ and every smooth function $\phi \in C^\infty(M)$ with $\phi(p) = 0, d\phi(p) \neq 0$ we have $L(\phi^l u)(p) \neq 0$. As we range over all ϕ we range over all ξ nonzero in the cotangent space. We can check in local coordinates that $L(\phi^l u)(p) = \sigma_l(L)(p,\xi)(u(p))$. This verifies the injectivity of the matrix $\sigma_l(L)(p,\xi)$. By our fiber dimension assumption it is an isomorphism on the fibers.

The importance of the ellipticity condition is as follows: from a formal Fourier perspective the application of a differential operator corresponds to taking the Fourier transform, multiplying by some polynomial (the symbol) and then inverting the Fourier transform. If differentiability corresponds to a polynomial growth condition on the Fourier transform, then multiplication by a polynomial of order l on the Fourier corresponds to losing l levels of regularity. Ellipticity guarantees that we have, locally in x, bounds of the form:

$$C_1|\xi|^m > |\sigma(L)(x,\xi)| > C_2|\xi|^m \tag{4.1}$$

for sufficiently large ξ. The latter inequality is derived from ellipticity; the former from the operator having order $\leq l$. This allows us to precisely relate polynomial growth conditions on $\hat{Lf}$ to those of $\hat{f}$, which suggests that if Lf is in H^s then f is in H^{s+l}.

4.2. **A Rough Program for Elliptic Regularity.** The goal of this section is to explain the machinery behind our main regularity result for elliptic operators. As stated in section 3.4 a linear differential operator $L : C^\infty(M; E) \to C^\infty(M; F)$ of order l extends to a continuous map $\mathscr{D}'(M; E) \to \mathscr{D}'(M; F)$ on distributions, and specifically to continuous linear maps $H^{s+l}(M; E) \to H^s(M; F)$. The key regularity result is the following.

Theorem 4.2. *Let $L : \mathscr{D}'(M; E) \to \mathscr{D}'(M; F)$ be the extension of an elliptic differential operator of order l between vector bundles on a compact, oriented manifold. Then there exists a pseudo-differential operator S of order $-l$ such that:*

$$SL = I - K_1 : \mathscr{D}'(M; E) \to \mathscr{D}'(M; E)$$
$$LS = I - K_2 : \mathscr{D}'(M; F) \to \mathscr{D}'(M; F).$$

where the maps K_1, K_2 send compactly supported distributions to smooth sections.

This neatly solves our regularity questions for compact manifolds. We list the important corollaries below.

Corollary 4.3. *Suppose $f \in H^s(M;F)$ (respectively smooth), and $u \in \mathscr{D}'(M;F)$ solves the equation*

$$Lu = f \tag{4.4}$$

Then $u \in H^{s+l}(M;F)$ (resp. u is smooth).

Proof. Apply S to the equation (4.4):

$$Sf = SLu = u - K_1 u \tag{4.5}$$

$Sf \in H^{s+l}(M;E)$ (resp. smooth) because S is of order $-l$, and since $u - Sf = K_1 u$ is smooth, u must lie in the same regularity class. □

Corollary 4.6. *The kernel of L is a finite dimensional space of smooth sections of E.*

Proof. From Corollary 4.3 every element of the kernel is smooth. By either the Rellich Lemma or Arzela-Ascoli, the operators K_1, K_2 are compact as maps on Sobolev space. Thus L, S form a Fredholm pair. Classical functional analysis says that the kernel of L is finite dimensional in Sobolev space. The kernel is composed of only smooth sections, which lie in all Sobolev spaces, so the kernel is finite dimensional in general. □

Corollary 4.7. *Given any s, we have the following estimate using Sobolev norms for some $C > 0$.*

$$\|u\|_{s+l} \leq C(\|Lu\|_s + \|u\|_s) \tag{4.8}$$

Proof. Write $u = SLu + K_1 u$, note that since K_1 is smoothing it can be factored to a bounded operator

$$H^s(M;E) \hookrightarrow \mathscr{D}'(M;E) \to C^\infty(M;E) \hookrightarrow H^{s+l}.$$

Both S, K are bounded $H^s(M;E) \to H^{s+l}(M;E)$ so we have the result. □

The next two sections describe the machinery of kernels and pseudo-differential operators used to prove theorem 4.2 although a full proof is beyond the scope of this paper.

