POLYNOMIALS IN THE DIRICHLET PROBLEM

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Abstract. The Dirichlet problem concerns solutions of the following system:

$$\begin{cases} \Delta u = 0 & \text{in } D \\ u = f & \text{on } \partial D \end{cases}$$

for a domain D with a sufficiently smooth boundary ∂D. While this problem has been solved generally in terms of Green's functions, we shall investigate specific conditions on the data function f and the domain D which permit alternative methods of solutions, using tools of algebra. We will give a brief overview for the case of polynomial data on ellipsoids, before approaching the problem of recovering harmonic functions from data on conic sections.

Contents

1. Introduction

The purpose of this paper is to consider the Dirichlet problem, posed in a non-traditional way on varieties in $\mathbb{R}^n$, and to determine conditions for existence and uniqueness. Classically, the Dirichlet problem is as follows:

Given a region D in $\mathbb{R}^n$ with boundary ∂D and function $f : \partial D \to \mathbb{R}$, is there a unique function $u : \bar{D} \to \mathbb{R}$ such that u is harmonic ($\Delta u = 0$) in the interior and $u = f$ on the boundary?

Given the utility of the laplacian to mathematical physics, analysis, and geometry, this question has been thoroughly investigated. General solutions exist in terms of elementary PDE techniques, given a sufficiently smooth boundary ∂D and continuous f, determined by the computation of an appropriate Green's function via solutions to Fredholm integral equations. Moreover, uniqueness follows easily by the maximum principle:

Theorem 1.1. *Solutions to the Dirichlet problem posed on some connected open subset $D \subseteq \mathbb{R}^n$ are unique.*

Proof. Suppose u, v solve the Dirichlet problem for some data function f on the boundary ∂D. It follows that $u - v$ is harmonic, and identically zero along ∂D. Moreover, the maximum principle guarantees that $u-v$ achieves its maximum along ∂D, so that $u - v \leq 0$ in the domain D. We have $u - v \geq 0$ from the corresponding minimum principle, yielding the equality $u - v = 0$ for all values inside D. Thus $u = v$, so uniqueness follows from existence. □

Existence and uniqueness aside, a common question is the nature of solutions which arise given the domain D and data function f. For example, if f is some polynomial and D is an open ball in $\mathbb{R}^n$, what can be said about the resulting solution u? This question is difficult to address through the obscurity of integral equations, so we instead turn to the algebra of *finite dimensional* vector spaces.

1.1. **Ellipsoids.** Given a polynomial ring $R = k[x_1, ..., x_n]$ over some field k, the homogeneous polynomials of degree d form a vector space R_d (under polynomial addition) for any nonnegative integer d. A convenient basis $\mathcal{B}_d$ for each space R_d is simply the collection of monomials of degree d, so that:

$$|\mathcal{B}_d| = \binom{n+d-1}{d} \tag{1.2}$$

It is clear that $\mathcal{B}_{d_1} \cap \mathcal{B}_{d_2} = \varnothing$ if $d_1 \neq d_2$ (because $R = \oplus_{d=0}^{\infty} R_d$ is a graded algebra). We define P_m to be the vector space of polynomials degree m and less.

$$P_m = \{f \in R : \deg f \leq m\} = \operatorname{span}_k(\cup_{d=0}^{m} \mathcal{B}_d) \tag{1.3}$$

We compute its dimension by evaluating the sum from 0 to m

$$\dim P_m = \sum_{d=0}^{m} |\mathcal{B}_d| = \sum_{d=0}^{m} \binom{n+d-1}{d} = \binom{n+m}{m} \tag{1.4}$$

Now consider the polynomial

$$q(x_1, ..., x_n) = \frac{{x_1}^2}{{\alpha_1}^2} + \cdots + \frac{{x_n}^2}{{\alpha_n}^2} - 1, \text{ for some constants } \alpha_i \in \mathbb{R}^+ \tag{1.5}$$

which allows us to define the domain $D \subseteq \mathbb{R}^n$ and elliptical boundary ∂D:

$$\begin{aligned} D &= \{(x_1, ..., x_n) \in \mathbb{R}^n : q(x_1, ..., x_n) < 0\} \\ \partial D &= \{(x_1, ..., x_n) \in \mathbb{R}^n : q(x_1, ..., x_n) = 0\} \end{aligned} \tag{1.6}$$

