INTRODUCTION TO THE KEISLER ORDER

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Abstract. In this paper, we introduce the basic definitions and concepts necessary to define the Keisler Order. We will prove the order is well-defined as well as the existence of a maximal class with respect to the order.

Contents

1. Introduction

The Keisler Order was first introduced by H. Jerome Keisler in 1967. Currently, this order is known to be a pre-order on (countable) first-order theories which, broadly speaking, ranks classes of theories by complexity. Stronger theorems have been proven for stable theories (e.g. the Keisler Order on stable theories is linear [6]), while the complete structure of the Keisler Order is still an open problem.

The classification of first-order theories is both a classic and modern program in model theory. Shelah's stability program, the most famous type of classification framework, organizes theories relative to the number of definable types over subsets of a model. While the stability program has had great success, the program also leaves unstable theories in some unclassifiable purgatory. Work on the Keisler Order has shed light on dividing lines between classes of unstable theories. Additionally, one of the major results in a paper by Malliaris and Shelah [4] shows that theories, which have the SOP_2-property, are maximal. This result was important in proving $\mathfrak{p} = \mathfrak{t}$, the oldest open problem in cardinal invariants.

We will begin with many definitions as well as examples to provide the reader with some intuition. We will leave most of the proofs which relate to the Keisler Order to the last two sections. The two big theorems we prove at the end can be found in Keisler's original paper on the topic [3].

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2. Notation and Basic Definitions

This paper will assume at least one course in basic first-order model theory. However, in this section, we will go over some of the necessary terminology and theorems required to understand this paper. A language $\mathscr{L} = \{f_1, f_2, ..., R_1, R_2, ..., c_1, c_2, ..\}$ is a collection of (n-ary) function, (n-ary) relation, and constant symbols (sometimes called *non−logical symbols*). Languages also contain logical symbols, i.e. $\wedge, \vee, \neg, \rightarrow$, equality, and parentheses, as well as (object-level quantification) $\forall, \exists$. A formula in a language is simply a *grammatically coherent* string of logical symbols which may or may not have free variables (for instance, $x = x$ or $(\exists x)(S(x) = y)$ where y is a free variables and x is bounded).

A theory T is a set of logical sentences with symbols from some fixed language $\mathscr{L}$. A complete theory is a maximally consistent set of sentences. A model, or a $\mathscr{L}$-structure, is some set-sized mathematical object with an interpretation for each non-logical symbol in the language. "$\models$" is a (semantic) binary relation between $\mathscr{L}$-structures and sentences in the language $\mathscr{L}$. We say a sentence φ is true in a model $\mathfrak{A}$ by writing $\mathfrak{A} \models \varphi$.

The following will be our notational habits. An arbitrary model will be denoted as $\mathfrak{A}$ or $\mathfrak{B}$. Usually, we will denote indexing sets as I, J, and cardinals as α, β, κ. Every theory T will have a corresponding fixed language $\mathscr{L}$ where the size of the underlying language is at most countable. The underlying set of a model $\mathfrak{A}$ is formally written as $dom(\mathfrak{A})$. However, we will usually write A for $dom(\mathfrak{A})$, B for $dom(\mathfrak{B})$. A set X (of sets) has the finite intersection property if and only if any finite intersection of elements is not empty. If A is a set, then $\mathcal{P}(A)$ is the power set of A and $\mathcal{P}_{\aleph_0}(A)$ is the collection of all finite subsets of A.

Finally, we have some more formal definitions and theorems which we will be referring to.

Definition 2.1. Let $\mathscr{L}$ be a language, $\mathfrak{A}$ be an $\mathscr{L}$-structure, and α be a cardinal. Then, we let $\mathscr{L}(\alpha)$ be the language with α-many new constant symbols, $\{c_\beta\}_{\beta<\alpha}$. Moreover, if $(a_\beta)_{\beta<\alpha}$ be a sequence of elements $\mathfrak{A}$, then $(\mathfrak{A}, \alpha)$ is naturally an $\mathscr{L}(\alpha)$-structure.

Definition 2.2. A collection of sentences, Δ, is said to be $satisfiable$ is there exists a model $\mathfrak{A}$ such that $\mathfrak{A} \models \Delta$. Δ is said to be finitely satisfiable if every finite subset of Δ is satisfiable.

Theorem 2.3. *(Completeness): A set of sentences Δ is consistent if and only if it is satisfiable.*

Theorem 2.4. *(Compactness): A set of sentences Δ is satisfiable if and only if it is finitely satisfiable.*

Definition 2.5. If $\mathfrak{A}$ and $\mathfrak{B}$ are two $\mathscr{L}$-structures, we say that $\mathfrak{A}$ is elementarily equivalent to $\mathfrak{B}$ (written $\mathfrak{A} \equiv \mathfrak{B}$) if for any sentence φ in the language $\mathscr{L}$, we have $\mathfrak{A} \models \varphi$ if any only if $\mathfrak{B} \models \varphi$.

Definition 2.6. Let $\mathfrak{A}$, $\mathfrak{B}$ be two $\mathscr{L}$-structures. We say that $\mathfrak{A}$ is isomorphic to $\mathfrak{B}$ if there exists a bijection $f : \mathfrak{A} \rightarrow \mathfrak{B}$ such that f preserves functions, relations, and constant symbols.

For a more detailed introduction, we refer the reader to the first few sections of any basic model theory text (e.g. Chang & Keisler [2]).

3. Ultrapowers

Ultrapower constructions are one of the two the central concepts necessary to understanding the Keisler Order. However, before we can define ultrapowers, we have to first get an intuition for ultrafilters and ultraproducts.

Definition 3.1. Let I be an indexing set. We say that D is a filter over I if D is a non-empty subset of $\mathcal{P}(I)$ with the following properties:
1. If $X \in D$ and $Z \supset X$, then $Z \in D$.
2. If $X, Y \in D$, then $X \cap Y \in D$.

