INTRODUCTION TO THE KEISLER ORDER

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Abstract. In this paper, we introduce the basic definitions and concepts necessary to define the Keisler Order. We will prove the order is well-defined as well as the existence of a maximal class with respect to the order.

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1. Introduction

The Keisler Order was first introduced by H. Jerome Keisler in 1967. Currently, this order is known to be a pre-order on (countable) first-order theories which, broadly speaking, ranks classes of theories by complexity. Stronger theorems have been proven for stable theories (e.g. the Keisler Order on stable theories is linear [5]), while the complete structure of the Keisler Order is still an open problem.

The classification of first-order theories is both a classic and modern program in model theory. Shelah’s stability program, the most famous type of classification framework, organizes theories relative to the number of definable types over subsets of a model. While the stability program has had great success, the program also leaves unstable theories in some unclassifiable purgatory. Work on the Keisler Order has shed light on dividing lines between classes of unstable theories. Additionally, one of the major results in a paper by Malliaris and Shelah [4] shows that theories, which have the $SOP_2$−property, are maximal. This result was important in proving $\mathfrak{p} = \mathfrak{t}$, the oldest open problem in cardinal invariants.

We will begin with many definitions as well as examples to provide the reader with some intuition. We will leave most of the proofs which relate to the Keisler Order to the last two sections. The two big theorems we prove at the end can be found in Keisler’s original paper on the topic [3].

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2. Notation and Basic Definitions

This paper will assume at least one course in basic first-order model theory. However, in this section, we will go over some of the necessary terminology and theorems required to understand this paper. A language \( L = \{ f_1, f_2, ..., R_1, R_2, ..., c_1, c_2, ... \} \) is a collection of (n-ary) function, (n-ary) relation, and constant symbols (sometimes called non-logical symbols). Languages also contain logical symbols, i.e. \( \land, \lor, \neg, \to \), equality, and parentheses, as well as (object-level quantification) \( \forall, \exists \). A formula in a language is simply a grammatically coherent string of logical symbols which may or may not have free variables (for instance, \( x = x \) or \((\exists x)(S(x) = \bar{y})\) where \( \bar{y} \) is a tuple of free variables and \( x \) is bounded).

A theory \( T \) is a set of logical sentences with symbols from some fixed language \( L \). A complete theory is a maximally consistent set of sentences. A model, or an \( L \)-structure, is some set-sized mathematical object with an interpretation for each non-logical symbol in the language. \( \models \) is a (semantic) binary relation between \( L \)-structures and sentences in the language \( L \). We say a sentence \( \varphi \) is true in a model \( A \) by writing \( A \models \varphi \).

The following will be our notational habits. An arbitrary model will be denoted as \( A \) or \( B \). Usually, we will denote indexing sets as \( I, J \), and cardinals as \( \alpha, \beta, \kappa \). Every theory \( T \) will have a corresponding fixed language \( L \) where the size of the underlying language is at most countable. The underlying set of a model \( A \) is formally written as \( \text{dom}(A) \). However, we will usually write \( A \) for \( \text{dom}(A) \), etc. The terms power and cardinality are interchangeable. A set \( X \) has the finite intersection property if and only if any finite intersection of elements is not empty. If \( A \) is a set, then \( P(A) \) is the power set of \( A \) and \( P_{\leq}(A) \) is the collection of all finite subsets of \( A \).

Finally, we have some more formal definitions and theorems which we will be referring to.

**Definition 2.1.** Let \( A \) be an \( L \)-structure. We let \( X \subset A \). Then, \( (A, X) \), sometimes written \( (A, x)_{x \in X} \), is a model (in the expanded language \( L \cup X \)) where we treat the elements of \( X \) as constant symbols. Formally, this is known as a diagram \(^1\).

**Definition 2.2.** A collection of sentences, \( \Delta \), is said to be satisfiable if there exists a model \( A \) such that \( A \models \Delta \). \( \Delta \) is said to be finitely satisfiable if every finite subset of \( \Delta \) is satisfiable.

**Theorem 2.3.** (Completeness): A set of sentences \( \Delta \) is consistent if and only if it is satisfiable.

**Theorem 2.4.** (Compactness): A set of sentences \( \Delta \) is satisfiable if and only if it is finitely satisfiable.

**Definition 2.5.** If \( A \) and \( B \) are two \( L \)-structures, we say that \( A \) is elementarily equivalent to \( B \) (written \( A \equiv B \)) if for any \( \varphi \) in \( L \), we have \( A \models \varphi \) if any only if \( B \models \varphi \).

**Definition 2.6.** Let \( A, B \) be two \( L \)-structures. We say that \( A \) is isomorphic to \( B \) if there exists a bijection \( f : A \to B \) such that \( f \) preserves functions, relations, and constant symbols.

\(^1\)When \( X \) is some arbitrary subset of some fixed size \( \alpha \), we will sometimes just write \( (A, \alpha) \) and \( L(\alpha) \) for the expanded model and language respectively.
For a more detailed introduction, we refer the reader to the first few sections of any basic model theory text (e.g. Chang & Keisler [2]).

3. Ultrapowers

Ultrapower constructions are one of the two central concepts necessary to understanding the Keisler Order. However, before we can define ultrapowers, we have to first get an intuition for ultrafilters and ultraproducts.