Our primary focus is the inversion of the operator $T : P_m \to P_m$, given by:

$$f \mapsto T(f) = \Delta(qf) \tag{1.7}$$

Lemma 1.8. *The operator T is linear and invertible.*

Proof. Linearity is clear; it suffices to show that T is injective. Suppose that $Tf = 0$, and set $u = qf$. It follows that u is harmonic, because $\Delta u = \Delta(qf) = Tf = 0$, and $u = 0$ automatically along the ellipse defined by q. Applying the maximum principle and minimum principle in turn shows u is identically zero, as before. Because the ring $\mathbb{R}[x_1, ..., x_m]$ is an integral domain, $qf = 0$ means $f = 0$. Thus $\ker T = \{0\}$. □

Theorem 1.9. *There exists a unique solution $u \in P_m$ to the Dirichlet problem.*

Proof. We claim $u = f - qT^{-1}\Delta f$ is a solution. This is verified by computing

$$\Delta u = \Delta f - \Delta(qT^{-1}\Delta f) = \Delta f - TT^{-1}\Delta f = \Delta f - \Delta f = 0 \tag{1.10}$$

and by noting that, on the ellipse defined by q, we have $u = f$. □

The computation above, while simplistic, is somewhat remarkable. Not only have we shown that the solution u can be computed directly – without the use of Green's functions – but we have shown that u is in fact a *polynomial* strictly by methods of linear algebra. Furthermore, we have the restriction $\deg u \leq \deg f$, simply by inspection of the solution.

Example 1.11. Take $q(x, y) = \frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} - 1$, for some $\alpha, \beta \in \mathbb{R}^+$ in the plane. Supposing that the data function f is some degree 2 polynomial, we may consider $T : P_2 \to P_2$. We now must determine the action of T upon each basis element (the collection of monomials degree two and lower in x and y):

$$T(1) = \Delta(q) = 2\left(\frac{1}{a^2} + \frac{1}{b^2}\right) \tag{1.12}$$

$$T(x) = \Delta(xq) = 2x\left(\frac{3}{a^2} + \frac{1}{b^2}\right) \tag{1.13}$$

$$T(y) = \Delta(yq) = 2y\left(\frac{1}{a^2} + \frac{3}{b^2}\right) \tag{1.14}$$

$$T(x^2) = \Delta(x^2q) = \frac{2(a^2 + 6b^2)}{a^2b^2}x^2 + \frac{2}{b^2}y^2 - 2 \tag{1.15}$$

$$T(xy) = \Delta(xyq) = 6xy\left(\frac{1}{a^2} + \frac{1}{b^2}\right) \tag{1.16}$$

$$T(y^2) = \Delta(y^2q) = \frac{2}{a^2}x^2 + \frac{2(6a^2 + b^2)}{a^2b^2}y^2 - 2 \tag{1.17}$$

Thus we have the matrix:

$$M_T = \begin{pmatrix} \frac{2}{a^2} + \frac{2}{b^2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{6}{a^2} + \frac{2}{b^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{a^2} + \frac{6}{b^2} & 0 & 0 & 0 \\ -2 & 0 & 0 & \frac{2(a^2+6b^2)}{a^2b^2} & 0 & \frac{2}{b^2} \\ 0 & 0 & 0 & 0 & \frac{6}{a^2} + \frac{6}{b^2} & 0 \\ -2 & 0 & 0 & \frac{2}{a^2} & 0 & \frac{2(6a^2+b^2)}{a^2b^2} \end{pmatrix} \tag{1.18}$$

To show invertibilty, we evaluate the determinant:

$$\det M_T = 48\left(\frac{3}{a^2} + \frac{1}{b^2}\right)\left(\frac{1}{a^2} + \frac{3}{b^2}\right)\left(\frac{2(a^2 + 6b^2)}{a^2b^2}\frac{2(6a^2 + b^2)}{a^2b^2} - \frac{2}{b^2}\frac{2}{a^2}\right)\left(\frac{1}{a^2} + \frac{1}{b^2}\right)^2$$

Nearly all the above terms are obviously nonzero. We only must consider the second to last term to ensure M_T is nondegenerate.

$$\left(\frac{2(a^2+6b^2)}{a^2b^2}\frac{2(6a^2+b^2)}{a^2b^2}-\frac{2}{b^2}\frac{2}{a^2}\right)=\frac{24}{a^4b^4}(a^4+6a^2b^2+b^4) \tag{1.19}$$

which is also clearly positive, upon partial expansion. Therefore the linear map T is invertible, for all $a, b \in \mathbb{R}$, and we conclude that there exists some polynomial $v \in P_2$ such that $v = T^{-1}f$ for any $f \in P_2$.