4.3. Schwartz Kernel Theorem, Smooth Kernels.

Before we explicitly construct the parametrix S, we need to know a few key results that will allow us to analyze the operators K_1, K_2. Let us assume we are working with oriented Riemannian manifolds and vector bundles with a fiberwise inner product.

Given a linear operator $\mathscr{K} : C_0^\infty(M;E) \to \mathscr{D}'(M;F)$ the Schwartz Kernel theorem tells us it corresponds uniquely to a distribution $K \in \mathscr{D}'(M \times M; \pi_1^* E \otimes \pi_2^* F)$ where π_1, π_2 are the projections onto the first and second variables. This correspondence is given by the formula

$$\langle \mathscr{K}\phi, \psi \rangle = \langle K, \phi \otimes \psi \rangle$$

In the above, $\phi \in C_0^\infty(M;E), \psi \in C_0^\infty(M;F)$ and where $\phi \otimes \psi \in C_0^\infty(M \times M; \pi_1^* E \otimes \pi_2^* F)$ is defined by

$$\phi \otimes \psi(x,y) = \phi(x) \otimes \psi(y)$$

In the context of elliptic regularity, the key result that comes from the study of kernels is that operators with smooth kernels are precisely those operators which can be extended to maps $\mathscr{E}'(M;E) \to C^\infty(M,F)$. We extend the correspondence by observing that if K is smooth $K(x,y)$ smoothly determines a map $E_x \to F_y$ if we identify $\mathrm{Hom}(E_x, F_y) \cong E_x \otimes F_y$ using the inner product structures on fibers. Thus for $\phi \in C_0^\infty(M;E)$ we have a well defined section:

$$(\mathscr{K}\phi)(y) = \int_M K(x,y)(\phi(x))dx \in F_y$$

More generally, if u is a compactly supported distribution in $\mathscr{E}'(M;E)$ then we may pair it with the section $K(\cdot, y)$ of $\mathrm{Hom}(E, F_y)$ to get an element of F_y. Note that this only extends to compactly supported distributions, as they are dual to all smooth sections, and thus may be paired with $K(\cdot, y)$ which may not have compact support. A local analysis of difference quotients in the above formula shows that we may, even in the distributional case, basically pass to differentiating under the integral, and get smoothness.

With these preliminaries in place, we see that to prove the regularity result we want, we need to construct S in such a way that the kernels of the maps K_1, K_2 are smooth.

4.4. **Parametrices, Pseudo-differential operators.** As an example, let $L = \sum a_\alpha \partial^\alpha$ be a differential operator $C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ with smoothly varying coefficients. And for $x, \xi \in \mathbb{R}^n$ let $L(x,\xi) = \sum a_\alpha(x)\xi^\alpha$. Then by utilizing a Fourier transform and inversion, we know that for $f \in C_0^\infty$ (or more generally Schwartz space) we have

$$Lf(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} L(x,\xi)\hat{f}(\xi)d\xi.$$

It can be checked that the appropriate kernel is

$$K(x,y) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle y-x,\xi\rangle} L(x,\xi)d\xi.$$

An operator $C_0^\infty(\mathbb{R}^n) \to C_0^\infty(\mathbb{R}^m)$ yields a similar formula with $L(x,\xi)$ now an $m \times n$ matrix of entries varying like a polynomial in ξ. If we hoped to invert such an operator, we might first hope to try something like:

$$Sf(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} \frac{1}{L(x,\xi)}\hat{f}(\xi)d\xi$$

In general this is a problem of singular integrals as $L(x,\xi)$ will almost always have zeroes or in the multivariable case the matrix will be singular, and thus $L(x,\xi)^{-1}$ is almost certainly not integrable.