Furthermore, we call D an ultrafilter if for any $X \subseteq I$, we have (exclusively) either $X \in D$ or $I - X \in D$. Intuitively, we can think of D as a mathematical object that makes decisions about which subsets of I are *large*. D thinks the entire set is large, any set containing a large set is large, and the intersection of any two large sets is large. Note, D may not think that the countable/uncountable intersection of large sets is large. We will see later that ultrafilters without the countable intersection property are valuable and are central to our study.

Definition 3.2. Let D be a filter over I. We say that D is a principal filter if there exists $X \subset I$ such that $D = \{Y \subseteq I : X \subset Y\}$.

We call any filter which is not principal a nonprincipal (or free) filter.

Example 3.3 (Principal Ultrafilter)**.** Let $I = \mathbb{N}$ and let $D = \{X \subseteq \mathbb{N} : 3 \in X\}$. Then, D is a principal ultrafilter over I.

Example 3.4 (Nonprincipal Ultrafilter)**.** It is provable that one cannot construct an example (since the existence of a nonprincipal ultrafilter is equivalent to a weak version of choice). There are models of ZF where there do not exist any nonprincipal ultrafilters. However, the constructions of these models of ZF not suitable for this paper. We refer the interested reader to [1].

For the remainder of this paper, every ultrafilter will be a nonprincipal ultrafilter. Furthermore, we will assume the full power of ZFC and thus never worry about the existence of ultrafilters (in general). The next two facts follow quickly from the definitions and are left unproven.

Proposition 3.5. *No free ultrafilter contains any finite sets.*

Proposition 3.6. *Let A be a collection of subsets of I such that A has the finite intersection property. Then A can be extended to an ultrafilter over I.*

Let I be an indexing set of cardinality α and let $\{\mathfrak{A}_i\}_{i\in I}$ be a collection of models. Let $\prod_{i\in I} \mathfrak{A}_i$ be the cartesian product of these models. Note that the elements of $\prod_{i\in I} \mathfrak{A}_i$ can be seen as function from I into $\{\mathfrak{A}_i\}_{i\in I}$ or as an α-termed sequence of elements where the ηth term (for $\eta < \alpha$) is an element of $\mathfrak{A}_\eta$. If f, g are elements of $\prod_{i\in I} \mathfrak{A}_i$, we say that f is D-equivalent to g (written as $f \equiv_D g$) if and only if f and g agree on a large set. Formally,

$$f \equiv_D g \iff \{i \in I : f(i) = g(i)\} \in D$$

Proposition 3.7. *If D is a filter, then $\equiv_D$ is an equivalence relation over $\prod_{i\in I} \mathfrak{A}_i$.*

Definition 3.8 (Ultraproduct)**.** Let I be an indexing set and let D be an ultrafilter over I. An ultraproduct of $\mathscr{L}$-structures is defined as,

$$\prod_{i\in I} \mathfrak{A}/D = \{f/D : f \in \prod_{i\in I} \mathfrak{A}_i\}.$$

For notational purposes, we will always have our I's fixed and so we will write $\prod_{i\in I}\mathfrak{A}_i$ as $\prod_D \mathfrak{A}_i$. In some sense, ultraproducts are similar to quotient spaces in topology. We are simply taking elements in our Cartesian product and gluing them together. Now, the following theorem demonstrates the strength of ultraproducts in model theory.

Theorem 3.9 (Łoś's Theorem)**.** *Let D be an ultrafilter over I. Then, for any $f_1, ..., f_n \in \prod_D \mathfrak{A}_i$, we have that*

$$\prod_D \mathfrak{A}_i \models \varphi(f_1, ..., f_n) \iff \{i \in I : \mathfrak{A}_i \models \varphi(f_1(i), ..., f_{n_i})\} \in D.$$

So what does this theorem actually mean? First of all, note that if each $\mathfrak{A}_i$ agrees on some (first-order) sentences in $\mathscr{L}$, then $\prod_D \mathfrak{A}_i$ also agrees on that sentence. In fact, if D thinks some subset of I is large, and all the models of the large set agree (disagree) on some sentence, then $\prod_D \mathfrak{A}_i$ also agrees (disagrees) on that sentence. Free ultraproducts can be thought of as averaging on an infinite set. They pick up on what is happening in general while forgetting about small perturbations and outliers. We will consider the following example to give an intuition on how ultraproducts work.

Example 3.10. First note that the axioms of an algebraically closed field are first-orderizable in the language $\mathscr{L} = \{0, 1, +, \times\}$. We will denote ACF to mean *algebraically closed field* while ACF_p will mean *algebraically closed field of characteristic p*. Let $\mathbb{P}$ denote the set of standard primes. Furthermore, let $\mathfrak{A}_p \models ACF_p$ and so each model, $\mathfrak{A}_p$ is an algebraically closed field of characteristic p. Let D be a nonprincipal ultrafilter of $\mathbb{P}$. Now, we consider the object $\prod_D \mathfrak{A}_i$. Note that since $\mathfrak{A}_i \models ACF$ for all $i \in \mathbb{P}$, it follows that $\prod_D \mathfrak{A}_i \models ACF$ and so $\prod_D \mathfrak{A}_i$ is an algebraically closed field. We will now find this field's characteristic. By proposition 3.5, there is no finite set in D. Since D is an ultrafilter, this means that D contains all cofinite sets. Define φ_i, for all $i \in \mathbb{N}$ as follows:

$$\varphi_i \equiv \neg(\underbrace{1 + 1 + 1... + 1}_{i} = 0).$$

For $i \in \mathbb{N}$, $\mathfrak{A}_j \models \varphi_i$ for $j \neq i$. Hence, we know that φ_i is true on a cofinite subset of $\mathbb{P}$. Therefore, $\prod_D \mathfrak{A}_i$ cannot have characteristic i for any $i \in \mathbb{N}$ (recall that fields cannot have composite characteristic anyway). Since $\prod_D \mathfrak{A}_i$ is a field and must have some characteristic, it has characteristic 0.