Showing $\det M_T$ is nonzero for all combinations of n and m would clearly be an exhaustive exercise. Recall, however, that the maximum principle *guarantees* $\ker T = \{0\}$. Moreover, because T is an endomorphism and P_m is finite dimensional, it follows that T is invertible, and thus we are guaranteed a unique solution. This series of implications will not be so trivial in later sections, and the $n, m = 2$ determinant approach will turn out to be useful in answering the question of general invertibility in the negative.

2. Conic Sections

Now we consider a different problem from Dirichlet's. Instead of finding a function u which is harmonic inside a given domain and matches perscribed conditions upon the boundary, we shall address the following:

Pseudo-Dirichlet Problem Let $q \in \mathbb{R}[x, y]$ define an algebraic set.

$$X = \{(x, y) \in \mathbb{R}^2 : q = 0\} \tag{2.1}$$

We seek a polynomial u satisfying the following:

$$\begin{cases} \Delta u = 0 & \text{for all } (x, y) \in \mathbb{R}^2 \\ u = f & \text{for all } (x, y) \in X \end{cases} \tag{2.2}$$

A word on the polynomial q is in order. It is inconvenient to consider a polynomial of degree higher than 2, otherwise the map $Tf = \Delta(qf)$ no longer maps $P_m \to P_m$. So the algebraic sets X in question are the *conic sections*. The natural question follows: for which conic sections do solutions to this new problem exist?

2.1. Determinant Approach.

First we shall examine the conic sections by considering the matrix M_T. While this will often not allow us to answer the question of which conic sections permit solutions to the pseudo-Dirichlet problem, we can at least begin to understand and conjecture by analyzing low-degree examples.

Example 2.3. Single Line

Consider $q = ax+by+1$, so the map T takes $P_m \to P_{m-1}$. Recall that $\dim P_m = \frac{1}{2}(m+1)(m+2)$, so the difference in dimension between domain and codomain is precisely $\frac{1}{2}(m+1)(m+2) - \frac{1}{2}m(m+1) = m+1$. We shall first consider $m = 2$, with the traditional basis of monomials yielding the following matrix:

$$M_T = \begin{pmatrix} 0 & 0 & 0 \\ 2a & 0 & 0 \\ 2b & 0 & 0 \\ 2c & 6a & 2b \\ 0 & 2b & 2a \\ 2c & 2a & 6b \end{pmatrix} \tag{2.4}$$

The columns are linearly independent, so the map $T : P_2 \to P_1$ is full rank. However, it is also clear that solutions are non-unique. We shall show both these statements are true in general.

Example 2.5. Hyperbola

Take $q = ax^2 + by^2 - 1$ and the same basis as before. We shall consider $m = 2$:

$$M_T = \begin{pmatrix} 2a+2b & 0 & 0 & 0 & 0 & 0 \\ 0 & 6a+2b & 0 & 0 & 0 & 0 \\ 0 & 0 & 2a+6b & 0 & 0 & 0 \\ -2 & 0 & 0 & 12a+2b & 0 & 2b \\ 0 & 0 & 0 & 0 & 6a+6b & 0 \\ -2 & 0 & 0 & 2a & 0 & 2a+12b \end{pmatrix} \tag{2.6}$$

$$\det M_T = 1152(a+b)^2(3a+b)(a+3b)(a^2+6ab+b^2)$$

We must have the sign of a opposite to that of b. We thus have the solutions:

$$\begin{gathered} a = -b \\ a = -3b \\ a = -\tfrac{1}{3}b \\ a = (2\sqrt{2}-3)b \\ a = -(2\sqrt{2}+3)b \end{gathered}$$

which yield a degenerate determinant. These indicate five distinct hyperbola for which T is not invertible. It follows that T is *not generally invertible for any* $m \geq 2$, because expanding our basis with higher degree monomials still yields a non-invertible matrix for the above critical values.

