ANALYSIS OF CHAOTIC SYSTEMS

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Abstract. This paper serves as an introduction to the analysis of chaotic systems, with techniques being developed by working through two famous examples. The first is the logistic map, a first-order discrete dynamical system, and the second is the Lorenz system, a three-dimensional system of differential equations.

Contents

1. Introduction

Chaos is an umbrella term for various complex behaviors of solutions to relatively simple, deterministic systems. The study of chaos came into its own in the 1970s, with the work of Edward Lorenz, which we will detail in Sections 5 and 6. Lorenz was developing a model of convection in the atmosphere and found a remarkable property - solutions through nearby points would diverge from each other extremely rapidly, but also remain confined within a certain space. This was later found to be characteristic of many other systems, thus beginning the study of chaotic dynamics. Before we discuss the Lorenz system in this paper, we will develop some definitions and techniques by considering the logistic map, a one-dimensional discrete model of population dynamics in Section 3. The goal of this paper is to demonstrate some of the remarkable properties of chaotic systems through these two examples and show in Section 4 that certain properties of chaotic behavior are actually universal and can actually be seen in many experimental systems. This paper is meant to

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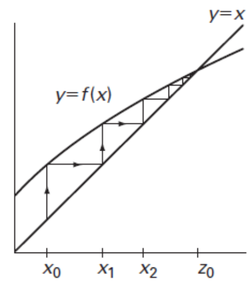

FIGURE 1. An Example of Graphical Iteration

give the reader only a flavor of the subject. A more complete treatment of chaotic systems can be found in the references.

2. Chaos in Discrete Dynamical Systems

Definition 2.1. 1) Let $x \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be a function that is C^∞. A *first-order discrete dynamical system* is a map of the form:

$$x_{n+1} = f(x_n)$$

2) The *orbit* of x is the sequence

$$x_0 = x, x_1 = f(x), \dots, x_n = f^n(x), \dots$$

3) We call x_0 a *fixed point* if $f(x_0) = x_0$.
4) We call x_0 a *periodic point of period* n if $f^n(x_0) = x_0$.

Figure 1 depicts the *graphical iteration* technique of visualizing orbits. If x_0 is an initial condition, we first start from the point (x_0, x_0) on the line $y = x$ and move vertically to intersect f at the point (x_0, x_1). Then, move horizontally to (x_1, x_1). Continuing in this manner, we can find the entire orbit.

Definition 2.2. Let $x_0 \in \mathbb{R}$ be a fixed point of f. If there is a neighborhood U of x_0 such that if $y \in U$, $f^n(y) \in U$ for all n and $\lim_{n\to\infty} f^n(y) = x_0$, then x_0 is *stable*. If for all $y \in U \backslash \{x_0\}$, $f^n(y)$ leaves U for some n, then x_0 is *unstable*.

The following proposition gives a simple test to determine the nature of the fixed points.

Proposition 2.3. *Let x_0 be a fixed point for f. Then:*
1) x_0 is stable if $|f'(x_0)| < 1$.
2) x_0 is unstable if $|f'(x_0)| > 1$.

Proof. 1) Let $|f'(x_0)| = v < 1$. Choose K such that $v < K < 1$. Since f' is continuous, there exists $\delta > 0$ such that $|f'(x)| < K$ for all $x \in [x_0 - \delta, x_0 + \delta]$. By the mean value theorem, for any $x \in [x_0 - \delta, x_0 + \delta]$, we have

$$\frac{f(x) - f(x_0)}{x - x_0} = \frac{f(x) - x_0}{x - x_0} = f'(c) \text{ for some } c \text{ between } x \text{ and } x_0.$$

Therefore, we have $|f(x) - x_0| < K|x - x_0| < |x - x_0|$ and by iteration, we find that $|f^n(x) - x_0| < K^n|x - x_0|$ so we have $\lim_{n\to\infty} f^n(x) = x_0$. The proof of (2) is similar. □

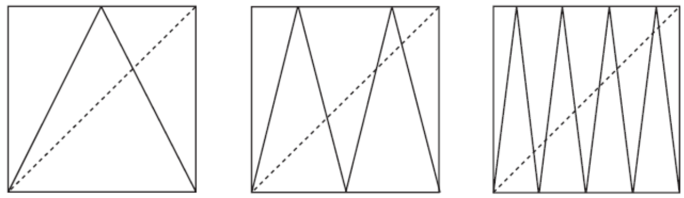

FIGURE 2. First, Second and Third Iterate of the Tent Map

Definition 2.4. $|f'(x_0)|$ is called the *stability coefficient* of x_0.

Next, we will define what it means to act chaotically. For simplicity, we will restrict the domain of f to act on an interval of $\mathbb{R}$.

Definition 2.5. Let f be a map from an interval $I = [a, b]$ to itself. f is *chaotic* if:

1) Periodic points of f are dense in I.

2) (Transitivity) Given any two open subintervals U_1 and U_2 in I, there is a point $x_0 \in U_1$ and an $n > 0$ such that $f^n(x_0) \in U_2$.

3) (Sensitive Dependence on Initial Conditions) There exists a *sensitivity constant*, which we we will denote as $\beta > 0$, such that for any $x_0 \in I$ and any open interval U about x_0, there is some $y_0 \in U$ and $n > 0$ such that $|f^n(x_0) - f^n(y_0)| > \beta$.

Proposition 2.6. *A dense orbit implies transitivity.*

Proof. Such an orbit will visit any open subinterval. $\square$

Example 2.7. The tent map $T : [0, 1] \to [0, 1]$ is a piecewise linear map given by

$$T(x) = \begin{cases} 2x & \text{if } 0 \leq x < \frac{1}{2} \\ -2x + 2 & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

The plot of T, under n compositions with itself, consists of 2^n lines alternating between slopes of 2 and -2. See Figure 2.

Proposition 2.8. *T is chaotic on* $[0, 1]$.

Proof. 1) (Density of Periodic Solutions) T^n maps each interval $\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]$ to $[0, 1]$ for $k = 0, 1, \ldots, 2^n - 1$. Therefore, T^n intersects the line $y = x$ once in each interval (this can easily be verified in Figure 2). As a result, each interval contains a fixed point of T^n or equivalently, a periodic point of T of period n. Therefore, periodic points of T are dense in $[0, 1]$.