However, if we have an ellipticity condition then locally for some $R > 0$, if $|\xi| > R$ then $L(x,\xi)$ will be nonsingular, as the homogenous terms will dominate. Thus, we choose a smooth map $q(x,\xi)$ that is equal to $L(x,\xi)^{-1}$ for large $|\xi|$, and consider the operator

$$Sf(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} q(x,\xi)\hat{f}(\xi)d\xi$$

The calculus of pseudo-differential operators gives substance to this argument. In the study of operators having local structure like the above, we can talk about the associated principal symbol $[q]$, which is in general an equivalence class of sections assigning to each $(x,\xi) \in T^*M$ a linear map $q(x,\xi) \in \mathrm{Hom}(E_x, F_x)$. We then ask that the symbol and its derivatives satisfy certain growth conditions in $|\xi|$. The symbol is of order l if it satisfies estimates of the form, in local coordinates

$$|\partial_\xi^\alpha \partial_x^\beta q(x,\xi)| \leq C_{\alpha,\beta}(1+|\xi|)^{l-|\alpha|}.$$

It is an exercise in Fourier analysis that this growth condition insures these operators map Sobolev spaces to Sobolev spaces in the appropriate way. The symbols do not transform well, so we need consider such symbols mod differences of a symbol of order $l-1$. Now for sufficiently nice operators L, S with principal symbols p, q that the principal symbols of LS, SL are pq, qp. Ellipticity of an operator L tells us that for the principal symbol p we can find a symbol q of order $-l$ such that $pq - I = 0 \mod S^{-1}$. We may thus find an operator S such that its principal symbol is q and then $LS = I - K$ for K pseudodifferential of order -1. Now, if it made sense, we would then define an operator

$$T = I + K + K^2 + K^3 \ldots.$$

We would then have:

$$L(ST) = I.$$

We are not quite so lucky, but we can assign a meaning to infinite sums like T up to an error of a smoothing operator. And thus we have, with K_1 smoothing:

$$L(ST) = I + K_1.$$

Theorem 4.2 and its corollaries follow immediately.

4.5. **Ellipticity of the Laplace Beltrami Operator.** In order to apply the machinery developed in the preceding sections we need to prove the Laplace-Beltrami operator is elliptic. We use the coordinate free definition adapted from Warner: checking that at any point x, for every p-form u with $\omega(x) \neq 0$ and every smooth function ϕ with $d\phi(x) \neq 0, \phi(x) = 0$ we have $\Delta(\phi^2\omega)(x) \neq 0$.

The proof, which can also be found in sections 6.34-6.35 of Warner [3] is a straightforward calculation that if $d\phi(x) = \xi \in TM_x^*$ then

$$\Delta(\phi^2\omega)(x) = -2[(-1)^{mp}{*\xi*\xi} + (-1)^{m(p-1)}\xi{*\xi*}](\alpha(m)) \tag{4.9}$$

One then studies the sequence

$$\Lambda^{p-1}TM_x^* \to \Lambda^p TM_x^* \to \Lambda^{p+1}TM_x^*.$$

The maps are given by left wedge product with ξ. This sequence is exact and we then use the fact that for an exact sequence of inner product spaces with maps A, B then the map $AA^* + B^*B$ is an isomorphism on the middle space. Computing the adjoints of multiplication by ξ shows that the term in brackets in equation 4.9 is $AA^* + B^*B$, thus an isomorphism, thus verifying ellipticity.

4.6. **Proof of the Hodge Decomposition.** Having established that $\mathcal{H}^p$ is finite dimensional, we have a decomposition

$$\Omega^p(M) = \mathcal{H}^p(M) \oplus (\mathcal{H}^p(M))^\perp \tag{4.10}$$

with respect to the L^2 inner product on forms. We thus need to show that $(H^p(M))^\perp = \Delta(\Omega^p(M))$. Because the Laplace-Beltrami operator is formally self-adjoint, the inclusion $\Delta\Omega^p(M) \subset (\mathcal{H}^p(M))^\perp$ is immediate.