Definition 3.11 (Ultrapower)**.** If I is an indexing set and D is an ultrafilter over I, then $\prod_D \mathfrak{A}_i$ is an ultrapower if for any $i, j \in I$, we have that $\mathfrak{A}_i \cong \mathfrak{A}_j$

Since the indexing of our models no longer provides a method of differentiation we will simply write ultrapowers as $\prod_D \mathfrak{A}$ when I is fixed. This construction, in relation to ultraproducts, might seem a little odd at first. We already know the exact set of first-order sentences that $\prod_D \mathfrak{A}$ satisfies. The proof that $\prod_D \mathfrak{A} \equiv \mathfrak{A}$ is a trivial corollary to Łoś's theorem. The following example begins to show how ultrapowers can be different from the models used to construct them.

Example 3.12. Let $\mathfrak{A} = (\mathbb{N}; \leq, S)$ where $\leq$ has its normal interpretation and S is interpreted as the unary successor function[1]. We let I be countable and let D

[1]Note that S can be defined in the language $\{\leq\}$. We have added S to our language to simplify our arguments.

be a nonprincipal ultrafilter over I. Note that $(\mathbb{N}; \leq, S)$ is well-ordered. We will show that $\prod_D \mathfrak{A}$ is not. Consider the element $f = (1, 2, 3, 4, ...) \in \prod_D \mathfrak{A}$. Notice that for every $m \in \mathbb{N}$, $m \leq f(i)$ is true on a cofinite set and as a result, true on a large set. Therefore, the element f is larger than every standard natural number. Furthermore, it is easy to show that $\mathbb{N} \models (\forall x)(x \neq 0 \rightarrow (\exists y)(S(y) = x))$. This statement simply reads: Every element not equal to 0 has a direct predecessor. Thus, we can find an infinite descending chain beginning with f. The chain begins like this:

$$\begin{gathered}(1, 2, 3, 4, 5, ...)\\(0, 1, 2, 3, 4, ...)\\(0, 0, 1, 2, 3, ...)\\\vdots\end{gathered}$$

We know that this chain does not terminate after finitely many steps, since if it did, then f would be some standard natural number. Since $\prod_D \mathfrak{A}$ has an infinite descending chain, we know that $\prod_D \mathfrak{A}$ is not well ordered.

Now, if you know some basic logic, you should be making a connection with the compactness theorem. Ultrapowers and ultraproducts are tools which apply the compactness theorem. However, while compactness simply proves that a certain model exists, ultrapowers and ultraproducts give us much more control over the models we are constructing.

Finally, in this section, we will define regular ultrafilters.

Definition 3.13. Let D be a nonprincipal ultrafilter over some infinite indexing set I. We say that D is a (β, α)-regular ultrafilter if there is a subset X of D such that

1. $|X| = \alpha$.
2. For any subset Y of X such that $|Y| = \beta$, we have that $\bigcap Y = \emptyset$.

We drop the (β, α) notation and just call an ultrafilter D *regular* if $\beta = \omega$ and $\alpha = |I|$. We also call X a *regular subset* of D if the above properties holds for X.

Since this is the type of ultrafilter we actually need for the definition of the Keisler Order, we will prove that D regular ultrafilters exist. The following comes from Proposition 4.3.5 in [2].

Lemma 3.14 (Regular Ultrafilter Existence)**.** *For every infinite cardinal κ, there exists a regular ultrafilter over κ.*

Proof. Let $\mathcal{P}_{\aleph_0}(\kappa)$ be the set of all finite subsets of κ. Note that $|\mathcal{P}_{\aleph_0}(\kappa)| = \kappa$. Let $f : \mathcal{P}_{\aleph_0}(\kappa) \rightarrow \kappa$ be a bijection and for each $\beta \in \kappa$, define $Y_\beta = \{\gamma \in \kappa\ : \beta \in f^{-1}(\gamma)\}$. Now, consider $A = \{Y_\beta : \beta \in \kappa\}$. It is clear that $|A| = \kappa$. Recall that if A has the finite intersection property, then A can be extended to an ultrafilter. Consider:

$$\bigcap_{j=1}^{n} Y_{\beta_j} = \bigcap_{j=1}^{n} \{\gamma \in I : \beta_j \in f^{-1}(\gamma)\}.$$

By definition, we have,

$$= \{\gamma \in \kappa : \beta_1, ..., \beta_n \in f^{-1}(\gamma)\} \neq \emptyset.$$

The inequality follows from the fact that f is a bijection and so if we set $\gamma_0 = f(\{\beta_1, ..., \beta_n\})$, we see that the set above is non-empty. Therefore, A has the finite intersection property and may be extended into an ultrafilter. It should also be clear that the intersection of countable subsets of A are empty and so A is our regular subset of its ultrafilter extension. □

4. Saturation and Satisfaction

While ultrapowers are necessary for understanding the Keisler Order, this topic alone is not sufficient. Another key ingredient of the Keisler Order is saturation. This concept, along with satisfaction, will bring the Keisler Order into view.

Definition 4.1. Let $\mathfrak{A}$ be a model in a language $\mathscr{L}$. Let $X \subseteq A$. We say that ρ is an n-type over X if

(1) ρ is a collection of formulas in n-free variables in the language $\mathscr{L}(X)$.
(2) For any finite subset ρ_0 of ρ, there is some $(c_1, ..., c_n) \in \mathfrak{A}^n$ such that for any $\varphi_i(y_1, .., y_n) \in \rho_0$, we have $\mathfrak{A} \models \varphi_i(c_1, ..., c_n)$.

We say that an n-type ρ is complete if and only if it is maximally consistent. Equivalently, ρ is a complete n-type if for any formaula $\psi(y_1, ..., y_n)$ in n-free variables, (exclusively) either $\psi(y_1, ..., y_n) \in \rho$ or $\neg\psi(y_1, ..., y_n) \in \rho$. Note that every element of a model has a corresponding complete 1-type (over $X \subset A$). In fact, every fixed n-tuple in any model has a corresponding complete n-type (over $X \subset A$). Consider the following: if we let $(a_1, ..., a_n)$ be a tuple of elements in $\mathfrak{A}^n$, let

$$\rho \equiv \{\phi(y_1, ..., y_n) : \mathfrak{A} \models \phi(a_1, ..., a_n)\}.$$

Definition 4.2 (Satisfaction of 1-Types)**.** Let ρ be a complete 1-type in one free variable. We say that ρ is satisfied/realized in $\mathfrak{A}$ if there exists an a in A such that $\mathfrak{A} \models \varphi(a)$ for all $\varphi(x) \in \rho$.