**Definition 3.1.** Let $I$ be an indexing set. We say that $D$ is a filter over $I$ if $D$ is a non-empty subset of $\mathcal{P}(I)$ with the following properties:

1. If $X \in D$ and $Z \supset X$, then $Z \in D$.
2. If $X, Y \in D$, then $X \cap Y \in D$.

Furthermore, we call $D$ an ultrafilter if for any $X \subseteq I$, we have (exclusively) either $X \in D$ or $I - X \in D$. Intuitively, we can think of $D$ as a mathematical object that makes decisions about which subsets of $I$ are large. $D$ thinks the entire set is large, any set containing a large set is large, and the intersection of any two large sets is large. Note, $D$ may not think that the countable/uncountable intersection of large sets is large. We will see later that ultrafilters without the countable intersection property are valuable and are central to our study.

**Definition 3.2.** Let $D$ be a filter over $I$. We say that $D$ is a principal filter if there exists $X \subset I$ such that $D = \{ Y \subseteq I : X \subset Y \}$.

We call any filter which is not principal a nonprincipal (or free) filter.

**Example 3.3.** (Principal Ultrafilter): Let $I = \mathbb{N}$ and let $D = \{ X \subseteq \mathbb{N} : 3 \in X \}$. Then, $D$ is a principal ultrafilter over $I$.

**Example 3.4.** (Nonprincipal Ultrafilter): It is provable that one cannot construct an example (since the existence of a nonprincipal ultrafilter is equivalent to a weak version of choice). There are models of $ZF$ where there do not exist any nonprincipal ultrafilters. However, the constructions of these models of $ZF$ require complex forcing arguments not suitable for this paper [1].

For the remainder of this paper, every ultrafilter will be a nonprincipal ultrafilter. Furthermore, we will assume the full power of $ZFC$ and thus never worry about the existence of ultrafilters (in general). The next two facts follow quickly from the definitions and are left unproven.

**Proposition 3.5.** No free ultrafilter contains any finite sets.

**Proposition 3.6.** Let $A$ be a collection of subsets of $I$ such that $A$ has the finite intersection property. Then $A$ can be extended to an ultrafilter over $I$.

Let $I$ be an indexing set of cardinality $\alpha$ and let $\{ \mathfrak{A} \}_{i \in I}$ be a collection of models. Let $\prod_{i \in I} \mathfrak{A}$ be the cartesian product of these models. Note that the elements of $\prod_{i \in I} \mathfrak{A}$ can be seen as functions from $I$ into $\{ \mathfrak{A} \}_{i \in I}$ or as an $\alpha$-termed sequences of elements where the $\eta$th term (for $\eta < \alpha$) is an element of $\mathfrak{A}_\eta$. If $f, g$ are elements of $\prod_{i \in I} \mathfrak{A}$, we say that $f$ is $D$–equivalent to $g$ (written as $f \equiv_D g$) if and only if $f$ and $g$ agree on a large set. Formally,

$$f \equiv_D g \iff \{ i \in I : f(i) = g(i) \} \in D$$

**Proposition 3.7.** If $D$ is a filter, then $\equiv_D$ is an equivalence relation over $\prod_{i \in I} \mathfrak{A}$. 

Definition 3.8. (Ultraproduct): Let $I$ be an indexing set and let $D$ be an ultrafilter over $I$. An ultraproduct of $L$–structures is defined as,

$$\prod_{i \in I} A_i / D = \{ f_D : f \in \prod_{i \in I} A_i \}$$

For notational purposes, we will always have our $I$’s fixed and so we will write $\prod_{i \in I} A_i / D$ as $\prod_D A_i$. In some sense, ultraproducts are similar to quotient spaces in topology. We are simply taking elements in our Cartesian product and gluing them together. Now, the following theorem demonstrates the strength of ultraproducts in model theory.

Theorem 3.9. (Los’s Theorem): Let $D$ be an ultrafilter over $I$. Then, for any $f_1, ..., f_n \in \prod_D A_i$, we have that

$$\prod_D A_i \models \varphi(f_1, ..., f_n) \iff \{ i \in I : A_i \models \varphi(f_1(i), ..., f_n(i)) \} \in D$$

So what does this theorem actually mean? First of all, note that if each $A_i$ agrees on some (first-order) sentences in $L$, then $\prod_D A_i$ also agrees on that sentence. In fact, if $D$ thinks some subset of $I$ is large, and all the models of the large set agree (disagree) on some sentence, then $\prod_D A_i$ also agrees (disagrees) on that sentence. Free ultraproducts can be thought of as averaging on an infinite set. They pick up on what is happening in general while forgetting about small perturbations and outliers. We will consider the following example to give an intuition on how ultraproducts work.

Example 3.10. First note that the axioms of an algebraically closed field are first-orderizable in the language $L = \{ 0, 1, +, \times \}$. We will denote $ACF$ to mean algebraically closed field while $ACF_p$ will mean algebraically closed field of characteristic $p$. Let $\mathcal{P}$ denote the set of standard primes. Furthermore, let $\mathfrak{A}_p \models ACF_p$ and so each model, $\mathfrak{A}_p$ is an algebraically closed field of characteristic $p$. Let $D$ be a nonprincipal ultrafilter of $\mathcal{P}$. Now, we consider the object $\prod_D A_i$. Note that since $\mathfrak{A}_i \models ACF$ for all $i \in \mathcal{P}$, it follows that $\prod_D A_i \models ACF$ and so $\prod_D A_i$ is an algebraically closed field. We will now find this field’s characteristic. By proposition 3.5, there is no finite set in $D$. Since $D$ is an ultrafilter, this means that $D$ contains all cofinite sets.