2) (Transitivity) Let U_1 and U_2 be open subintervals of $[0, 1]$. For n sufficiently large and for some k, U_1 contains an interval of the form $\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]$. Therefore, T^n maps U_1 to $[0, 1]$ which contains U_2.

3) (Sensitive Dependence on Initial Conditions) Let $x_0 \in [0, 1]$. We will show that a sensitivity constant $\beta = \frac{1}{2}$ works. As in (2), any open interval U about x_0 is mapped by T^n to $[0, 1]$ for some sufficiently large n. Therefore, there exists $y_0 \in U$ such that $|f^n(x_0) - f^n(y_0)| \geq \frac{1}{2} = \beta$. $\square$

Definition 2.9. Let a discrete dynamic system depend on a parameter c. A *bifurcation* occurs when there is a "significant" change in the orbits of the system as c

varies. This informal definition will suffice for the purposes of this paper. Specific types of bifurcations will be described in greater detail when they arise.

Example 2.10. Let $f_c(x) = x^2 + c$. Solving $x^2 + c = x$ to find the fixed points yields the solutions $x_\pm = \frac{1 \pm \sqrt{1-4c}}{2}$. Hence, there are no fixed points if $c > \frac{1}{4}$, one fixed point when $c = \frac{1}{4}$ and two fixed points when $c < \frac{1}{4}$. $c = \frac{1}{4}$ is called a *saddle-node* or *tangent bifurcation* because two fixed points coincide and then annihilate each other.

3. The Logistic Map

The logistic population model is often used as an example in the study of ordinary differential equations and one that is simple to solve explicitly. Alternatively, we can consider how populations change from generation to generation. As we will show, the discretized form of the logistic equation displays much more complex behavior than its continuous analogue.

Definition 3.1. The *logistic map* is given by:

$$f(x_n) = x_{n+1} = rx_n(1 - x_n)$$

where $r \geq 0$ is a parameter that captures the growth rate of the population. We will investigate how the behavior of this system changes as r is varied.

3.1. $0 \leq r \leq 4$.

Proposition 3.2. *For $0 \leq r \leq 4$, the logistic map sends $[0, 1]$ to itself.*

Proof. In $[0, 1]$, the logistic equation attains maximum $\frac{r}{4}$ at $x = \frac{1}{2}$. If $0 \leq r \leq 4$, for any initial condition $x_0 \in [0, 1]$, $x_n \in [0, 1]$ for all n. $\square$

First, we will find the fixed points of f.

Proposition 3.3. *For $r < 1$, 0 is the only fixed point and it is stable for $r < 1$ and unstable for $r > 1$. For $r > 1$, there is an additional fixed point $x = 1 - \frac{1}{r}$ that is stable for $1 < r < 3$ and unstable for $r > 3$.*

Proof. Solving $rx(1 - x) = x$ to find the fixed points yields $x_1 = 0$ or $x_2 = 1 - \frac{1}{r}$. The latter is inside $[0, 1]$ for $r \geq 1$ (the two points coincide at $r = 1$).

We will use Proposition 2.3 to examine the stability of these points. Taking the derivative, we find that $f'(x) = r(1 - 2x)$ and plugging in the first fixed point, we get $f'(0) = r$. Therefore, 0 is stable for $r < 1$ and unstable for $r > 1$.

For the other fixed point, we have $f'\left(1 - \frac{1}{r}\right) = 2 - r$. Therefore, $1 - \frac{1}{r}$ is stable for $1 < r < 3$ and unstable for $r > 3$. $\square$

Proposition 3.4. *For $r > 3$, the logistic map has a 2-cycle that is stable if $r < 1 + \sqrt{6}$.*

Proof. A 2-cycle is a set of points $p \neq q \in [0, 1]$ such that $f(p) = q$ and $f(q) = p$. Therefore, $f^2(p) = p$ and $f^2(q) = q$ so p and q are fixed points of f^2. Solving $f^2(x) = r^2x(1 - x)[1 - rx(1 - x)] = x$ yields four solutions:

$$x_1 = 0, x_2 = 1 - \frac{1}{r}, x_3 = \frac{r + 1 + \sqrt{(r-3)(r+1)}}{2r} \text{ and } x_4 = \frac{r + 1 - \sqrt{(r-3)(r+1)}}{2r}.$$

The first two are the fixed points of f. The last two solutions are new and form a 2-cycle if $r > 3$.

Showing that a 2-cycle is stable is equivalent to showing that p and q are stable fixed points of f^2. This means that all points in $[0,1] \setminus \left\{0, 1-\frac{1}{r}\right\}$ tend to this 2-cycle.

$$\text{Let } \lambda = \frac{d}{dx}(f(f(x))|_{x=p} = f'(f(p))f'(p) = f'(q)f'(p) \text{ by the chain rule.}$$

By symmetry, this equals $\frac{d}{dx}(f(f(x))|_{x=q}$. Plugging in p and q gives:

$$\lambda = r^2(1-2q)(1-2p) = r^2[1-2(p+q)+4pq] = r^2\left[1 - 2\frac{r+1}{r} + 4\frac{r+1}{r^2}\right] = 4+2r-r^2.$$

Solving $|\lambda| < 1$ shows that the 2-cycle is stable for $3 < r < 1+\sqrt{6}$. □

Propositions 3.3 and 3.4 show that as r increases, the stability coefficient of the fixed point at $x = 1 - \frac{1}{r}$ decreases from 1 at $r = 1$ to -1 at $r = 3$, at which point it becomes unstable. At the same time, a stable 2-cycle is born. For this reason, $r = 3$ is called a *period doubling bifurcation*. By the same proof, the 2-cycle also becomes unstable leading to the creation of a stable 4-cycle at $r = 1+\sqrt{6}$. Therefore, a 2^n cycle appears for all n. We will denote r_n as the smallest r for which a 2^n cycle first appears. Computer approximations have yielded the following values:

$$\begin{aligned} r_1 &= 3 \\ r_2 &= 3.449\ldots \\ r_3 &= 3.54409\ldots \\ r_4 &= 3.5644\ldots \\ r_5 &= 3.568759\ldots \end{aligned}$$

The sequence converges to a limiting value $r_\infty = 3.569946\ldots$. More will be said on the nature of this convergence in Section 4. As r increases past r_∞, the behavior of solutions becomes chaotic. One technique for visualizing this transition to chaos is the *orbit diagram* that is constructed by choosing a random initial condition x_0 and then constructing the orbit of x_0 for some value of r. We plot the points of the orbit, discarding the first several hundred points so that we capture the eventual behavior of the orbit and not the transient initial behavior. Repeating this for many values of r yields Figure 3.