We let $S_1(X)$ be the collection of all (consistent) complete 1-types over $X \subset A$.

Remark 4.3. Note that the above definition can be clearly extended to n-types and has corresponding collections, $S_n(X)$.

Definition 4.4 (κ-Saturation)**.** Let κ be some infinite cardinal. We say that a structure $\mathfrak{A}$ is κ-saturated if for every $X \subseteq A$ with $|X| < \kappa$, all the types in $S_1(X)$ are realized in $\mathfrak{A}$.

Proposition 4.5. *One can show, by induction, that is $\mathfrak{A}$ is κ-saturated and $|X| < \kappa$, then every type in $S_n(X)$ is realized in $\mathfrak{A}$.*

Proposition 4.6. *Not every theory has a saturated model in every cardinality.*

Before we go any further with our definition building, we will give an indepth example.

Example 4.7 (($\mathbb{Q}$; <))**.** Let us consider $\mathbb{Q}$ in the language $\{<\}$. We will show that there does not exist an $\aleph_1$-saturated model of size $\aleph_\beta$ for $\aleph_0 \leq \aleph_\beta < 2^{\aleph_0}$. The problem here is that we have continuum many 1-types over $\mathbb{Q}$. Let $\mathbb{E}$ be any dense linear order of cardinality less than $2^{\aleph_0}$. One can find an isomorphic copy of $\mathbb{Q}$ inside $\mathbb{E}$. Consider any two (distinct, irrational) real numberes, s and r. We will write ρ_r and ρ_s as their complete corresponding 1-types over $\mathbb{Q}$. It is not difficult to show that ρ_r and ρ_s are finitely satisfiable, and so they are valid types. Without loss of generality, we assume that $s < r$. Since $s \neq r$ and $\mathbb{Q}$ is dense inside the reals, we have that there exists $q \in \mathbb{Q}$ such that $(x < q) \in \rho_s$ and $(x > q) \in \rho_r$. Hence, every real number corresponds to a different (complete) 1-type over a countable subset of the model (where the countable subset is the entire model itself). Therefore, any $\aleph_1$-saturated model is at least size $2^{\aleph_0}$ (since $|\mathbb{R}| = 2^{\aleph_0}$).

However, $(\mathbb{R}; <)$ is **not** a $\aleph_1$-saturated model of the theory of dense linear orderings. Let $(\mathbb{D}, <)$ be a $\aleph_1$-saturated model containing $\mathbb{Q}$. We will show that $(\mathbb{D}, <)$ realizes a type over $\mathbb{Q}$ that $(\mathbb{R}, <)$ does not realize. Consider the type $\rho_\xi = \{x < q : q \in \mathbb{Q}\} \cup \{x > 0\}$. First note, that since the rationals are dense in themselves, no elements of $\mathbb{Q}$ realize this type. However, one can easily show that ρ is finitely satisfiable. By the compactness theorem, we note that ρ_ξ is consistent. Therefore, it must be satisfied in our $\aleph_1$-saturated model (since this partial type is over a countable parameter set). Let a be the element of $\mathbb{D}$ which satisfies this type. Note that for any $q \in \mathbb{Q}$, we have that $q \leq 0$ or $q > a$. Suppose that $r \in \mathbb{R}$ and $r = a$. Since the rationals are dense in the reals, then there must be some $p \in \mathbb{Q}$ such that $0 < p < r$. But this a contradiction. Hence, $(\mathbb{R}, <)$ is not $\aleph_1$-saturated.

5. An Early Application

We have just defined a lot of new machinery, but it is probably still unclear how ultrapowers and saturation relate to one another. This section is dedicated to exhibiting the interaction of the two.

Definition 5.1 (Countably Incomplete Ultrafilter)**.** An ultrafilter is said to be countably incomplete if there exists a subset X of D such that $|X| = \aleph_0$ and $\bigcap X = \emptyset$.

We are going to show that any ultrapower, using a countably incomplete ultrafilter, is $\aleph_1$-saturated. However, we will need the following lemma first.

Lemma 5.2. *Let D be a countably incomplete ultrafilter. Then, there exists a countable descending chain $I = I_0 \subset I_1 \subset I_2 \subset \ldots$, such that $\bigcap_{n\in\omega} I_n = \emptyset$.*

Proof. Since D is countably incomplete, we know that there exists a set $X \subset D$ such that $|X| = \aleph_0$ and $\bigcap X = \emptyset$. Let $\{Y_1, \ldots, Y_n, \ldots\}$ be a well-ordering of the elements of X. Define,

$$J_n = \bigcap_{i=1}^{n} Y_i.$$

Since D is a filter, it is closed under finite intersection. Therefore, $J_n \in D$ for all $n < \omega$. Furthermore, it is clear that $J_n \supseteq J_{n+1}$ and that we have the following equality,

$$\bigcap_{i\in\omega} J_n = \emptyset.$$

We also know that for each J_n, there exists some J_m such that $J_n \supsetneq J_m$, for some $m > n$. If this was not the case, then $\bigcap_{i\in\omega} J_i = J_m$. Now, we can choose a subsequence of $\{J_i\}_{i\in\omega}$ such that J_{i+1} is a proper subset of J_i for each i. By well-ordering this set in the obvious way, we have found the collection that we are looking for. □

The following theorem comes from Theorem 6.1.1 in [2].

Theorem 5.3. *Let $\mathscr{L}$ be countable and let D be a countably incomplete ultrafilter over some infinite set I. Then, for any collection $\{\mathfrak{A}_i\}_{i\in I}$ of $\mathscr{L}$-structures, we have that $\prod_D \mathfrak{A}_i$ is $\aleph_1$-saturated.*

Proof. Let $C \subset \prod_D \mathfrak{A}_i$ be a countable set and let $\Delta(x)$ be a set of formulas (with one free variable) in the language $\mathscr{L}_1 = \mathscr{L}(C)$. It suffices to find an $f/D \in \prod_D \mathfrak{A}_i$ such that f/D satisfies every finite subset of $\Delta(x)$.