Define $\varphi_i$, for all $i \in \mathbb{N}$ as follows:

$$\varphi_i \equiv \neg(\underbrace{1 + 1 + 1 ... + 1}_{i} = 0)$$

For $i \in \mathbb{N}$, $\mathfrak{A}_j \models \varphi_i$, for $j \neq i$. Hence, we know that $\varphi_i$ is true on a cofinite subset of $\mathcal{P}$. Therefore, $\prod_D A_i$ cannot have characteristic $i$ for any $i \in \mathbb{N}$. Since $\prod_D A_i$ is a field and must have some characteristic, it has characteristic $0$.

Definition 3.11. (Ultrapower): If $I$ is an indexing set and $D$ is an ultrafilter over $I$, then $\prod_D A_i$ is an ultrapower if for any $i, j \in I$, we have that $\mathfrak{A}_i \cong \mathfrak{A}_j$.

Since the indexing of our models no longer provides a method of differentiation we will simply write ultrapowers as $\prod_D \mathfrak{A}$ when $I$ is fixed. This construction, in relation to ultraproducts, might seem a little odd at first. We already know the exact set of first-order sentences that $\prod_D \mathfrak{A}$ satisfies. The proof that $\prod_D \mathfrak{A} \equiv \mathfrak{A}$

2Recall that fields cannot have composite characteristic anyway.
is a trivial corollary to Los’s theorem. The following example begins to show how ultrapowers can be different from the models used to construct them.

**Example 3.12.** Let $\mathfrak{A} = (\mathbb{N}; \leq, S)$ where $\leq$ has its normal interpretation and $S$ is interpreted as the unary successor function. We let $I$ be countable and let $D$ be a nonprincipal ultrafilter over $I$. Note that $(\mathbb{N}; \leq, S)$ is well-ordered. We will show that $\prod_D \mathfrak{A}$ is not. Consider the element $f = (1, 2, 3, 4, ...) \in \prod_D \mathfrak{A}$. Notice that for every $m \in \mathbb{N}$, $m \leq f(i)$ is true on a cofinite set and as a result, true on a large set. Therefore, the element $f$ is larger than every standard natural number.

Furthermore, it is easy to show that $\mathbb{N} \models (\forall x)(x \neq 0 \rightarrow (\exists y)(S(y) = x))$. This statement simply reads: Every element not equal to 0 has a direct predecessor. Thus, we can find an infinite descending chain beginning with $f$. The chain begins like this:

\[
(1, 2, 3, 4, 5, ...)
(0, 1, 2, 3, 4, ...)
(0, 0, 1, 2, 3, ...)
\]

We know that this chain does not terminate after finitely many steps, since if it did, then $f$ would be some standard natural number. Since $\prod_D \mathfrak{A}$ has an infinite descending chain, we know that $\prod_D \mathfrak{A}$ is not well ordered.

Now, if you know some basic logic, you should be making a connection with the compactness theorem. Ultrapowers and ultraproducts are tools which apply the compactness theorem. However, while compactness simply proves that a certain model exists, ultrapowers and ultraproducts give us much more control over the models we are constructing.

Finally, in this section, we will define regular ultrafilters.

**Definition 3.13.** Let $D$ be a nonprincipal ultrafilter over some infinite indexing set $I$. We say that $D$ is a $(\beta, \alpha)$–regular ultrafilter if there is a subset $X$ of $D$ such that

1. $|X| = \alpha$.
2. For any subset $Y$ of $X$ such that $|Y| = \beta$, we have that $\bigcap Y = \emptyset$.

We drop the $(\beta, \alpha)$ notation and just call an ultrafilter $D$ regular if $\beta = \omega$ and $\alpha = |I|$. We also call $X$ a regular subset of $D$ if the above properties holds for $X$.

Since this is the type of ultrafilter we actually need for the definition of the Keisler Order, we will prove that $D$ regular ultrafilters exist.

**Lemma 3.14.** *(Regular Ultrafilter Existence):* For every infinite cardinal $\kappa$, there exists a regular ultrafilter over $\kappa$.

**Proof.** Let $\mathcal{P}_{\mathbb{R}_0}(\kappa)$ be the set of all finite subsets of $\kappa$. Note that $|\mathcal{P}_{\mathbb{R}_0}(\kappa)| = \kappa$. Let $f : \mathcal{P}_{\mathbb{R}_0}(\kappa) \rightarrow \kappa$ be a bijection and for each $\beta \in \kappa$, define $Y_\beta = \{i \in I : \beta = f^{-1}(i)\}$. Now, consider $A = \{Y_\beta : \beta \in \kappa\}$. It is clear that $|A| = \kappa$. Recall that if $A$ has the finite intersection property, then $A$ can be extended to an ultrafilter. Consider:

\[
\bigcap_{j=1}^{n} Y_{\beta_j} = \bigcap_{j=1}^{n} \{i \in I : \beta_j \in f^{-1}(i)\}
\]

3Note that $S$ can be defined in the language $\{\leq\}$. We have added $S$ to our language to simplify our arguments.
By definition, we have
\[
\{ i \in I : \beta_1, \ldots, \beta_n \in f^{-1}(i) \} \neq \emptyset
\]
The inequality follows from the fact that \( f \) is a bijection from \( \mathcal{P}_{\aleph_0}(\kappa) \) to \( \kappa \). Therefore, \( A \) has the finite intersection property and may be extended into an ultrafilter. It should also be clear that the intersection of countable subsets of \( A \) are empty and so \( A \) is our regular subset of its ultrafilter extension.

\[
\square
\]

4. Saturation and Satisfaction

While ultrapowers are necessary for understanding the Keisler Order, this topic alone is not sufficient. Another key ingredient of the Keisler Order is saturation. This concept, along with satisfaction, will bring the Keisler Order into view.