One of the most remarkable features of the orbit diagram is that it is not entirely chaotic for $r > r_\infty$. As we will see in the next proposition, a stable 3-cycle is born around $r = 3.8284\ldots$. These intervals of stability are called *periodic windows*.

Proposition 3.5. *For* $3.8284\ldots \leq r \leq 3.8415\ldots$, *there is a stable 3-cycle.*

Proof. The proof is nearly the same as that of Proposition 3.4, but with f^3 in the place of f^2. □

When the 3-cycle becomes unstable, a period doubling bifurcation occurs and a stable 6-cycle is born. As before, we can find a stable (3×2^n)-cycle for all n. As a result, if one branch of the stable 3-cycle is magnified, the orbit diagram reappears in miniature. Said another way, the orbit diagram exhibits a fractal structure.

We will now consider choosing r just below the periodic window. Figure 4 shows a typical orbit for $r = 3.8282$. The orbit nearly forms a 3-cycle with interspersed periods of chaos. This phenomenon is called *intermittency*.

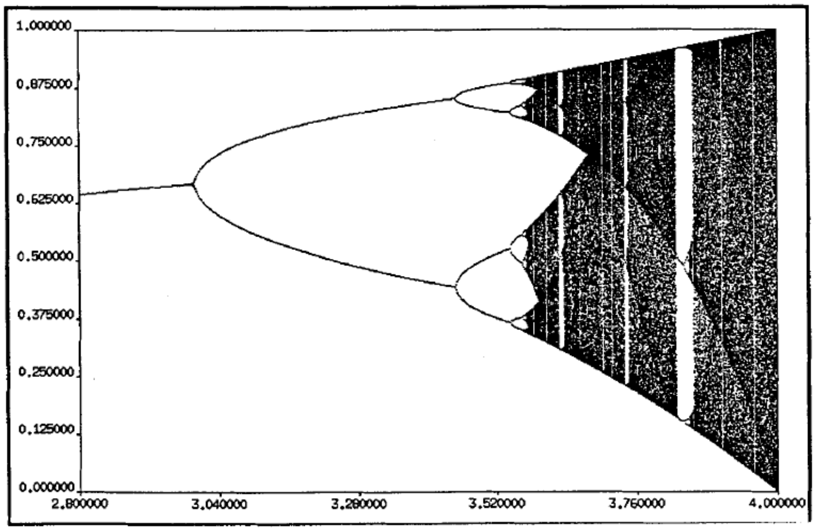

FIGURE 3

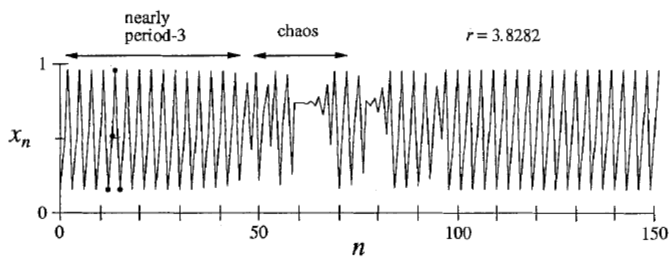

FIGURE 4

We will offer a pictoral explanation for this phenomenon. Figure 5(a) is the plot of f^3. Figure 5(b) depicts the magnification of the boxed section of Figure 5(a). There is a narrow channel between f^3 and the line $y = x$ so an orbit passing through requires a large number of iterations and during the crossing, we have $f^3(x_n) \approx x_n$.

As r moves further away from the periodic window, we expect to see less and less intermittency until fully chaotic behavior is once again achieved. This is called the *intermittency route to chaos.*

For $r > r_\infty$, there are unstable 2^n-cycles for all n. Therefore, the periodic points of f are dense. From the orbit diagram, we can see that a "typical" orbit appears to be dense on some interval smaller than $[0, 1]$ for $r < 4$. The final component of chaotic behavior which we have not yet discussed is sensitive dependence on initial conditions.

Definition 3.6. Let x_0 be some initial condition and consider the orbit of a nearby point $x_0 + \delta_0$ where δ_0 is very small. Let δ_n be the separation in the two orbits after n iterates. If $|\delta_n| \approx |\delta_0| e^{n\lambda}$, then λ is called the *Liapunov exponent.* A positive

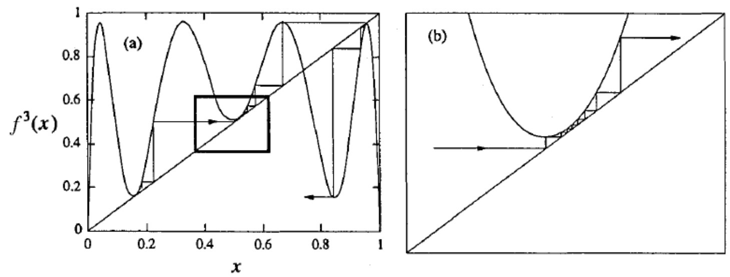

FIGURE 5

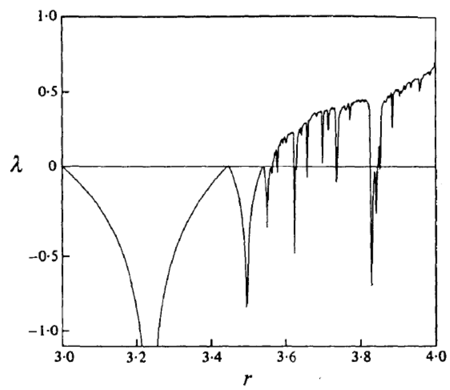

FIGURE 6

Liapunov exponent means that orbits separate exponentially fast, which is sufficient to show that the system displays sensitive dependence on initial conditions.

Figure 6 shows a plot of the Liapunov exponent for the logistic map for different values of r. Note that for $r < r_\infty$, we have $\lambda < 0$ since we don't expect behavior to be chaotic. The negative spikes for $r > r_\infty$ correspond to the periodic windows.