Next we apply Corollary 4.7 and the compactness of the inclusion $L^2 = H^0 \hookrightarrow H^{-2}$ to see that if we have a bounded sequence of smooth forms $\{\omega_i\}$ with bounded Laplacians (both with respect to

L^2 norm) then a subsequence of the ω_i is Cauchy. It follows that we have an estimate using L^2 norms of the form

$$\|\omega\| \leq c\|\Delta\omega\| \tag{4.11}$$

for $\omega \in (\mathcal{H}^p)^\perp$. If not we would have a sequence with $\|\omega_i\|_{L^2} = 1$ in $(\mathcal{H}^p)^\perp$ with $\|\Delta\omega_i\|_{L^2} \to 0$. We may assume the sequence is Cauchy (it has a Cauchy subsequence by the above). Then the L^2 limit of the sequence is some form ω satisfying the equation

$$\langle \omega, \Delta\eta \rangle = 0 \tag{4.12}$$

for all η. The self adjointness of Δ in the L^2 pairing implies that $\Delta^t = \Delta$ in the sense of section 3.3. This implies that in the distributional sense $\Delta\omega = 0$. Further, $\omega \in \mathcal{H}^p$. This is impossible, as ω should have norm 1 under the quotient map $L^2/\mathcal{H}^p$. Finally we let $\omega \in (\mathcal{H}^p)^\perp$ and define a functional l on L^2 forms by first defining it on the range of Δ (a closed subspace)

$$l(\Delta\alpha) = \langle \omega, \alpha \rangle.$$

This is bounded by (4.10). We extend to a bounded functional on all of L^2 by Hahn-Banach which is represented by some $\eta \in L^2$ by Riesz representation, and $\Delta\eta = \omega$ in the distribution sense, thus in the classical sense by Corollary 4.3. It follows that $(\mathcal{H}^p)^\perp = \Delta\Omega^p$ and the Hodge decomposition is justified.

5. Applications

To show the power of the Hodge decomposition and associated machinery, we now turn to a few applications. The first is a quick proof of Poincaré Duality.

Theorem 5.1. *(Poincaré Duality) Let M be a compact, connected, oriented manifold of dimension m. There is a non-degenerate pairing $H^k_{dR}(M) \otimes H^{m-k}_{dR}(M) \to \mathbb{R}$.*

Proof. We endow M with some Riemannian metric. The Hodge decomposition theorem implies that the deRham cohomology groups are finite dimensional. Consider the pairing $H^k_{\mathrm{dR}} \otimes H^{m-k}_{\mathrm{dR}} \to \mathbb{R}$ given by wedge product and integration on forms

$$(\omega, \eta) \mapsto \int_M \omega \wedge \eta. \tag{5.2}$$

This pairing induces a well-defined pairing on cohomology. We may restrict to ω, η harmonic. We then claim that $*\Delta = \Delta*$, which is an exercise in the properties of the $*$ operator, so that the Hodge star preserves harmonic forms. Letting $\eta = *\omega$, we have $(\omega, \eta) \mapsto \|\omega\|_2^2$ which shows the pairing is non-degenerate, inducing an isomorphism on the vector spaces. $\square$

Our second application proves that we may reduce computations of the cohomology of a compact, connected Lie group to a computation about certain representations of their Lie algebra.

Theorem 5.3. *Let G be compact connected Lie group (of dimension m). Denote by $B^\bullet(G)$ the subcomplex of bi-invariant forms on G, that is forms invariant under both left and right multiplication by elements of G. Then the inclusion of complexes $B^\bullet(G) \hookrightarrow \Omega^\bullet(G)$ induces an isomorphism of graded-commutative algebras $B^\bullet(G) \cong H^\bullet_{dR}(G)$*

Proof. Any Lie group G is oriented, so we show that for an appropriate metric $B^\bullet(G) = \mathcal{H}^\bullet(G)$. Let $I_{g_1,g_2} : G \to G$ be the map ${}_{g_1,g_2}(h) = g_1 h g_2^{-1}$. Give G the unique two-sided Haar measure μ with total mass 1 and an arbitrary metric m_0. Then define a metric m by

$$m(X,Y) = \int_{G\times G} (I^*_{g_1,g_2} m_0)(X,Y) d(\mu(g_1) \times \mu(g_2)) \tag{5.4}$$

The new metric m is bi-invariant, and we claim that with respect to any bi-invariant metric the harmonic forms are precisely the bi-invariant ones.