Definition 4.1. Let \( \mathfrak{A} \) be a model in a language \( \mathcal{L} \). Let \( X \subseteq A \). We say that \( \rho \) is an \( n \)-type over \( X \) in \( \mathfrak{A} \) if
1. \( \rho \) is a collection of formulas in \( n \) free variables in the language \( \mathcal{L} \cup X \) (i.e. \( \rho \) is of the form \( \bigcup_{i \in I} \{ \varphi_i(y_1, \ldots, y_n) \} \) where \( \varphi_i(y_1, \ldots, y_n) \) are in \( \mathcal{L} \cup X \)).
2. For any finite subset \( \rho_0 \) of \( \rho \), there is some \( (c_1, \ldots, c_n) \in \mathfrak{A} \) such that \( \mathfrak{A} \models \varphi_i(c_1, \ldots, c_n) \) for all \( \varphi \in \rho_0 \).

We say that an \( n \)-type \( \rho \) is complete if and only if it is maximally consistent. Equivalently, \( \rho \) is a complete \( n \)-type if for any formula \( \psi(y_1, \ldots, y_n) \) in \( n \) free variables, (exclusively) either \( \psi(y_1, \ldots, y_n) \in \rho \) or \( \neg \psi(y_1, \ldots, y_n) \in \rho \). Note that every element of a model has a corresponding complete 1-type (over \( X \)) of elements in \( \mathfrak{A} \), let
\[
\rho \equiv \{ \phi(y_1, \ldots, y_n) : (\mathfrak{A}, x, a_1, \ldots, a_n) \models \phi(a_1, \ldots, a_n) \}
\]

Definition 4.2. (Satisfaction of 1-Types): Let \( \rho \) be a complete 1-type in one free variable. We say that \( \rho \) is satisfied/realized in \( \mathfrak{A} \) if there exists an \( a \) in \( \mathfrak{A} \) such that \( (\mathfrak{A}, a) \models \varphi(a) \) for all \( \varphi(x) \in \rho \).

We let \( S_1(X) \) be the collection of all (consistent) complete 1-types over \( X \subseteq A \).

Remark 4.3. Note that the above definition can be clearly extended to \( n \)-types and has corresponding collections, \( S_n(X) \).

Definition 4.4. (\( \kappa \)-Saturation): Let \( \kappa \) be some infinite cardinal. We say that a structure \( \mathfrak{A} \) is \( \kappa \)-saturated if for every \( X \subseteq A \) with \( |X| < \kappa \), all the types in \( S_1(X) \) are realized in \( \mathfrak{A} \).

Proposition 4.5. If \( \mathfrak{A} \) is \( \kappa \)-saturated and \( |X| < \kappa \), then every type in \( S_n(X) \) is realized in \( \mathfrak{A} \).

Proposition 4.6. Not every theory has a saturated model in every cardinality.

Before we go any further with our definition building, we will give an indepth example.
Example 4.7. \((\mathbb{Q}, <)\): Let us consider \(\mathbb{Q}\) in the language \(\{<\}\). We will find that there does not exist an \(\aleph_1\)-saturated model of size \(\aleph_0\) for \(\aleph_0 \leq \aleph_0 < 2^{\aleph_0}\). The problem here is that we have continuum many 1-types which are definable in \(L\) over \(\mathbb{Q}\). Better yet, we are showing that \(S_1(\mathbb{Q}) = 2^{\aleph_0}\). Consider any two (distinct, irrational) real numbers, \(s\) and \(r\). We will write \(\rho_s\) and \(\rho_r\) as their complete corresponding 1-types over \(\mathbb{Q}\). Without loss of generality, we assume that \(s < r\). Since \(s \neq r\) and \(\mathbb{Q}\) is dense inside the reals, we have that there exists \(q \in \mathbb{Q}\) such that \((x < q) \in \rho_s\) and \((x > q) \in \rho_r\). Hence, every real number corresponds to a different (complete) 1-type over a countable subset of the model (where the countable subset is the entire model itself). Therefore, the \(\aleph_1\)-saturated model is at least size \(2^{\aleph_0}\) (since \(|\mathbb{R}| = 2^{\aleph_0}\)).

However, \(\langle \mathbb{R}, < \rangle\) is not a \(\aleph_1\)-saturated model of the theory of \(Th(\mathbb{Q}, <)\). Let \(\langle \mathbb{D}, < \rangle\) be a \(\aleph_1\)-saturated model. We will show that \(\langle \mathbb{D}, < \rangle\) realizes a type over \(\mathbb{Q}\) that \(\langle \mathbb{R}, < \rangle\) does not realize. Consider the type \(\rho_t = \{x < q : q \in \mathbb{Q}\} \cup \{x > 0\}\). First note, that since the rationals are dense in themselves, no elements of \(\mathbb{Q}\) realize this type. However, one can easily show that \(\rho\) is finitely satisfiable. By the compactness theorem, we note that \(\rho_t\) is consistent. Therefore, it must be satisfied in our \(\aleph_1\)-saturated model (since it is definable over a countable subset of our underlying set). Let \(a\) be the element of \(\mathbb{D}\) which satisfies this type. Note that for any \(q \in \mathbb{Q}\), we have that \(q < 0\) or \(q > a\). Suppose that \(r \in \mathbb{R}\) and \(r = a\). Since the rationals are dense in the reals, then there must be some \(p \in \mathbb{Q}\) such that \(0 < p < r\). But this a contradiction. Hence, \(\langle \mathbb{R}, < \rangle\) is not \(\aleph_1\)-saturated.