We claim that the "amount" of hyperbolas for which the determinant is singular is in some way "small." In fact, we can view the determinant of M_T as a homogeneous polynomial in the coefficients a and b, neither of which is permitted to be zero, and thus the solutions $\det M_T = 0$ can be counted on the projective line $\mathbb{R}P^1$:

$$\begin{gathered} (1:-1) \\ (1:-3) \\ (3:-1) \\ (1:2\sqrt{2}-3) \\ (1:-2\sqrt{2}-3) \end{gathered} \tag{2.7}$$

The degree of $\det M_T$ is precisely $\frac{1}{2}(m+1)(m+2)$, placing a upper bound on the space of hyperbolas uncooperative with the pseudo-Dirichlet problem for any given $f \in P_m$. We shall show that these singularities only arise in special cases, irregardless of m.

Example 2.8. Crossing Lines

We have $q = ax^2 + by^2$. The only difference between this and the hyperbola is in the lower-order terms resulting from the constant term in the previous example. This discrepancy is undetected by the determinant, however, as seen when we take $m = 2$ and compare with the previous matrix:

$$M_T = \begin{pmatrix} 2a+2b & 0 & 0 & 0 & 0 & 0 \\ 0 & 6a+2b & 0 & 0 & 0 & 0 \\ 0 & 0 & 2a+6b & 0 & 0 & 0 \\ 0 & 0 & 0 & 12a+2b & 0 & 2b \\ 0 & 0 & 0 & 0 & 6a+6b & 0 \\ 0 & 0 & 0 & 2a & 0 & 2a+12b \end{pmatrix} \tag{2.9}$$

These matrices have the same determinant in general, so uncooperative cases of the crossing lines correspond directly to the pathological hyperbolas (and vice versa).

Example 2.10. Parabola

Take $q = ax + by^2 - 1$ and the basis $\{1, x, y, x^2, xy, y^2\}$, which gives the matrix:

$$M_T = \begin{pmatrix} 2b & 0 & 0 & 0 & 0 & 0 \\ 2a & 2b & 0 & 0 & 0 & 0 \\ 0 & 0 & 6b & 0 & 0 & 0 \\ -2 & 6a & 0 & 2b & 0 & 2b \\ 0 & 0 & 2a & 0 & 6b & 0 \\ -2 & 2a & 0 & 0 & 0 & 12b \end{pmatrix} \tag{2.11}$$

We evaluate the determinant:

$$\det M_T = 3456b^6$$

which is nondegenerate. This leads us to conjecture that parabolas, like ellipses, always offer a solution to the pseudo-Dirichlet problem, which we hope is unique.

Example 2.12. Parallel Lines

Without loss of generality, take $q = by^2 - 1$, so that the determinant of the corresponding matrix matches that of the parabolic case. Thus the case of parallel lines goes hand in hand with parabolas.

We have pushed the determinant approach to its limit, short of programming a computer to investigate determinants for larger values of m in some Collatz-esque fashion, and must now turn to a different method for answering the pseudo-Dirichlet problem. Next we shall take the question "what do our solutions look like?" more seriously, and investigate the polynomials which comprise them.

2.2. **Harmonic Polynomials.** Suppose we have some second-degree polynomial $q \in \mathbb{R}[x, y]$ defining the algebraic set X, on which we consider the pseudo-Dirichlet problem. Suppose, further, that q is chosen so that a (unique) solution u exists for any given f. We know that such a q can be chosen – any ellipse will do, for example. Now define the map $P : \mathbb{R}[x, y] \to \mathbb{R}[x, y]$ as the map specified by $Pf = u$.

Proposition 2.13. *The map P is linear. Moreover, P is a projection.*

Proof. Linearity is known from previous sections, and $P^2 = P$ follows from the fact that a harmonic data function f trivially satisfies the pseudo-Dirichlet problem (that is, $Pf = f$ if $\Delta f = 0$).