3.2. $r \geq 4$.
The goal of this section will be to prove that the logistic map is chaotic for $r > 4$. This case is complicated by the fact that orbits of points in $[0, 1]$ can leave $[0, 1]$. Therefore, we will need a new technique called conjugacy.

Definition 3.7. Let I, J be intervals and $f : I \to I$, $g : J \to J$ be maps. f and g are *conjugate* if there exists a homeomorphism $h : I \to J$ that satisfies the *conjugacy equation* $h \circ f = g \circ h$. f and g are *semiconjugate* if h is at most $n - to - 1$.

Remark 3.8. h maps orbits of f to orbits of g. If x_0 is an initial condition in I, then $h(x_0)$ is an initial condition in J and by iterating the conjugacy equation, we have $h(f^n(x_0)) = g^n(h(x_0))$.

Example 3.9. The tent map defined in Example 2.9 is semiconjugate to the logistic equation for $r = 4$ via $h(x) = \frac{1}{2}(1 - \cos(2\pi x))$ since $h(T(x)) = \frac{1}{2}(1 - cos(4\pi x) =$

$\frac{1}{2} - \frac{1}{2}(2cos^2(2\pi x - 1) = 1 - cos^2(2\pi x) = 4\frac{1}{2}(1 - cos(2\pi x))\frac{1}{2}(1 + cos(2\pi x)) = 4h(x)(1 - h(x)) = f(h(x))$.

Theorem 3.10. *Let f and g be conjugate via h. If f is chaotic on I, then g is chaotic on J.*

Proof. 1) (Density of Periodic Points) Let U be an open subinterval of J and consider $h^{-1}(U) \subset I$. Since periodic points are dense in I, there exists a periodic point of period n, which we will denote $x \in h^{-1}(U)$. By the conjugacy equation, $g^n(h(x)) = h(f^n(x)) = h(x)$. Therefore, $h(x)$ is a periodic point of period n in U and periodic points are dense in J.

2) (Transitivity) Let U and V be open subintervals of J. By continuity, $h^{-1}(U)$ and $h^{-1}(V)$ are open subintervals of I. By the transitivity condition for f, there exists $x_0 \in h^{-1}(U)$ such that $f^m(x_0) \in h^{-1}(V)$ for some m. Then, $h(x_0) \in U$ and $g^m(h(x_0)) = h(f^m(x_0)) \in V$.

3) (Sensitive Dependence on Initial Conditions) Let f have sensitivity constant β. Let $I = [\alpha_0, \alpha_1]$. Assume $\beta < \alpha_1 - \alpha_0$. For all $x \in [\alpha_0, \alpha_1 - \beta]$, consider the function $|h(x + \beta) - h(x)|$. This function is continuous and positive, so it has minimum value, which we will denote β^*. Therefore, h maps intervals of length β to intervals of length at least β^*. We claim g has sensitivity constant β^*. Let $x_0 \in J$ and V be an open interval about x_0. Then, $h^{-1}(V)$ is an open interval about $h^{-1}(x_0)$. By sensitive dependence on initial conditions, there exists $y_0 \in h^{-1}(V)$ and $n > 0$ such that $|f^n(h^{-1}(x_0)) - f^n(y_0)| > \beta$. Then,

$$|h(f^n(h^{-1}(x_0)) - h(f^n(y_0)| = |g^n(x_0) - g^n(h(y_0))| > \beta^*.$$

□

Remark 3.11. The above proof does not require that the preimages of any of the intervals be unique and therefore, also works for semiconjugacies.

Corollary 3.12. *The logistic map is chaotic for $r = 4$ by the semiconjugacy in Example 3.9 (recall that we have already shown that the tent map is chaotic in Proposition 2.8).*

Definition 3.13. 1) Let f be the logistic map for $r > 4$. Let $A_0 \subset [0, 1]$ denote the open interval on which we have $f > 1$. The orbits of points in A_0 will be mapped to a point > 1, then to a point < 0 and then will tend to $-\infty$. The preimage of A_0 under f consists of two open intervals, one on each side of A_0, the union of which we will denote A_1. Continuing in this fashion, we denote A_n as the 2^n disjoint open intervals in $[0, 1]$ whose nth iterate lies in A_0.

2) Let Λ denote the initial conditions whose orbits stay in $[0, 1]$ for all time. Since any point that lies in an A_n eventually leaves $[0, 1]$, we have:

$$\Lambda = I \setminus \bigcup_{n=0}^{\infty} A_n.$$

3) Let $I_0 = \Lambda \cap \left[0, \frac{1}{2}\right]$ and $I_1 = \Lambda \cap \left[\frac{1}{2}, 1\right]$. Let $x_0 \in \Lambda$. Then, the orbit of x_0 lies in $I_0 \cup I_1$ for all time. The *itinerary* of x_0 is the sequence $S(x_0) = (s_0 s_1 s_2 \ldots)$ where $s_j = k \Leftrightarrow f^j(x_0) \in I_k$.

The methodology is to show that the logistic map is conjugate to a chaotic map and then use Theorem 3.10 but first, we will require a few preliminary definitions and propositions.

Definition 3.14. Let Σ be the set of all sequences of 0's and 1's. Let $s, t \in \Sigma$. Let

$$d(s,t) = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i}.$$

Remark 3.15. As expected, $d(s,t)$ is a metric on Σ. We will not need this fact for the purposes of the paper but the proof is fairly straightforward.

Proposition 3.16. *$d(s,t)$ converges for all $s, t \in \Sigma$.*

Proof. If $s_i \neq t_i$ for all i, then $d(s,t) = \sum\limits_{i=0}^{\infty} \frac{1}{2^i} = 2$. By the comparison test, the sum converges for all $s, t \in \Sigma$. □

Proposition 3.17. *Let $s, t \in \Sigma$. $s_j = t_j$ for $j = 0, 1, \ldots, n \Leftrightarrow d(s,t) \leq \frac{1}{2^n}$.*

Proof. 1) ($\Rightarrow$) $d(s,t) = \sum\limits_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i} = 0 + \sum\limits_{i=n+1}^{\infty} \frac{|s_i - t_i|}{2^i} \leq \frac{1}{2^{n+1}} \sum\limits_{i=0}^{\infty} \frac{1}{2^i} = \frac{1}{2^n}$.