5. An Early Application

We have just defined a lot of new machinery, but it is probably still unclear how ultrapowers and saturation relate to one another. This section is dedicated to exhibiting the interaction of the two.

Definition 5.1. (Countably Incomplete Ultrafilter): An ultrafilter is said to be countably incomplete if there exists a subset \(X\) of \(D\) such that \(|X| = \aleph_0\) and \(\bigcap X = \emptyset\).

These ultrafilters are much weaker than regular filters, so our theorem will be very general. We are going to show that any ultrapower, using a countably incomplete ultrafilter, is \(\aleph_1\)-saturated. However, we will need the following lemma first.

Lemma 5.2. Let \(D\) be a countably incomplete ultrafilter. Then, there exists a countable descending chain \(I = I_0 \supset I_1 \supset I_2 \supset \ldots\), such that \(\bigcap_{n \in \omega} I_n = \emptyset\).

Proof. Since \(D\) is an ultrafilter, we know that \(I \in D\). Since \(D\) is countably incomplete, we know that there exists a set \(X \subset D\) such that \(|X| = \aleph_0\) and \(\bigcap X = \emptyset\). Let \(\{Y_1, \ldots, Y_n, \ldots\}\) be a well-ordering of the elements of \(X\).

Define

\[ J_n = \bigcap_{i=1}^{n} Y_i. \]

\(^4\)We are actually showing that \(S_1(\mathbb{Q}) \geq 2^{\aleph_0}\). The other direction follows from the fact that there are at most \(2^{\aleph_0}\)—many definable 1-types over a countable set in a countable language.
Since $D$ is closed under intersection, it is closed under finite intersection. Therefore, $J_n \in D$ for all $n < \omega$. Furthermore, it is clear that $J_n \supseteq J_{n+1}$ and that we have the following equality
$$\bigcap_{i \in \omega} J_n = \emptyset$$

We also know that for each $J_n$, there exists some $J_m$ such that $J_n \supseteq J_m$, for some $m > n$. If this was not the case, then $\bigcap_{i \in \omega} J_i = J_m$. Now, we can choose a subsequence of $\{J_i\}_{i \in \omega}$ such that $J_{i+1}$ is a proper subset of $J_i$ for each $i$. By well-ordering this set in the obvious way, we have found the collection that we are looking for.

\[\square\]

**Theorem 5.3.** Let $\mathcal{L}$ be countable and let $D$ be a countably incomplete ultrafilter over some infinite set $I$. Then, for any collection $\{\mathfrak{A}_i\}_{i \in I}$ of $\mathcal{L}$-structures, we have that $\prod_D \mathfrak{A}_i$ is $\aleph_1$-saturated.

**Proof.** Let $\Delta(x)$ be a set of formulas (with one free variable) in the language $\mathcal{L}_1 = \mathcal{L}(\aleph_0)$. It suffices to show that if each finite subset of $\Delta(x)$ is satisfied in $\prod_D \mathfrak{A}_i$, then $\Delta(x)$ is realized in $\prod_D \mathfrak{A}_i$.

Suppose that each finite subset of $\Delta(x)$ is realized in $\prod_D \mathfrak{A}_i$. Because $\mathcal{L}_1$ is countable, we know that $\Delta(x)$ is countable. Therefore, we can well order our elements of $\Delta(x)$ as follows;
$$\Delta(x) = \{\delta_1(x), \delta_2(x), \ldots\}$$

Since $D$ is countably incomplete, we know that there exists a descending chain $I = I_0 \supseteq I_1 \supseteq I_2 \supseteq \ldots$ such that $\bigcap_{n \in \omega} I_n = \emptyset$. Now, we let $X_0 = I_0$ and define
$$X_n = I_n \cap \{i \in I : \mathfrak{A}_i \models (\exists x)(\delta_1 \land \ldots \land \delta_n)\}$$

Recall that we assumed that every finite subset of $\Delta(x)$ is satisfied in $\prod_D \mathfrak{A}_i$. Therefore, by Los’s Theorem, we have that $\{i \in I : \mathfrak{A}_i \models (\exists x)(\delta_1 \land \ldots \land \delta_n)\}$ is large (hence, it is in $D$). Since $I_n$ is also in $D$, we know by definition of a filter that $X_n$ is in $D$ for each $n \in \mathbb{N}$. Further note that $\bigcap_{n \in \omega} X_n = \emptyset$ because of the following equality
$$\bigcap_{n \in \omega} X_n = \bigcap_{n \in \omega} (I_n \cap \{i \in I : \mathfrak{A}_i \models (\exists x)(\delta_1 \land \ldots \land \delta_n)\})$$
\[\begin{align*}
= & \bigcap_{n \in \omega} (I_n) \cap \bigcap_{n \in \omega} (\{i \in I : \mathfrak{A}_i \models (\exists x)(\delta_1 \land \ldots \land \delta_n)\}) \\
= & \emptyset \cap \bigcap_{n \in \omega} (\{i \in I : \mathfrak{A}_i \models (\exists x)(\delta_1 \land \ldots \land \delta_n)\}) = \emptyset
\end{align*}\]