Action by multiplication preserves orientation, metric, and commutes with exterior derivative. Thus I_{g_1,g_2} commutes with the Hodge star and the Laplace-Beltrami operator. So it maps harmonic forms to harmonic forms. Further, since G is connected and thus smoothly path connected, and $G \times G$ is as well, we have a smooth path from (e, e) to (g_1, g_2) inducing a smooth homotopy between I_{g_1,g_2} and the identity. Let ω be a harmonic k-form, $I^*_{g_1,g_2}\omega = \omega$ on the level of cohomology, but both are harmonic, so the equality holds as forms. Thus ω is bi-invariant.

Let $f : G \to G$ be the inversion map. f^* maps left-invariant forms to right-invariant forms and vice versa, thus preserving bi-invariance. Since $Df(e)$ is the negative identity on the tangent space at the identity of G, for a bi-invariant k-form ω we have $f^*\omega = (-1)^k\omega$. This leads to the following:

$$(-1)^{k+1} d\omega = f^*(d\omega) = df^*\omega = (-1)^k d\omega \tag{5.5}$$

So ω is automatically closed. Finally, suppose $\omega = d\eta$ is exact. Then we have by differentiation under the integral sign:

$$\omega = \int_{G\times G} d(I^*_{g_1,g_2}\eta)\ d(\mu\times\mu) = d\left(\int_{G\times G} I^*_{g_1,g_2}\eta\ d(\mu\times\mu)\right) \tag{5.6}$$

that ω is the exterior derivative of a bi-invariant form, thus is zero. This shows the uniqueness of bi-invariant representatives of cohomology classes, completing the proof. $\square$

The above theorem reduces the computation of $H^\bullet_{\mathrm{dR}}(G)$ to an algebraic problem about the Lie algebra $\mathfrak{g}$ and its adjoint representation. Indentifying exterior dual powers $\Lambda^k\mathfrak{g}^*$ with left-invariant k-forms, copies of the trivial representation in exterior dual powers of the adjoint representation correspond to conjugation-invariant k-forms. A left-invariant form is bi-invariant if and only if it is conjugation invariant. It follows that the multiplicity of the trivial representation is the dimension of $H^k_{\mathrm{dR}}(G)$.

6. Conclusion

As exhibited in Section 5, the Hodge decomposition for a compact, oriented, Riemannian manifold can simplify proofs by allowing one to reduce a question about cohomology to a question about the finite dimensional space of harmonic forms. In addition, the machinery built here may be generalized in a number of directions. When studying complex vector bundles over a complex manifold, similar techniques decompose the deRham complex by holomorphic and antiholomorphic forms. One application is the finite dimensionality of the space of holomorphic sections of a complex vector bundle E.

Another generalization of this machinery is the index problem for elliptic operators on compact manifolds. The machinery of section 4 establishes that an elliptic operator L between vector bundles over a compact oriented manifold is Fredholm as a map on appropriate Sobolev spaces. Equivalent to the defining condition that a Fredholm operator is invertible up to compact error is the condition that it have finite dimensional kernel, cokernel and closed range. Thus we may ask what the index of L is where index is defined as

$$\mathrm{ind}L := \dim\ker L - \dim\mathrm{coker}L \tag{6.1}$$

.

A preliminary result, which can be found in Section 19.2 of Hörmander [2] is that $\mathrm{ind}L$, or equivalently $\dim\mathrm{coker}L$ does not depend on which Sobolev space we work in. In a final generalization, this line of inquiry eventually lead to the Atiyah-Singer index theorem which relates the above analytic index to a topological index defined in terms of certain cohomology classes of the vector bundles.

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