2) ($\Leftarrow$) If $s_j \neq t_j$ for some $j \leq n$, then $d(s,t) \geq \frac{1}{2^j} \geq \frac{1}{2^n}$. □

Next, we will define the shift map on Σ and show that it is chaotic.

Definition 3.18. The *shift map* $\sigma : \Sigma \to \Sigma$ is defined by $\sigma(s_0 s_1 s_2 \ldots) = (s_1 s_2 s_3 \ldots)$.

Theorem 3.19. *σ is chaotic on Σ.*

Proof. 1) (Density of Periodic Points) Let $s = (s_0 s_1 s_2 \ldots)$. Let $\epsilon > 0$ and choose n sufficiently large such that $\frac{1}{2^n} < \epsilon$. Construct $t = (s_0 s_1 s_2 \ldots s_n s_0 s_1 s_2 \ldots)$. Then, t is periodic with period $n+1$ and is within ϵ of s by Proposition 3.17.

2) (Transitivity) Construct $s^* = (0\ 1\ 00\ 01\ 10\ 11\ 000 \ldots)$ by successively listing each block of length 1, then 2, and so on. We claim that the orbit of s^* is dense, which implies transitivity by Proposition 2.6. If $s \in \Sigma$, the orbit of s^* will eventually share the first n elements with s for any n.

3) (Sensitive Dependence on Initial Conditions) We will show that a sensitivity constant $\beta = 2$ works. Let $\widehat{s_n}$ denote the opposite of s_n. Let $s \in \Sigma$ and construct $s' = \left(s_0 s_1 \ldots s_n \widehat{s_{n+1}} \widehat{s_{n+2}} \ldots\right)$. Then, we have $d(s, s') = \frac{1}{2^n}$ and $d(\sigma^n(s), \sigma^n(s')) = 2 = \beta$. □

Finally, we will show that the shift map is conjugate to the logistic map via the itinerary function.

Theorem 3.20. *The itinerary function $S : \Lambda \to \Sigma$ is a homeomorphism if $r > 2 + \sqrt{5}$.*

Proof. For these values of r, we have $|f'(x)| > K > 1$ for some K and for all $x \in I_0 \cup I_1$.

1) (One-to-One) Let $x, y \in \Lambda$ and suppose $S(x) = S(y)$. This means that $f^n(x)$ and $f^n(y)$ lie on the same side of $\frac{1}{2}$ for all n. Assume $x \neq y$. Since we have $|f'| > K > 1$, each iteration expands the distance between $f^n(x)$ and $f^n(y)$ by a factor of at least K, so eventually they cannot both lie on the same side of $\frac{1}{2}$. This contradicts the fact that they have the same itinerary.

2) (Onto) Let $s = (s_0 s_1 s_2 \ldots) \in \Sigma$. We will define

$$I_{s_0 s_1 \ldots s_n} = \{x \in I | x \in I_{s_0}, f(x) \in I_{s_1}, \ldots, f^n(x) \in I_{s_n}\} = I_{s_0} \cap f^{-1}(I_{s_1}) \cap \ldots \cap f^{-n}(I_{s_n}).$$

In other words, $I_{s_0 s_1 \ldots s_n}$ contains the initial conditions whose itineraries are $(s_0 s_1 \ldots s_n \ldots)$. Note that $I_{s_0 s_1 \ldots s_n} = I_{s_0} \cap f^{-1}(I_{s_1 \ldots s_n})$. By induction, we may assume $I_{s_1 \ldots s_n}$ is a nonempty, closed subinterval, so $f^{-1}(I_{s_1\ldots s_n})$ consists of two closed subintervals, one in I_0 and one in I_1. Hence, $I_{s_0} \cap I_{s_1\ldots s_n}$ consists of a single, nonempty closed subinterval. Furthermore, the $I_{s_0 s_1\ldots s_n}$ are nested since we have:

$$I_{s_0 s_1\ldots s_n} = I_{s_0 s_1\ldots s_{n-1}} \cap f^{-n}(I_{s_n}) \subset I_{s_0 s_1\ldots s_{n-1}}.$$

By the Nested Interval Theorem, $\bigcap\limits_{n=0}^{\infty} I_{s_0 s_1\ldots s_n}$ is nonempty and therefore, there exists an $x \in \Lambda$ such that $S(x) = s$.

3) (Continuous) Let $x \in \Lambda$ and suppose $S(x) = (s_0 s_1 \ldots)$. Let $\epsilon > 0$ and pick n sufficiently large so that we have $\frac{1}{2^n} < \epsilon$. Consider the closed subinterval $I_{s_0 s_1\ldots s_n}$ defined in (2). Then, there exists $\delta > 0$ such that if $|x-y| < \delta$, then $y \in I_{s_0 s_1\ldots s_n}$. Then, we have $d(S(x), S(y)) \leq \frac{1}{2^n} < \epsilon$ by Proposition 3.17 since $S(x)$ and $S(y)$ agree in the first $n+1$ terms.

4) (Continuous Inverse) The proof is very similar to (3) and will be omitted. $\square$

Theorem 3.21. *σ is conjugate to f via the itinerary function S.*

Proof. We have shown in Theorem 3.20 that S is a homeomorphism, so all that is left to show is that the conjugacy equation is satisfied. Let $x_0 \in \Lambda$ and $S(x_0) = (s_0 s_1 s_2 \ldots)$. Then, we have $x_0 \in I_{s_0}, f(x_0) \in I_{s_1}, f^2(x_0) \in I_{s_2}$ and so on. Then, $S(f(x_0)) = (s_1 s_2 s_3 \ldots) = \sigma(S(x_0))$. $\square$

Corollary 3.22. *The logistic map is chaotic for $r > 2 + \sqrt{5}$.*

4. Universality

While the chaotic behavior of the logistic map is interesting on its own, the results are significantly less powerful if they cannot be generalized. The purpose of this section is to examine some of the universal features present in the logistic map, from both a qualitative and a quantitative perspective.

4.1. The U-Sequence.

Definition 4.1. A map is *unimodal* if it is smooth, concave down and has a single maximum.