□

It is sensible, then, to imagine that P projects onto the space of harmonic polynomials in $\mathbb{R}[x,y]$. Denote by R_n the polynomials of degree n, so that $\mathbb{R}[x,y] = \bigoplus_{n=0}^{\infty} R_n$ can be regarded as a graded ring, and note that $\dim R_n = n+1$. Consider the laplacian restricted to the polynomials of degree n:

$$\begin{gathered} \Delta_0 : R_0 \to 0 \\ \Delta_1 : R_1 \to 0 \\ \Delta_n : R_n \to R_{n-2} \text{ for } n \geq 2 \end{gathered} \tag{2.14}$$

Because we can write a polynomial f as the sum of its homogeneous components, $f = \sum_{n=0}^{\infty} f_n$, we can reconstruct the laplacian as $\Delta = \sum_{n=0}^{\infty} \Delta_n$, with the obvious inclusion maps to $\mathbb{R}[x,y]$ omitted. Notice that $\dim R_n - \dim R_{n-2} = (n+1)-(n-1) = 2$ for any $n \geq 2$, so it follows that $\operatorname{nul} \Delta_n \geq 2$.

Example 2.15. $\Delta_5 : R_5 \to R_3$ has nullity 2.

Proof. We choose the bases $\{x^5, x^4y, x^3y^2, x^2y^3, xy^4, y^5\}$ and $\{x^3, x^2y, xy^2, y^3\}$:

$$\begin{gathered} x^5 \mapsto 20x^3 \\ x^4y \mapsto 12x^2y \\ x^3y^2 \mapsto 6x^3 + 2xy^2 \\ x^2y^3 \mapsto 2y^3 + 6x^2y \\ xy^4 \mapsto 12xy^2 \\ y^5 \mapsto 20y^3 \end{gathered} \tag{2.16}$$

This yields the matrix

$$\begin{pmatrix} 20 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 \\ 6 & 0 & 2 & 0 \\ 0 & 6 & 0 & 2 \\ 0 & 0 & 12 & 0 \\ 0 & 0 & 0 & 20 \end{pmatrix}$$

which is full rank. Thus we have $\operatorname{rank} \Delta_5 = 4$ and $\operatorname{nul} \Delta_5 = 2$. □

Theorem 2.17. *Δ_n is onto, for all $n \geq 0$.*

Proof. The claim is trivial for $n = 0, 1$, because $\Delta_n : R_n \to 0$. Now suppose that $n \geq 2$. We choose the basis $B_n = \{x^n, x^{n-1}y, x^{n-2}y^2, ..., x^2y^{n-2}, xy^{n-1}, y^n\}$ of monomials. Acting Δ_n on the basis elements, we have:

$$\begin{gathered} x^n \mapsto n(n-1)x^{n-2} \\ x^{n-1}y \mapsto (n-1)(n-2)x^{n-3}y \\ x^{n-2}y^2 \mapsto (n-2)(n-3)x^{n-4}y^2 + 2x^{n-2} \\ \vdots \\ x^{n-k}y^k \mapsto (n-k)(n-k-1)x^{n-k-2}y^k + k(k-1)x^{n-k}y^{k-2} \\ \vdots \\ x^2y^{n-2} \mapsto (n-2)(n-3)x^2y^{n-4} + 2y^{n-2} \\ xy^{n-1} \mapsto (n-1)(n-2)xy^{n-3} \\ y^n \mapsto n(n-1)y^{n-2} \end{gathered} \tag{2.18}$$

The representative matrix is full-rank, so Δ_n is onto. Moreover, nul $\Delta_n = \begin{cases} 1 & n = 0 \\ 2 & n \geq 1 \end{cases}$.

□

Corollary 2.19. *The harmonic subspace of P_n has dimension $2n + 1$.*

Thus the operator P taking f to u, when restricted to polynomials of degree n or less, takes vectors in a space of dimension $\frac{1}{2}(n + 1)(n + 2)$ to a $2n + 1$-dimensional subspace. The following question naturally arises: what does a basis for this subspace look like?