Furthermore, we also know that $X_n \supseteq X_{n+1}$. Now, for all $i \in I$, there is a largest $n(i) < \omega$ such that $i \in X_{n(i)}$. Now, we find an element $\prod_D \mathfrak{A}_i$, which satisfies $\Delta(x)$. We are constructing a function $f$. If $n(i) = 0$, let $f(i)$ be any arbitrary a in $A_i$. If $n(i) < 0$, choose $f(i) \in A_i$ such that
$$\mathfrak{A}_i \models \delta_1(f(i)) \land \ldots \land \delta_{n(i)}(f(i))$$

Note that for any $i \in X_n$, we have that $n \leq n(i)$ and therefore $\mathfrak{A}_i \models \delta(f(i))$. Since this is a large set, by Los’s theorem, we have that $\prod_D \mathfrak{A}_i \models \delta(f_D)$ for all $n > 0$. Therefore, $f_D$ satisfies $\Delta(x)$ in $\prod_D \mathfrak{A}_i$. We have finished the proof.

\[\square\]

**Remark 5.4.** Note that every regular ultrafilter is countably incomplete.

Corollary 5.5. If $D$ is a nonprincipal ultrafilter over $\mathbb{N}$ and $\{\mathcal{A}_i\}_{i \in \mathbb{N}}$ is a collection of $\mathcal{L}$-structures, then $\prod_D \mathcal{A}$ is $\aleph_1$-saturated.

Proof. By the theorem above, it suffices to show that every nonprincipal ultrafilter over $\mathbb{N}$ is countably incomplete. Recall that since $D$ is nonprincipal, it contains all cofinite sets. Let $\mathcal{C}$ be the collection of all cofinite sets in $D$. Recall also that $|\mathcal{P}_{\aleph_0}(\mathbb{N})| = \aleph_0$. We have a natural bijection $f : \mathcal{P}_{\aleph_0}(\mathbb{N}) \to \mathcal{C}$ defined by:

$$f(S) = \mathbb{N} - S$$

Now, let $I_0 = \mathbb{N}$ and define

$$I_{n+1} = I_n - n$$

It is clear that $I_n \in D$ and that $I_n \supset I_{n+1}$ for $n \in \omega$. Finally, for sake of contradiction, suppose that

$$\bigcap_{n \in \omega} I_n \neq \emptyset$$

Then, there exists some $x$ in the natural numbers such that $x \in \bigcap_{n \in \omega} I_n$. However, consider $I_{x+1}$. By definition, $I_{x+1} = I_x - x$ which shows that $x \notin \bigcap_{n \in \omega} I_n$. Hence, we have a contradiction and so $\bigcap_{n \in \omega} I_n = \emptyset$.

Therefore, $D$ must be countably incomplete and so $\prod_D \mathcal{A}$ is $\aleph_1$-saturated by the previous theorem. \qed

6. The Order

Now that we have all the necessary definitions in place, we can finally define the Keisler Order.

Definition 6.1. (Keisler Order): We say that a theory $T_1 \subseteq \kappa T_2$ if for any $\mathcal{A}_1 \models T_1$, $\mathcal{A}_2 \models T_2$, and regular ultrafilter $D$ over $\kappa$, we have that if $\prod_D \mathcal{A}_1$ is $\kappa^+\text{-saturated}$ then $\prod_D \mathcal{A}_1$ must be $\kappa^+\text{-saturated}$. Now we say that $T_1 \subseteq \kappa T_2$ if for every infinite cardinal, $\kappa$, we have that $T \subseteq \kappa T_2$. This second definition, $\subseteq$, is the Keisler Order.

Note that the ultrafilter construction is model theoretic while the order is on the theories of the models. Therefore, we still need to show that this definition is well defined (i.e. that it is not dependent on our choice of model).

Theorem 6.2. Fix some language $\mathcal{L}$ and some indexing set $I$. Furthermore, suppose that $\mathcal{A} \equiv \mathcal{B}$ over $\mathcal{L}$ and $D$ is a regular ultrafilter over $I$. Then we have that $\prod_D \mathcal{A}$ is $\alpha^+\text{-saturated}$ if and only if $\prod_D \mathcal{B}$ is $\alpha^+\text{-saturated}$.

Proof. First note that the two directions have the same proof. Assume, without loss of generality, that $\prod_D \mathcal{A}$ is $\alpha^+\text{-saturated}$. Let $X = \{Y_i\}_{i \in I}$ be a regular subset of $D$. Let $\Delta$ be a collection of formulas in one free variable (in the expanded language $\mathcal{L}(\alpha)$) such that $\Delta$ is finitely satisfiable in $(\prod_D \mathcal{A}, \alpha)$ and $|\Delta| \leq \alpha$. It suffices to show that $\Delta$ is realized in $(\prod_D \mathcal{A}, \alpha)$.

Since $|\Delta| \leq |X| = \alpha$, let $h$ be an injection from $\Sigma$ into $X$. We define

$$\Delta(i) = \{ \delta \in \Delta : i \in h(\delta) \}$$

and

$$X(i) = h(\Delta(i)) = \{ h(\delta) : \delta \in \Delta(i) \}$$

Note that $\Delta(i)$ is finite. If $\Delta(i)$ were infinite, then we could find an infinite collection of elements in $X$ such that their intersection would be non-empty. Finding this collection would contradict the regularity of $X$. 