Metropolis et. al (1973) considered maps of the form $x_{n+1} = rf(x_n)$ on $[0,1]$, where f is unimodal and $f(0) = f(1) = 0$. They proved that as r is varied, the order in which stable periodic solutions appear is independent of the choice of unimodal map. This gives rise to a sequence, called the *U-sequence*, that the periodic attractors occur in. Omitting periods higher than 6, the U-sequence is 1, 2, 4, 6, 5, 3, 6, 5, 6, 4, 6, 5, 6. The first three elements of the sequence represent the initial period-doubling route to chaos. The element 3 represents the beginning of the periodic window we found in Proposition 3.5.

Example 4.2. The sine map $x_{n+1} = r\sin(\pi x_n)$ for $0 \leq r \leq 1$ is unimodal on $[0,1]$. The orbit diagram is shown in Figure 7. At a quick glance, it looks remarkably similar to the orbit diagram for the logistic map.

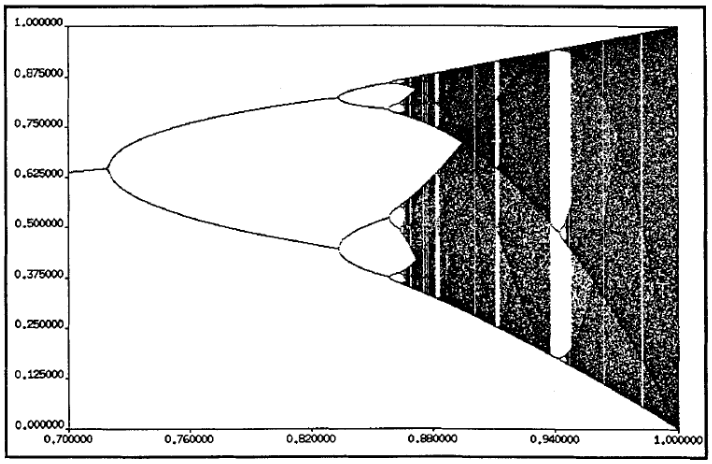

FIGURE 7

4.2. **The Feigenbaum Numbers.** Recall that we had previously defined r_n as the lowest value of r for which the logistic map has a 2^n-cycle.

Definition 4.3. Let $\delta_n = \frac{r_n - r_{n-1}}{r_{n+1} - r_n}$. Then, $\delta = \lim_{n\to\infty} \delta_n = 4.66920161\ldots$ is called the *Feigenbaum delta*.

This constant convergence rate is independent of the choice of unimodal map. Experiments in fluid convection, diode circuits, optical bistability and acoustic waves have measured values of δ within about 20% of the true value. The difficulty is that the distance between successive r_n quickly shrinks and it becomes difficult to distinguish between the r_n's after $n = 3$ or 4.

Remark 4.4. One experimental use of δ is to estimate the value of r_∞ by the following approximation, which remarkably only requires the accurate measurements of r_1 and r_2.

$\delta \approx \frac{r_2 - r_1}{r_3 - r_2} \Rightarrow r_3 \approx r_2 + \frac{r_2 - r_1}{\delta}$.

Similarly, $r_4 \approx r_3 + \frac{r_3 - r_2}{\delta} = r_2 + \frac{r_2 - r_1}{\delta} + \frac{r_2 - r_1}{\delta^2} = r_2 + (r_2 - r_1)\left(\frac{1}{\delta} + \frac{1}{\delta^2}\right)$.

In general, we have $r_n \approx r_2 + (r_2 - r_1)\sum_{i=1}^{n-2} \frac{1}{\delta^i}$ and by definition, we have $r_\infty = \lim_{n\to\infty} r_n = r_2 + (r_2 - r_1)\left(\frac{1}{\delta - 1}\right)$.

Definition 4.5. Let x_m denote the maximum of f and let d_n denote the distance from x_m to the nearest point in a 2^n-cycle (see Figure 8). Then, $\alpha = \lim_{n\to\infty} \frac{d_n}{d_{n+1}} = -2.5029\ldots$ is called the *Feigenbaum alpha*. As with the Feigenbaum delta, this constant is also independent of the choice of unimodal map.

5. THE LORENZ SYSTEM

In 1963, Edward Lorenz, a meteorologist, developed a simple model of convection in the atmosphere. By holding most of the variables constant, he was able to reduce the system to three-dimensional state space and he found the first chaotic system.

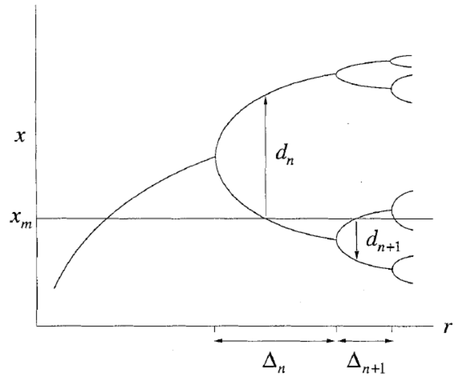

FIGURE 8

In this section, we will present a few elementary properties of the Lorenz system in order to try to visualize the geometry of the solutions.

Definition 5.1. 1) A *n-dimensional, autonomous system of ordinary differential equations* is a system

$$\frac{dx_1}{dt} = f_1(x_1(t), x_2(t), \ldots, x_n(t))$$
$$\vdots$$
$$\frac{dx_n}{dt} = f_n(x_1(t), x_2(t), \ldots, x_n(t))$$

We will adopt the shorthand $X = F(X)$ where X is the vector $(x_1, x_2, \ldots, x_n)$.

2) $X = (x_1, \ldots, x_n)$ is an *equilibrium point* if $f_i(x_1, \ldots, x_n) = 0$ for all i. X is *stable* or a *sink* if there is a neighborhood U of X such that if $Y \in U$, the solution through Y stays in U for all time and tends to X. X is *unstable* or a *source* if for all $Y \in U \backslash \{X\}$, the solution through Y eventually leaves U. Otherwise, X is a *saddle-node.*

Remark 5.2. This section assumes familiarity with n-dimensional linear systems of ordinary differential equations and in particular, how to determine if an equilibrium point of a linear system is a source, sink or saddle-node from the eigenvalues. An excellent introductory text is [1]. We will also make use of the following lemma. The details of the proof can be found in [1, ch.8].