Suppose that each finite subset of $\Delta(x)$ is realized in $\prod_D \mathfrak{A}_i$. Because $\mathscr{L}_1$ is countable, we know that $\Delta(x)$ is countable. Therefore, we can well order our elements of $\Delta(x)$ as $\{\delta_1(x), \delta_2(x), ...\}$. Since D is countably incomplete, we know that there exists a descending chain $I = I_0 \supset I_1 \supset I_2 \supset ...$ such that $\bigcap_{n\in\omega} I_n = \emptyset$. Now, we let $X_0 = I_0$ and define

$$X_n = I_n \cap \{i \in I : \mathfrak{A}_i \models (\exists x)(\delta_1(x) \wedge ... \wedge \delta_n(x))\}.$$

Since every finite subset of $\Delta(x)$ is satisfied in $\prod_D \mathfrak{A}_i$ we may apply Łoś's Theorem and we know that $\{i \in I : \mathfrak{A}_i \models (\exists x)(\delta_1(x) \wedge ... \wedge \delta_n(x))\}$ is large (hence, it is in D). Since I_n is also in D, we know by definition of a filter that each X_n is in D for each $n \in \mathbb{N}$. Furthermore, note that $\bigcap_{n\in\omega} X_n = \emptyset$ because of the following computation,

$$\bigcap_{n\in\omega} X_n = \bigcap_{n\in\omega} I_n \cap \{i \in I : \mathfrak{A}_i \models (\exists x)(\delta_1(x) \wedge ... \wedge \delta_n(x))\}$$

$$= \bigcap_{n\in\omega} I_n \cap \bigcap_{n\in\omega} \{i \in I : \mathfrak{A}_i \models (\exists x)(\delta_1(x) \wedge ... \wedge \delta_n(x))\}$$

$$\emptyset \cap \bigcap_{n\in\omega} \{i \in I : \mathfrak{A}_i \models (\exists x)(\delta_1(x) \wedge ... \wedge \delta_n(x))\} = \emptyset$$

By construction, we have $X_n \supset X_{n+1}$ for each n. Now, for all $i \in I$, we let $n_i = \max\{n : i \in X_n\}$. Since $\bigcap X_n = \emptyset$, n_i is some finite natural number for each i. We will now construct an element, f/D, in $\prod_D \mathfrak{A}_i$ which satisfies $\Delta(x)$. If $n_i = 0$, let $f(i)$ be any arbitrary a in A_i. If $n_i > 0$, choose $f(i) \in A_i$ such that,

$$\mathfrak{A}_i \models \delta_1(f(i)) \wedge ... \wedge \delta_{n_i}(f(i)).$$

Note that for any $i \in X_n$, we have that $n \leq n_i$ (by definition) and therefore $\mathfrak{A}_i \models \delta_n(f(i))$. Since X_n is a large set, by Łoś's theorem, we have that $\prod_D \mathfrak{A}_i \models \delta_n(f/D)$ for every n. So f/D satisfies $\Delta(x)$.

□

Remark 5.4. Note that every regular ultrafilter is countably incomplete.

Corollary 5.5. *If D is a nonprincipal ultrafilter over $\mathbb{N}$ and $\{\mathfrak{A}_i\}_{i\in\mathbb{N}}$ is a collection of $\mathscr{L}$-structures, then $\prod_D \mathfrak{A}$ is $\aleph_1$-saturated.*

Proof. By the theorem above, it suffices to show that every nonprincipal ultrafilter over $\mathbb{N}$ is countably incomplete. Recall that since D is nonprincipal, it contains all cofinite sets. Now, let $I_0 = \mathbb{N}$ and define,

$$I_{n+1} = I_n - \{n\}.$$

It is clear that $I_n \in D$ as well as the fact that $I_n \supset I_{n+1}$ for $n \in \omega$. It is also clear that $\bigcap_{n\in\mathbb{N}} I_n$ is empty by construction. Therefore, every ultrafilter over $\mathbb{N}$ is countable incomplete. Hence, $\prod_D \mathfrak{A}_i$ is $\aleph_1$-saturated by the previous theorem. □

6. The Order

Now that we have all the necessary definitions in place, we can finally define the Keisler Order.

Definition 6.1 (Keisler Order)**.** We say that a theory $T_1 \trianglelefteq_\kappa T_2$ is for any $\mathfrak{A}_1 \models T_1$, $\mathfrak{A}_2 \models T_2$, and regular ultrafitler D over κ, we have that if $\prod_D \mathfrak{A}_2$ is κ^+-saturated then $\prod_D \mathfrak{A}_1$ must be κ^+-saturated. Now we say that $T_1 \trianglelefteq T_2$ if for every infinite cardinal, κ, we have that $T \trianglelefteq_\kappa T_2$. This second definition, $\trianglelefteq$, is the Keisler Order.

Note that the ultraproduct construction is on models while the order is on theories. Therefore, we still need to show that this definition is well defined (i.e. that it is not dependent on our choice of model). The following theorem is from Corollary 2.1a of [3].

Theorem 6.2. *Fix some language $\mathscr{L}$ and some indexing set I. Furthermore, suppose that $\mathfrak{A} \equiv \mathfrak{B}$ over $\mathscr{L}$ and D is a regular ultrafilter over I. Then we have that $\prod_D \mathfrak{A}$ is α^+-saturated if and only if $\prod_D \mathfrak{B}$ is α^+-saturated.*

Proof. We need a method of translation between the two ultrapowers. Usually, one can get away with confusing parameters and constant symbols in the language. In this argument, we will need to be careful and make use of the $\mathscr{L}(\alpha)$ language discussed in the introduction. Note that the two directions have identical proof. Assume, without loss of generality, that $\prod_D \mathfrak{B}$ is α^+-saturated. Let $X = \{Y_i\}_{i \in I}$ be a regular subset of D. Fix $C \subset \prod_D \mathfrak{A}$ where $|C| = \alpha$.