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Also, since $X(i)$ is the image of an injection from a finite set, we have that $|X(i)| = |\Delta(i)| < \omega$.

Let $a = \{a_i\}_{i \in I}/f D$, and for each $i$ let $a(i) = a_i$. Let $\Gamma(i)$ be the set of all sentences of $\mathcal{L}(\alpha)$ of the form

$$\{ (\exists x) \bigwedge_{j \in s} \delta_j : s \neq 0, s \in P(\Delta(i)), \delta_j \in \Delta(i) \}$$

Note that $\Gamma(i)$ is a valid collection of first-order sentences since $\Delta(i)$ is finite. $\Gamma(i)$ is simply every possible subcollection of sentences in $\Delta(i)$. Since $\Gamma(i)$ is finite and $\mathfrak{A}_i \equiv \mathfrak{B}_i$, we can choose $b(i)$ in $\mathfrak{B}_i$ such that

$$\Gamma(i) \cap Th(\mathfrak{A}_i, a(i)) = \Gamma(i) \cap Th(\mathfrak{B}_i, b(i))$$

for all $i \in I$. Therefore, we have chosen our $b(i)$’s such that each subset of $\Delta(i)$ realized in $(\mathfrak{A}_i, a(i))$ if and only if it is realized in $(\mathfrak{B}_i, b(i))$.

Let $b = \{b_i\}_{i \in I}/D$. Therefore, $b$ is an element of $\prod_D \mathfrak{B}$. We will now show that $\Delta$ is finitely satisfiable in $(\prod_D \mathfrak{B}, b)$. Let $\delta_1, \ldots, \delta_n \in \Delta$ and let

$$\varphi = (\exists x) \bigwedge_{1 \leq j \leq n} \delta_j$$

Note that $\{i \in I : \varphi \in \Gamma(i)\} \in D$. Since $\Delta$ is finitely satisfiable in $(\prod_D \mathfrak{A}, a)$, $\varphi$ holds in $\prod_D \mathfrak{A}$ and therefore, $\varphi$ holds in $(\mathfrak{A}_i, a(i))$ on a large set. Now, from above, we have that for every $i \in I$ such that $(\mathfrak{A}_i, a(i)) \models \varphi$, we have that $(\mathfrak{B}_i, b(i)) \models \varphi$. Therefore, $\varphi$ holds in $(\mathfrak{B}_i, b(i))$ for a large subset of $D$. Then $\{i \in I : (\mathfrak{B}_i, b(i)) \models \varphi\}$ is a large set and so $\varphi$ holds in $(\prod_D \mathfrak{B}, b)$.

Because $\Delta$ is finitely satisfiable in $(\prod_D \mathfrak{B}, b)$, $\Delta$ has power at most $\alpha$, and $\prod_D \mathfrak{B}$ is $\alpha^+-$saturated, there exists an element $g/D \in \prod_D \mathfrak{B}$ such that $g/D$ realizes $\Delta$.

For each $i \in I$, we let $T(i)$ be the set of all $\delta \in \Delta(i)$ such that $g/D$ satisfies $\delta$ in $(\mathfrak{B}_i, b(i))$. Since $T(i)$ is finitely satisfiable in $(\mathfrak{B}_i, b(i))$, it is also finitely satisfiable in $(\mathfrak{A}_i, a(i))$. Note that since $T(i)$ is finite, we can find an element $f(i) \in A_i$ such that $f(i)$ realizes $T(i)$ in $(\mathfrak{A}_i, a(i))$.

Finally, we must now show that $f/D = \{f(i)\}_{i \in I}/D$ satisfies $\Delta$ in $(\prod_D \mathfrak{A}, a)$. Let $\delta \in \Delta$. Then $\delta \in \Delta(i)$ on a large set. Also, $g/D$ satisfies $\delta$ in $(\prod_D \mathfrak{B}, b)$, so $g(i)$ satisfies $\delta$ in $(\mathfrak{B}_i, b(i))$ for on a large subset of $D$. Thus, we have that $\delta \in T(i)$ for a large subset of $D$. It follows that $f(i)$ satisfies $\delta$ in $(\mathfrak{A}_i, a(i))$ for a large subset of $D$, and therefore, $f/D$ satisfies $\Delta$ in $\prod_D \mathfrak{A}$.

\section{Existence of a Maximal Class}

In this section, we will prove the existence of a maximal class. The theories we will show are maximal, in some sense, can define the concept of saturation. The theories encode the idea of saturation. Note that the following proof provides a sufficient condition for maximality.

\textbf{Definition 7.1.} (Weak Ideals): Let $n \in \mathbb{N}$. We say that $T$ is a weak ideal over $n$ if

1. $T \subseteq P(n)$ and $T \neq \emptyset$.
2. If $t \in T$ and $\emptyset \neq s \subset t$, then $s \in T$.

The following example is the key concept to keep in mind when understanding why weak ideals are important.
Example 7.2. Let $\Delta = \{\delta_1, ..., \delta_n\}$. Suppose that $\mathfrak{A} \models \delta_1 \land ... \land \delta_m$ where $m \leq n$. Then, if we let,

$$T = \{ s \in \mathcal{P}(n) : \mathfrak{A} \models \bigwedge_{k \in s} \delta_k \}$$

then, $T$ is a weak ideal over $n$.