Lemma 5.3. *Let X_0 be an equilibrium solution to $X = F(X)$. The* linearization *about X is the system $Y' = DF_{X_0}Y$ where DF_{X_0} refers to the Jacobian matrix evaluated at the point X_0. Linearization preserves the nature of equilibria.*

Definition 5.4. The Lorenz system is the 3-dimensional, autonomous system of ordinary differential equations given by:

$$x' = \sigma(y - x)$$
$$y' = rx - y - xz$$
$$z' = xy - bz$$

where $\sigma > 0$ is the Prandtl number, $r > 0$ is the Rayleigh number and $b > 0$ is a constant related to the size of the system. We also require $\sigma > b + 1$. The derivation of the Lorenz system and an explanation of the constants is contained in [3, Sec.1.5].

We will now list and prove some of the properties of the Lorenz system. First, we will show that solutions to the system are either symmetric or have a symmetric partner.

Proposition 5.5. *If $(x(t), y(t), z(t))$ is a solution, so is $(-x(t), -y(t), z(t))$ (i.e. reflection through the z-axis preserves the vector field).*

Proof. Let $(x(t), y(t), z(t))$ be a solution. Then, we have

$$x'(t) = \sigma(y(t) - x(t)) \Rightarrow -x'(t) = \sigma(-y(t) - (-x(t)).$$
$$y'(t) = rx(t) - y(t) - x(t)z(t) \Rightarrow -y'(t) = r(-x(t)) - (-y(t)) - (-x(t))z(t).$$
$$z'(t) = x(t)y(t) - bz(t) \Rightarrow z'(t) = (-x(t))(-y(t)) - bz(t).$$

Therefore, $(-x(t), -y(t), z(t))$ is also a solution. □

Next, we will determine the nature of equilibria solutions.

Proposition 5.6. *The origin is an equilibrium point that is a sink for $0 \le r < 1$.*

Proof. It is clear that if $x = y = z = 0$, then $x' = y' = z' = 0$. By the lemma above, the linearization about the origin gives the system $Y' = \begin{pmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{pmatrix} Y$ which has eigenvalues $-b, \lambda_\pm = \frac{-(\sigma+1)\pm\sqrt{(\sigma+1)^2-4\sigma(1-r)}}{2}$. If $0 \le r < 1$, then $\lambda_\pm < 0$ so the origin is a sink. In fact, all solutions will tend to the origin, so the origin is called a *global attractor.* □

Proposition 5.7. *If $r > 1$, there are two additional equilibria at $Q_\pm = (\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1)$. $Q_\pm$ are sinks if $1 < r < \sigma\left(\frac{\sigma+b+3}{\sigma-b-1}\right)$.*

Proof. It is easy to check that these are equilibria. By linearizing around $Q_\pm$, we find that the eigenvalues satisfy the polynomial:

$$f(\lambda) = \lambda^3 + (1 + b + \sigma)\lambda^2 + b(\sigma + r)\lambda + 2b\sigma(r - 1) = 0.$$

When $r = 1$, the roots are $0, -b$ and $-\sigma - 1$. For $r > 1$, we expect the roots to be close to these values and in particular, to be real. Noting that if $r > 1$ and one of the roots $\lambda \ge 0$, then $f(\lambda) > 0$. Therefore, the three roots of f must be negative. This proves the lower bound. To prove the upper bound, $Q_\pm$ become unstable when one of the eigenvalues has zero real part. This cannot happen for real roots, so the roots must take the form $\pm i\omega$ with $\omega \neq 0$. Solving $f(i\omega) = 0$ yields the desired result. □

These equilibria are surrounded by an unstable limit cycle visualized in Figure 9. As $r \to \sigma\left(\frac{\sigma+b+3}{\sigma-b-1}\right)$, the limit cycle shrinks until it is absorbed by the fixed point, which then becomes a saddle-node. This is known as a *Hopf bifurcation.* A partial bifurcation diagram is shown in Figure 10, where solid lines indicate a stable periodic solution and dashed lines indicate an unstable periodic solution. We see that there is no stable periodic solution for $r > \sigma\left(\frac{\sigma+b+3}{\sigma-b-1}\right)$. One may naively guess

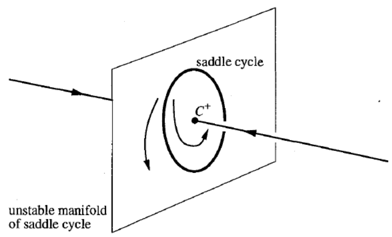

FIGURE 9

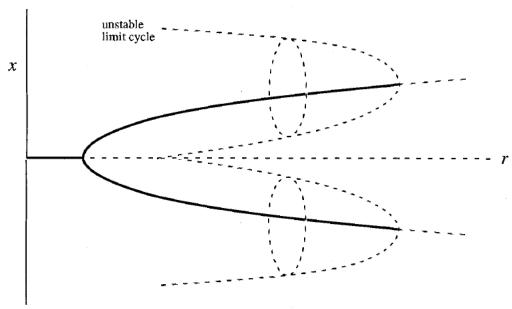

FIGURE 10. Bifurcation Diagram

that trajectories after this point are repelled out to infinity, but this is not the case, as we will see in the next proposition.

Proposition 5.8. *Let* $V(x, y, z) = rx^2 + \sigma y^2 + \sigma(z - 2r)^2 = v$ *be the ellipsoid centered at* $(0, 0, 2r)$. *There exists* v^* *such that any solution that starts outside* $V(x, y, z) = v^*$ *eventually enters and stays there for all time.*

Proof. We want to show that for $v \geq v^*$, we have $V' < 0$.

$$V' = -2\sigma(rx^2 + y^2 + b(z^2 - 2rz)) = -2\sigma(rx^2 + y^2 + b(z - r)^2 - br^2).$$

The equation $rx^2 + y^2 + b(z - r)^2 = 2br^2$ is an ellipsoid. Choose v^* large enough so that the ellipsoid $V = v^*$ strictly contains this ellipsoid in its interior. □

6. Chaos on the Lorenz Attractor

In the previous section, we saw that all solutions eventually congregate into a space which we will define as the Lorenz attractor. In this section, we will use some of the techniques developed for the logistic map to describe the chaotic behavior of solutions.

Definition 6.1. A closed set A is an *attractor* if:

1) A is invariant. 2) There is an open set U containing A such that if $X \in U$, then the distance from X to A tends to zero as $t \to \infty$. In other words, A attracts an open set of initial conditions. 3) A is minimal (i.e. there is no proper subset of A that satisfies both (1) and (2)).