Theorem 2.20. *Let $\varphi_k = \mathrm{Re}[(x + iy)^k]$ and $\psi_k = \mathrm{Im}[(x + iy)^k]$, for any positive integer k. Then the collection of polynomials $\{1, \varphi_1, \psi_1, \varphi_2, \psi_2, ...\}$ is a basis for the harmonic subspace of $\mathbb{R}[x, y]$.*

Proof. Firstly, $\varphi_1 = x$ and $\psi_1 = y$, which are obviously linearly independent. Next is $\varphi_2 = x^2 - y^2$ and $\psi_2 = xy$, and so on. We shall proceed by induction: if the collection $\{1, \varphi_1, \psi_1, ..., \varphi_n, \psi_n\}$ is linearly independent, we claim that adding φ_{n+1} and ψ_{n+1} to the set does not spoil independence. Indeed, it is clear that φ_{n+1} and ψ_{n+1} cannot be constructed by the previous terms, by arguments of degree; we must show a nontrivial combination $\alpha\varphi_{n+1} + \beta\psi_{n+1}$ cannot yield a lower degree polynomial to show independence. But this is clear, because φ_{n+1} contains a term x^{n+1} and ψ_{n+1} does not.

Secondly, the claim that φ_k and ψ_k are harmonic follow as a consequence of complex analysis. Recall that a differentiable function $f : \mathbb{C} \to \mathbb{C}$ satisfies the Cauchy-Riemann equations, and moreover has harmonic real and imaginary part. The map z^k is differentiable for all positive k, so φ_k and ψ_k are harmonic.

Lastly, the collections $\{1, \varphi_1, \psi_1, ..., \varphi_n, \psi_n\}$ has cardinality $2n + 1$, precisely matching our computation for the subspace of harmonic polynomials in P_n.

□

2.3. **Constructing Solutions.**

Proposition 2.21. *The single line yields a (non-unique) solution to the pseudo-Dirichlet problem.*

Proof. We have three cases: $x = c$, $y = mx + b$, and $y = c$. In the latter two instances, evaluating the data function f on the given line gives $f(x, y(x)) = g(x)$, for some polynomial g of degree $\deg f$, and the first case follows from the third by symmetry. Thus we can consider X to be the line $y = 0$ without loss of generality.

In this case, we have the immediate solution $u(x, y) = \mathrm{Re}[g(x + iy)] = \mathrm{Re}[f(x + iy, 0)]$. The harmonic properties of u follow because $g(z) = g(x + iy)$ is analytic, and we have $u|_X = u(x, 0) = \mathrm{Re}[g(x)] = g(x) = f(x, 0) = f|_X$ This solution is not unique; for any polynomial $h(x)$ with $h(0) = 0$, take $u_h(x, y) = \mathrm{Im}[h(x+iy)]$. Then u_h has only odd powers of y, so evaluating $u_h|_X = u_h(x, 0) = 0$, and $u + u_h$ also solves the pseudo-Dirichlet problem.

□

Proposition 2.22. *The parallel lines and parabola yield a unique solution to the pseudo-Dirichlet problem.*

Proof. We shall give a method to compute the solution u in the case of parallel lines.

Consider the parallel lines given by the zero set of $y^2 - a$ for some $a \in \mathbb{R}^+$, along with a data polynomial f of degree n. The canonical homomorphism into the ring of polynomials $\mathbb{R}[x, y]/\langle y^2 - a\rangle$ takes f to an equivalence class $[\tilde{f}]$, where $\tilde{f}$ is the polynomial f with y^2 identified to a. That is, $\tilde{f}$ is a polynomial with only linear terms in y, such that $f = \tilde{f}$ when restricted to the parallel lines $y = \pm\sqrt{a}$. Therefore the highest degree homogeneous component of $\tilde{f}$ is exactly some linear combination of x^n and $x^{n-1}y$. Similarily, the highest degree term of the $\tilde{\varphi}_n$ and $\tilde{\psi}_n$ is x^n and $nx^{n-1}y$, respectively. It follows that there exists a combination $\alpha_n\tilde{\varphi}_n + \beta_n\tilde{\psi}_n$ which matches the highest degree terms of $\tilde{f}$, so their difference is of degree $n - 1$.

We now have a new polynomial, $\tilde{f} - (\alpha_n\tilde{\varphi}_n + \beta_n\tilde{\psi}_n)$, with degree less than n. Repeating this process of stripping away highest order terms by combinations of φ_k and ψ_k ultimately yields a linear combination differing from f by a constant, which we shall define as α_0. We then have the harmonic polynomial

$$u(x, y) = \alpha_0 + \sum_{k=1}^{n} (\alpha_k\varphi_k + \beta_k\psi_k) \tag{2.23}$$

so that $\tilde{u} = \tilde{f}$ (meaning $u = f$ along $y = \pm\sqrt{a}$). Uniqueness follows by the previous rank-nullity arguments.