For clarity, for each element f/D in $\prod_D \mathfrak{A}_i$, we fix a section map. In particular, we fix some function f' in $\prod_i \mathfrak{A}_i$ such that $f'/D = f/D$. Let $\mathfrak{a} = (f/D_\beta : \beta < \alpha)$ be an α-enumeration the elements of C. Then, if we fix some index i we have a sequence of elements in $\mathfrak{A}_i$ of length α (with possible repetitions) via $\mathfrak{a}(i) = (f'_\beta(i))_{\beta<\alpha}$ where f'_β is our fixed section map for f/D_β. Then both $(\prod_D \mathfrak{A}, \mathfrak{a})$ and, for any fixed index i in I, $(\mathfrak{A}, \mathfrak{a}(i))$ are $\mathscr{L}(\alpha)$-structure. Let $\Delta(x)$ be a collection of formulas in one free variable (in the expanded language $\mathscr{L}(\alpha)$) such that $\Delta(x)$ is finitely satisfiable in $(\prod_D \mathfrak{A}, \mathfrak{a})$. It suffices to find some f/D in $\prod_D \mathfrak{A}_i$ such that f/D realizes $\Delta(x)$. Now, since $|\Delta| \leq |X| = \alpha$, we let h be an injection from Δ into X. We define,

$$\Delta(i) = \{\delta(x) \in \Delta(x) : i \in h(\delta(x))\},$$

and,

$$X_i = h(\Delta(i)) = \{h(\delta(x)) : \delta(x) \in \Delta(i)\}.$$

Note that $\Delta(i)$ is finite. If $\Delta(i)$ were infinite, then we could find a infinite collection of elements in X such that their intersection would be non-empty which would contradict the regularity of X. We let $\Gamma(i)$ be the collection of sentences in $\mathscr{L}(\alpha)$ with the following description;

$$\Gamma(i) = \{(\exists x) \bigwedge_{\delta(x) \in s} \delta(x) | s \in \mathcal{P}(\Delta(i)) - \emptyset\}$$

Note that $\Gamma(i)$ is a valid collection of first-order formulas since $\Delta(i)$ is finite. $\Gamma(i)$ is simply every possible subcollection of sentences in $\Delta(i)$. Since $\Gamma(i)$ is finite and $\mathfrak{A}_i \equiv \mathfrak{B}_i$ we can construct a sequence of points, $(b_i^\beta)_{\beta<\alpha}$, in $\mathfrak{B}_i$ such that,

$$\Gamma(i) \cap Th((\mathfrak{A}_i, \mathfrak{a}(i)) = \Gamma(i) \cap Th((\mathfrak{B}_i, (b_i^\beta)_{\beta<\alpha}))$$

This follows from the fact that every n-type over a finite set is realized in every model. By construction, any subset of $\Gamma(i)$ is finitely satisfiable in $(\mathfrak{A}_i, \mathfrak{a}(i))$ if and only if it is finitely satisfiable in $(\mathfrak{B}_i, (b_i^\beta)_{\beta<\alpha})$. Notice that now we have turned $\mathfrak{B}_i$ into a $\mathscr{L}(\alpha)$-structure by adding the sequence of points $\mathfrak{b}(i)$. We now wish to construct a sequence in $\prod_D \mathfrak{B}_i$. In particular, we want to have some $\mathfrak{b}$ similar to $\mathfrak{a}$. Let g_β be the element in $\prod_{i\in I} \mathfrak{B}_i$ given by $g_\beta(i) = b_i^\beta$. Then, we let $\mathfrak{b} = (g_\beta/D)_{\beta<\alpha}$. Hence, $(\prod_D \mathfrak{B}_i, \mathfrak{b})$ is naturally a $\mathscr{L}(\alpha)$-structure.

We claim that $\Delta(x)$ is finitely satisfiable in $(\prod_D \mathfrak{B}_i, \mathfrak{b})$. Fix $\delta_1(x), ..., \delta_n(x)$ in $\Delta(x)$. Then, let $\varphi = (\exists x) \bigwedge_{j=1}^n \delta_j(x)$. One must first check that $\{i \in I : \varphi \in \Gamma(i)\}$ is in D. By construction, we have that for each $j \leq n$, $\{i \in I : i \in h(\delta_j(x))\}$ is in D by construction. The intersection of these sets is also in D. Notice,

$$\bigcap_{j=1}^{n} \{i \in I : i \in h(\delta_j(x))\} = \{i \in I : \delta_1(x), ..., \delta_n(x) \in \Delta(i)\}$$

$$= \{i \in I : (\exists x) \bigwedge_{j=1}^{n} \delta_j \in \Gamma(i)\} = \{i \in I : \varphi \in \Gamma(i)\}.$$

Since $\Delta(x)$ is finitely satisfiable in $(\prod_D \mathfrak{A}, \mathfrak{a})$, we know that $(\prod_D \mathfrak{A}_i, \mathfrak{a}) \models \varphi$. By Łoś's Theorem, $(\mathfrak{A}_i, \mathfrak{a}(i)) \models \varphi$ for D-almost all i (call this indexing collection J). Then, $(\mathfrak{B}_i, (b_i^\beta)_{\beta<\alpha}) \models \varphi$ for any i in the collection $\{i \in I : \varphi \in \Gamma(i)\} \cap J$. This set is the intersection of two large sets and so by Łoś's Theorem, we have that $(\prod_D \mathfrak{B}, \mathfrak{b}) \models \varphi$.