An attractor is strange if it has a fractal structure. An attractor is chaotic if it exhibits sensitive dependence on initial conditions.

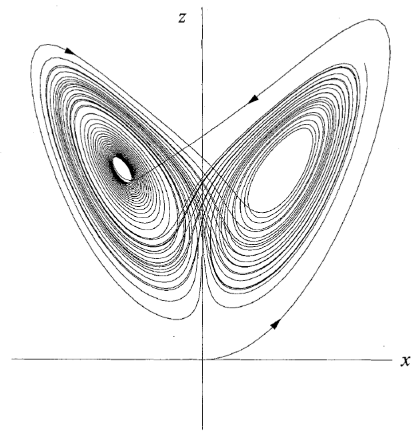

Figure 11. The Lorenz Attractor

Figure 11 displays one solution that has been projected onto the xz-plane, which gives us a sense of the geometrical shape of the Lorenz attractor. The trajectory spirals around Q_- several times before crossing over to spiral around Q_+. It continues to switch back and forth for all time, with the number of circuits made on each side varying as if random.

The first interesting property of the Lorenz attractor is that it has no volume, which we will demonstrate in the following proposition.

Proposition 6.2. *The Lorenz system contracts volumes.*

Proof. Suppose D is a region in $\mathbb{R}^3$ with a smooth boundary and let $D(t)$ denote the image of D as it evolves according to the Lorenz equation. Let $V(t)$ be the volume of $D(t)$. By Liouoville's theorem, we have:

$$\frac{dV}{dt} = \int\limits_{D(t)} \operatorname{div} F \, dx \, dy \, dz$$

where the divergence of a vector field $F(X)$ is given by div $F = \sum\limits_{i=1}^{3} \frac{\partial F_i}{\partial x_i}(X)$.

Plugging in the divergence for the Lorenz system, we have:

$$\frac{dV}{dt} = -(\sigma + 1 + b)V \Rightarrow V(t) = e^{-(\sigma+1+b)t}V(0).$$

Therefore, any volume contracts exponentially fast to 0. A system that contracts volumes is called *dissipative.* □

Corollary 6.3. *The volume of the Lorenz attractor is 0.*

Next, we will show that the Lorenz attractor is chaotic by redefining Liapunov exponents for an n-dimensional system of differential equations.

Definition 6.4. Consider the evolution of an infinitesimal sphere of initial conditions. The sphere becomes an infinitesimal ellipsoid. Let $\delta_k(t)$ denote the length of the kth principal axis of the ellipsoid. If $\delta_k(t) \approx \delta_k(0)e^{\lambda_k t}$, then the λ_k are called the *Liapunov exponents.* Let $\lambda = \max\{\lambda_k\}$.

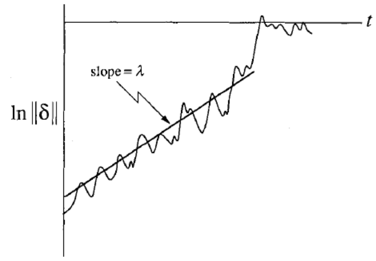

FIGURE 12

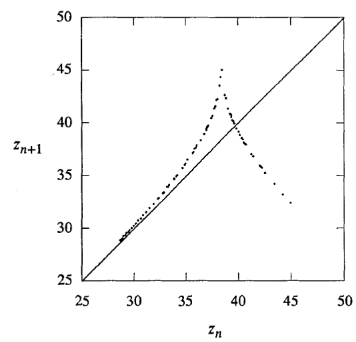

FIGURE 13. The Lorenz Map

Remark 6.5. Let $x(0)$ be a point on the attractor and let $x(0) + \delta(0)$ be a nearby point. Let $\delta(t)$ be the separation in the two orbits at time t. Then, $|\delta(t)| \approx |\delta(0)|e^{\lambda t}$. A plot of $\ln|\delta(t)|$ versus t is shown in Figure 12. The initial slope is $\lambda \approx 0.9$. Eventually, the plot saturates, or levels off, since the maximum distance possible on the attractor is achieved. As before, a positive λ indicates sensitive dependence on initial conditions. Therefore, the Lorenz attractor is chaotic.

Next, we will examine the Lorenz map, a beautiful result that emerges out of the chaotic behavior.

Definition 6.6. Lorenz noted that the trajectory appeared to only leave one spiral after reaching a threshold distance from the center. Moreover, the amount by which the threshold is exceeded seemed to determine the number of spirals in the next circuit. Lorenz defined z_n to be the nth local maximum of $z(t)$ and plotted z_{n+1} versus z_n. The result is plotted in Figure 13. We will denote this function f and define the *Lorenz map* as $z_{n+1} = f(z_n)$.

It is difficult to distinguish between chaotic behavior and a trajectory that eventually settles down and becomes periodic. No argument has yet been made that has shown that the latter case does not occur for the Lorenz system. However, we can show, in following proposition, that if the Lorenz system has any closed orbits, they must necessarily be unstable.

Proposition 6.7. *If the Lorenz system has any closed orbits, they are unstable.*

Proof. Consider the sequence of $\{z_n\}$ corresponding to a closed orbit. The sequence must eventually repeat, so let $z_{n+p} = z_n$. Consider the fate of a perturbation to the closed orbit that passes through $z_n + \eta_n$. Let η_t denote the distance from the closed orbit. Then, by linearization around z_n, we have:

$$z_{n+1} + \eta_{n+1} = f(z_n + \eta_n) \approx f(z_n) + f'(z_n)\eta_n = z_{n+1} + f'(z_n)\eta_n \Rightarrow \eta_{n+1} \approx f'(z_n)\eta_n.$$

Similarly, we have:

$$\eta_{n+2} \approx f'(z_{n+1})\eta_{n+1} = f'(z_{n+1})f'(z_n)\eta_n$$

After p iterations, we have:

$$\eta_{n+p} \approx \left[\prod_{i=0}^{p-1} f'(z_{n+i}\right] \eta_n$$

From the plot of the Lorenz map,we can see that $|f'(z)| > 1$ for all z. Then, $|\eta_{n+p}| > |\eta_n|$, which proves that the closed orbit is unstable. □

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