□

Remember that we have shown parabolas and parallel lines behave similarly with respect to solvability of the pseudo-Dirichlet problem. The same is true of crossing lines and hyperbola, a fact to keep in mind for the following proof:

Proposition 2.24. *The crossing lines and hyperbola do not generally admit a solution to the pseudo-Dirichlet problem.*

Proof. Consider crossing lines of the form $q(x, y) = ax^2 - y^2$, for some $a \in \mathbb{R}^+$. Again, when considering a data function f restricted to the zero set of q, we can consider some related polynomial $\tilde{f}$ with all quadratic terms in y replaced with ax^2, via the canonical homomorphism $\mathbb{R}[x, y] \to \mathbb{R}[x, y]/\langle ax^2 - y^2\rangle$. So the highest order term of $\tilde{f}$ is some combination of x^n and $x^{n-1}y$.

Note that φ_n is even in y. In fact,

(2.25)

$$\begin{aligned}\varphi_n = \operatorname{Re}\left[(x + iy)^n\right] = \operatorname{Re}\left[\sum_{k=0}^{n}\binom{n}{k}x^{n-k}(iy)^k\right] &= \sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\binom{n}{2k}x^{n-2k}y^{2k} \\ &\equiv x^n\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}(-a)^k \mod q\end{aligned}$$

So $\tilde{\varphi}_n$ is some multiple of the monomial x^n. If we suppose $f = x^n$, it follows that $u = \alpha\varphi_n$ is the only sensible solution, for some α. Notice the crossing lines

can be written as $re^{\pm i\theta}$, for some $\theta \in (0, \pi)$. We have $x^n = \alpha \operatorname{Re}[(x+iy)^n]$ along these lines, which can be rewritten in terms of θ as:

$$\begin{aligned} r^n \cos^n(\theta) = \alpha \operatorname{Re}\left[r^n e^{in\theta}\right] = \alpha r^n \cos(n\theta) \\ \cos^n(\theta) = \alpha \cos(n\theta) \end{aligned} \tag{2.26}$$

That is, α can be determined if and only if $n\theta \neq (m + \frac{1}{2})\pi$.

Now ψ_n is odd in y, so it follows that ψ_n is equivalent to some multiple of $x^{n-1}y$ mod q. If we suppose $f = x^{n-1}y$ in the same spirit as before, then we expect $u = \beta\psi_n$ for some β. So $x^{n-1}y = \beta \operatorname{Im}[(x+iy)^n]$:

$$\begin{aligned} r^n \cos^{n-1}(\theta)\sin(\theta) = \beta \operatorname{Im}\left[r^n e^{in\theta}\right] = \beta r^n \sin(n\theta) \\ \cos^{n-1}(\theta)\sin(\theta) = \beta \sin(n\theta) \end{aligned} \tag{2.27}$$

So β can be determined if and only if $n\theta \neq m\pi$.

Thus the crossing line (and corresponding hyperbola) admits a unique solution to the pseudo-Dirichlet problem if the angle θ between the lines and x-axis is not a rational multiple of π. So, measured with respect to this angle $\theta \in (0, \pi)$, the problem posed is solvable almost always. □

Remember the bizarre results from example 2.2, in which we identified the singular values in the hyperbola and crossing lines case

$$\begin{gathered} a = -b \\ a = -3b \\ 3a = -b \\ a = (2\sqrt{2} - 3)b \\ a = -(2\sqrt{2} + 3)b \end{gathered}$$

for the equation $ax^2 + by^2 = 0$ with $f \in P_2$. These values are singular because

$$\begin{gathered} \arctan(1) = \tfrac{1}{4}\pi \\ \arctan(\sqrt{3}) = \tfrac{1}{3}\pi \\ \arctan(\tfrac{1}{\sqrt{3}}) = \tfrac{1}{6}\pi \\ \arctan(\sqrt{3 - 2\sqrt{2}}) = \tfrac{1}{8}\pi \\ \arctan(\sqrt{3 + 2\sqrt{2}}) = \tfrac{3}{8}\pi \end{gathered}$$

This proof completes the characterization of conic sections.

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