Since $\prod_D \mathfrak{B}$ is α^+-saturated and the size of $\Delta(x)$ is less than α, we may conclude that $\Delta(x)$ in realized in $(\prod_D \mathfrak{B}_i, \mathfrak{b})$. Let $g \in \prod_I \mathfrak{B}$ such that g/D realizes $\Delta(x)$. Now, we just need to use our machinary to move g/D back to $\prod_D \mathfrak{A}_i$. For each $i \in I$, we let $\Theta(i)$ be the set of $\delta(x) \in \Delta(i)$ such that $(\mathfrak{B}_i, (b_i^\beta)_{\beta<\alpha}) \models \delta(g(i))$. By definition, $\Theta(i)$ is finitely satisfiable in $(\mathfrak{B}_i, (b_i^\beta)_{\beta<\alpha})$. Therefore, by construction, $\Theta(i)$ is also finitely satisfiable in $(\mathfrak{A}, \mathfrak{a}(i))$. Note that since $\Theta(i)$ is finite, we can find an element $f(i) \in A_i$ such that $\mathfrak{A} \models \delta(f(i))$ for $\delta(x) \in \Theta(i)$. Finally, we can now show that f/D satisfies $\Delta(x)$ in $(\prod_D \mathfrak{A}_i, \mathfrak{a})$. Let $\delta(x) \in \Delta$. Then $\{i : \delta(x) \in \Delta(i)\}$ is large. Now, g/D satisfies $\delta(x)$ in $(\prod_D \mathfrak{B}, b)$, so $g(i)$ satisfies $\delta(x)$ in $(\mathfrak{B}_i, b(i))$ for D-almost all i. Thus, we have that $\delta(x) \in \Theta(i)$ for D-almost all i. It follows that $f(i)$ satisfies $\delta(x)$ in $(\mathfrak{A}_i, \mathfrak{a}(i))$ for D-almost all i, and therefore, f/D satisfies $\delta(x)$ in $(\prod_D \mathfrak{A}, \mathfrak{a})$. Since $\delta(x)$ was arbitrary, it follows that f/D realizes the type $\Delta(x)$. $\square$

7. Existence of a Maximal Class

In this section, we will prove the existence of a maximal class. The theories we will show are maximal, in some sense, understand the concept of saturation. The theories *encode* the idea of saturation. Note that the main proof in this section provides a sufficient condition for maximality.

Definition 7.1. (Weak Ideals): Let $n \in \mathbb{N}$. We say that T is a weak ideal over n if

1. $T \subseteq \mathcal{P}(n)$ and $T \neq \emptyset$.
2. If $t \in T$ and $\emptyset \neq s \subset t$, then $s \in T$.

The following example is the key concept to keep in mind when understanding why weak ideals are important.

Example 7.2. Let $\Delta = \{\delta_1, ..., \delta_n\}$ be a finite collection of sentences. Suppose that $\mathfrak{A} \models \delta_{i_1} \wedge ... \wedge \delta_{i_m}$ where $m \leq n$. Then, if we let,

$$T = \{s \in \mathcal{P}(n) : \mathfrak{A} \models \bigwedge_{k \in s} \delta_k\}$$

then, T is a weak ideal over n.

Definition 7.3. (Versatile Formula): Let $\varphi(x, \overline{y})$ be some formula in a language $\mathscr{L}$. Then, for every n and weak ideal T over n, we define the formula,

$$\theta_T(\overline{y}_1, ..., \overline{y}_n) = \Big([\bigwedge_{t \in T} (\exists x) \bigwedge_{m \in t} \varphi(x, \overline{y}_m)] \wedge [\bigwedge_{t \notin T} \neg(\exists x) \bigwedge_{m \in t} \varphi(x, \overline{y}_m)] \Big).$$

We say that $\varphi(x, \overline{y})$ is a versatile formula if for every n and every weak ideal T over n, we have that $\mathfrak{A} \models \exists \overline{y}_1, ..., \overline{y}_n \theta_T(\overline{y}_1, ..., \overline{y}_n)$.

At first glance, the versatile formula might seem a little daunting. Note that in the standard model of arithmetic, $(\mathbb{N}; +, \times, 0, 1)$, the formula

$$\varphi(x, y) \equiv (\exists z)[(z \times x = y) \wedge (x \neq 1)],$$

is a versatile formula. φ just states that x is a non-trivial divisor of y. The following is from Theorem 3.1 in [3].

Theorem 7.4. *There exists a maximal class with respect to the Keisler Order.*

Proof. We show that if $\mathfrak{A}$ has a versatile formula, then $\mathfrak{A}$ is maximal. Let D be a regular ultrafilter and suppose that $\prod_D \mathfrak{A}$ is α^+-saturated. It suffices to show that for any (countable) language $\mathscr{L}'$ and any $\mathscr{L}'$-structure $\mathfrak{B}$, $\prod_D \mathfrak{B}$ is α^+-saturated.

Let W be a subset of $\prod_D \mathfrak{B}_i$ of size α. Let $\mathfrak{b}$ be some α-enumeration of W. Let $\Delta(x)$ be a collection of formulas in one free variable (in the language $\mathscr{L}(\alpha)$). Suppose that $\Delta(x)$ is finitely satisfiable in $(\prod_D \mathfrak{B}, \mathfrak{b})$. We want to show that $\Delta(x)$ is realized in $(\prod_D \mathfrak{B}, \mathfrak{b})$. For each i in I we fix $\mathfrak{b}(i)$, an α-enumeration of elements in $\mathfrak{B}_i$ such that

$$\Big(\prod_D \mathfrak{B}, \mathfrak{b}\Big) \cong \prod_D (\mathfrak{B}_i, \mathfrak{b}(i)).$$

This can by done by by Łoś's Theorem. Let X be a regular subset of D. Let $h : \Delta(x) \to X$ be an injection. As in the proof of Theorem 6.2, we define

$$\Delta(i) = \{\delta(x) \in \Delta(x) : i \in h(\delta(x))\}.$$

Notice that each $\Delta(i)$ is finite for the same reason as Theorem 6.2. Now, for $i \in I$ we can write $\Delta(i) = \{\delta_1(x), ..., \delta_n(x)\}$. We define,

$$T(i) = \{t \subset n : t \neq \emptyset, \Big(\prod_D \mathfrak{B}, \mathfrak{b}\Big) \models (\exists x) \bigwedge_{m \in t} \delta_m(x)\}.$$