Definition 7.3. (Versatile Formula): Let $\varphi(x_0, \bar{x})$ be some formula in a language $\mathcal{L}$. We say that $\varphi(x_0, \bar{x})$ is a versatile formula in some $\mathcal{L}$-structure, $\mathfrak{A}$, if for every $n \in \mathbb{N}$ and every weak ideal $T$ over $n$, we have that

$$\mathfrak{A} \models (\exists x_0, \bar{x}) \left( \left[ \bigwedge_{t \in T} (\exists x_0) \bigwedge_{m \in t} \varphi(x_0, \bar{x}) \right] \land \left[ \bigwedge_{t \in \overline{T}} \neg(\exists x_0) \bigwedge_{m \in t} \varphi(x_0, \bar{x}) \right] \right)$$

At first glance, the versatile formula might seem a little daunting. Note that in the standard model of arithmetic, $(\mathbb{N}; +, \times, 0, 1)$, the formula,

$$\varphi(x_0, x_1) \equiv (\exists z)[(z \times x_0 = x_1) \land (x_0 \neq 1)]$$

is a versatile formula. $\varphi$ just states that $x_0$ is a non-trivial divisor of $x_1$.

Theorem 7.4. There exists a maximal class with respect to the Keisler Order.

Proof. We show that if $\mathfrak{A}$ has a versatile formula, the $\mathfrak{A}$ is maximal.

Let $D$ be a regular ultrafilter and suppose that $\prod_D \mathfrak{A}$ is $\alpha^+$-saturated. It suffice to show that for any (countable) language $\mathcal{L}'$ and any $\mathcal{L}'$-structure $\mathfrak{B}$, $\prod_D \mathfrak{B}$ is $\alpha^+$-saturated.

Let $\mathfrak{b}$ be an $\alpha$-term sequence in $\prod_D \mathfrak{B}$. Let $\Delta$ be a collection of formulas in one free variable (in the language $\mathcal{L}'(\alpha)$) such that $|\Delta| \leq \alpha$. Also, suppose that $\Delta$ is finitely satisfiable in $\prod_D (\mathfrak{B}, b)$.

Let $X$ be a regular subset of $D$. Let $h : \Delta \to X$ be an injection. Again, set

$$\Delta(i) = \{ \delta \in \Delta : i \in h(\delta) \}$$

Notice that each $\Delta(i)$ is finite for the same reason as the previous theorem. Note also that

$$\prod_D (\mathfrak{B}_i, b) = \prod_D (\mathfrak{B}_i, b(i))$$

Now, for $i \in I$ we can write $\Delta(i) = \{ \delta_1, ..., \delta_n \}$. Let

$$T = \{ t \subset n : t \neq 0, (\mathfrak{B}_i, b(i)) \models (\exists x) \bigwedge_{m \in t} (\exists x_0) \delta_m \}$$

Then, we know that $T$ is a weak ideal over $n$. Let $\varphi(x_0, \bar{x})$ be a versatile formula for $\mathfrak{A}$. Now, for each $i \in I$, we let $\bar{a}(\delta_m, i)$ be the tuple of elements of $\mathfrak{A}_i$ which satisfy $\bar{x}$. Now, for every $\delta \in \Delta$, choose $f_\delta$ to be a function from $I$ into $\prod_D \mathfrak{A}$ such that whenever $\delta \in \Delta(i)$, $f_\delta = \bar{a}(\delta_i, i)$. Finally, we set $\overline{\prod} = \overline{f_\delta}/D$.

Since $\Delta$ has size $\alpha$, the elements $\overline{\prod}$ can be put into correspondence with the constants in $\mathcal{L}(\alpha)$. For every $\delta \in \Delta$, let $\varphi_\delta$ be the one-variable formula of $\mathcal{L}(\alpha)$ generated by replacing $\bar{x}$ by $\overline{\prod}$.

It is now the case that for any $i \in I$ and $\delta_1, ..., \delta_n \in \Delta(i)$, $\delta_1 \land ... \land \delta_m$ holds in $(\mathfrak{B}, b(i))$ if and only if $\varphi_{\delta_1} \land ... \land \varphi_{\delta_m}$ holds $(\mathfrak{A}, a(i))$. Since $\delta_1 \land ... \land \delta_m$ holds on a large subset of $D$, we also know that $\varphi_{\delta_1} \land ... \land \varphi_{\delta_m}$ holds on a large subset of $D$. Therefore, $\varphi_{\delta_1} \land ... \land \varphi_{\delta_m}$ is satisfied in $\prod_D (\mathfrak{A}, a)$ and the set $\Phi = \{ \varphi_\delta : \delta \in \Delta \}$ is finitely satisfiable in $\prod_D (\mathfrak{A}, a)$. Since $\prod_D \mathfrak{A}$ is $\alpha^+$-saturated, $\Phi$ is realized in $\prod_D \mathfrak{A}$. Let $g/D$ satisfy $\Phi$. Now choose, for each $i$, and element $f(i) \in B$ which
satisfies $\delta$ whenever $\delta \in \Delta(i)$ and $g(i)$ satisfies $\phi_i$. Then for each $\delta \in \Delta$, $f(i)$ satisfies $\delta$ on a large set, therefore $f/D$ satisfies $\Delta$ in $(\prod_D B, b)$. Hence, $\prod_D B$ is $\alpha^+-$saturated.

Note that by our theorem in the last section, we now know that the theory of arithmetic is maximal with respect to the Keisler Order. However, a necessary and sufficient condition for maximality is still unknown.

As stated at the beginning of the paper, the complete order type of the Keisler Order is still unknown. It is known that there exists a minimal class, at least two non-minimal and non-maximal classes, and a maximal class.

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**References**