Since $\Delta(x)$ is finitely satisfiable, we know that $T(i)$ is a weak ideal over n. Let $\varphi(x, \overline{y})$ be a versatile formula for $\mathfrak{A}$. Assume that $|\overline{y}| = k$, i.e. $\overline{y} = (y_1, ..., y_k)$. Then, for any $n > 0$ and F a weak ideal over n, we have that the formula $\theta_F(\overline{y}_1, ..., \overline{y}_n)$ is finitely satisfiable in $\mathfrak{A}$. In particular, $\theta_{T(i)}(\overline{y}_1, ..., \overline{y}_n)$ is finitely satisfiable in $\mathfrak{A}$. We let $\overline{a}_{1,i}, ..., \overline{a}_{n,i}$ be a sequence of elements in $\mathfrak{A}^{|\overline{y}|}$ which satisfy the formula $\theta_{T(i)}(\overline{y}_1, ..., \overline{y}_n)$ in the obvious way. Now, for fixed $\delta(x) \in \Delta(x)$, we choose a function f_δ from I to $\mathfrak{A}^{|\overline{y}|}$ such that for any i such that $\delta(x) \in \Delta(i)$ (and so $\delta(x) = \delta_l(x)$ for some l less than or equal to $|\Delta(i)|$), we have that $f_\delta(i) = \overline{a}_{l,i}$. Moreover, since $|\overline{y}| =$

k, we can split f_δ into functions which map from I into $\mathfrak{A}$. In particular, for $l \leq k$, we let $f_{\delta,l}$ map from I to $\mathfrak{A}$ such that $f_\delta(i) = (f_{\delta,1}(i), ..., f_{\delta,k}(i))$. Now, for every $l \leq k$, we define $g_{\delta,l}$ as $f_{\delta,1}/D$. We notice that the cardinality of $\{g_{\delta,l} : \delta(x) \in \Delta(x), l \leq k\}$ is equal to α. We let $\mathfrak{a}$ be an enumeration of this set. Therefore, $(\mathfrak{A}, \mathfrak{a})$ is a $\mathscr{L}(\alpha)$ structure. Notice that for any index j, we have well-defined $\mathscr{L}(\alpha)$ structure, $(\mathfrak{A}_i, \mathfrak{a}(i))$ where $\mathfrak{a}(i)$ is induced enumeration of $\{f_{\delta,l}(i) : \delta(x) \in \Delta(x), l \leq k\}$.

Now, for every $\delta(x)$ in $\Delta(x)$, we consider the formula $\sigma_\delta(x) = \varphi(x, c_{\delta,1}, ..., c_{\delta,k})$ where $c_{\delta,l}$ is the constant in $\mathscr{L}(\alpha)$ corresponding to the element $g_{\delta,l}$. By construction, for any $i \in I$ and $\delta_1(x), ..., \delta_n(x) \in \Delta(i)$, we have that $\delta_1(x) \wedge ... \wedge \delta_m(x)$ holds in $(\mathfrak{B}, \mathfrak{b}(i))$ if and only if $\sigma_{\delta_1}(x) \wedge ... \wedge \sigma_{\delta_n}(x)$ holds in $(\mathfrak{A}, \mathfrak{a}(i))$. Since $(\exists x)\delta_1(x) \wedge ... \wedge \delta_m(x)$ holds on a large subset of D, we also know that $\exists(x)\sigma_{\delta_1}(x) \wedge ... \wedge \sigma_{\delta_n}(x)$ holds on a large subset of D. Therefore, $\sigma_{\delta_1}(x) \wedge ... \wedge \sigma_{\delta_n}(x)$ is satisfied in $(\prod_D \mathfrak{A}, \mathfrak{a})$ and the set $\Phi = \{\varphi_\delta : \delta \in \Delta\}$ is finitely satisfiable in $(\prod_D \mathfrak{A}, \mathfrak{a})$. Since $\prod_D \mathfrak{A}$ is α^+-saturated and $|\Phi| \leq \alpha$, Φ is realized in $\prod_D \mathfrak{A}$. Let g/D satisfy Φ.

We now construct a map from I into $\mathfrak{B}$. For each i in I, we let $f(i) \in \mathfrak{B}$ be an element which satisfies $\delta(x)$ whenever $\delta(x) \in \Delta(i)$ and $g(i)$ satisfies $\phi_\delta(x)$. Notice that $f(i)$ satisfies $\delta(x)$ on a large set because $\{i \in I : \delta(x) \in \Delta(i)\}$ and $\{i \in I : \mathfrak{A}_i \models \sigma_\delta(g(i))\}$ are both large sets. Therefore, f/D satisfies Δ in $(\prod_D \mathfrak{B}, b)$. Hence, $\prod_D \mathfrak{B}$ is α^+-saturated.

□

Note that by our theorem in the last section, we now know that the theory of arithmetic is maximal with respect to the Keisler Order. However, a necessary and sufficient condition for maximality is still unknown.

As stated at the beginning of the paper, the complete order type of the Keisler Order is still unknown. It is known that there exists a minimal class, at least two non-minimal and non-maximal classes, and a maximal class. At the time of this editing, it is also known that the Keisler order is infinite has contains an infinite descending chain of classes [5].

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References

[1] Blass, A. "A Model Without Ultrafilters." *Bull. Acad. Polon. Sci., Ser. Sci. Math. Astronom. Pyhs.* 25 (1977), no. 4, 329-331.
[2] Chang, C. C., and H. J. Keisler. *Model Theory.* Amserdam: North-Holland Pub., 1973. Print.
[3] Keisler, H. J. "Ultraproducts which are not Saturated." *The Journal of Symbolic Logic*, Vol. 32, No. 1 (Mar., 1967), pp. 23-46
[4] Malliaris, M. and S. Shelah. "General Topology Meets Model Theory, on P and T." *Proceedings of the National Academy of Sciences* 110.33 (2013): 13300-3305. Web.
[5] Malliaris, Maryanthe, and Saharon Shelah. "Keislers order has infinitely many classes." *Israel Journal of Mathematics* 224.1 (2018): 189-230.
[6] S. Shelah. *Classification theory and the number of non-isomorphic model*, rev. ed. North-Holland, 1990.