DISTRIBUTION THEORY AND APPLICATIONS TO PDE

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Abstract. We introduce the theory of distributions and examine their relation to the Fourier transform. We then use this machinery to find solutions to linear partial differential equations, in particular, fundamental solutions to partial differential operators. Finally, we develop Sobolev spaces in order to study the relationship between the regularity of a partial differential equation and its solution, namely elliptic regularity.

Contents

1. Distribution Theory

1.1. **Introduction to Distributions.** Distributions are an important tool in modern analysis, especially in the field of partial differential equations, as we shall see later in the paper, in addition to being very useful in physics and engineering. The utility of distributions arises from the fact that they are generalized functions, which allows for operations, such as differentiation and convolution, on objects that fail to be functions. Distributions also have the 'nice' property that they act on a space of test functions whose elements are smooth and zero outside of some closed and bounded set. The purpose of this paper is to demonstrate some of the interesting properties that such generalized functions possess and then use these properties to prove some major results in partial differential equations.

Due to the breadth of basic tools in analysis that are employed in the study of distributions and partial differential equations, we assume knowledge of such tools, in particular functional analysis, Fourier analysis, L^p spaces, and point-set topology. Also, unless otherwise specified, all functions are defined on $\mathbb{R}^n$. It should be noted

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that the proofs of some lemmas, propositions and theorems have been omitted due to their length and their lack of relevance to the focus of the paper.

For the theory of distributions, we follow Gerald B. Folland's *Real Analysis* [4]. To begin, we first study the properties of the space of functions C_c^∞. For $E \subset \mathbb{R}^n$, we define $C_c^\infty(E)$ to be the set of all functions such that are infinitely continuously differentiable and whose support is compact and contained in E. For C_c^∞, we can define the following norms:

Definition 1.1. We define the uniform norm for f on $E \subset \mathbb{R}^n$ by

$$\|f\|_u = \sup_{x \in E} |f(x)|.$$

For norms for C_c^∞, we apply the uniform norm for all partial derivatives with respect to the multi-index α.

Definition 1.2. For $E \subset \mathbb{R}^n$ and $\phi \in C_c^\infty$, the norms are given by

$$\|\phi\|_{[\alpha]} = \|\partial^\alpha \phi(x)\|_u.$$

Thus, we say a sequence $\{\phi_k\}$ converges in C_c^∞ to ϕ if and only if $\partial^\alpha \phi_k$ converges uniformly to $\partial^\alpha \phi$ for all α. This set of norms allows the space C_c^∞ to be a Fréchet space, i.e it is a complete, metrizable, locally convex topological vector space.

Now, we define a distribution as follows:

Definition 1.3. For $U \subset \mathbb{R}^n$, a *distribution* on U is a continuous linear functional F on $C_c^\infty(U)$ such that for every compact $K \subset U$, $F|C_c^\infty(U)$ is continuous with respect to the topology defined by the norms on C_c^∞.

For a distribution F, we use the notation $< F, \phi >$, where $\phi \in C_c^\infty$, so that it is understood that F acts on test functions from the space C_c^∞.

Example 1.4. For $U \subset \mathbb{R}^n$, if f is integrable on every compact $K \subset U$, then the functional $\phi \to \int f\phi$ is a distribution.

We denote the set of all distributions on a set $U \subset \mathbb{R}^n$ by $\mathcal{D}'(U)$. For linear mappings from distributions on an open set U to distributions on an open set V, we use a continuous linear map, which we shall now define.

Definition 1.5. A linear map $T : C_c^\infty(U) \to C_c^\infty(V)$ is *continuous* if for each compact $K \subset U$, there is a compact $K' \subset V$ such that $T(C_c^\infty(K)) \subset C_c^\infty(K')$.

Thus, we can use a pair of linear maps $T : \mathcal{D}'(U) \to \mathcal{D}'(V)$, $T' : C_c^\infty(U) \to C_c^\infty(V)$ to map $\mathcal{D}'(U)$ to $\mathcal{D}'(V)$ as follows:

$$\langle TF, \phi \rangle := \langle F, T'\phi \rangle$$

where $F \in \mathcal{D}'(U)$ and $\phi \in C_c^\infty(U)$. Linear maps now allow us to define some properties of distributions under certain transformations.

1.2. **Properties of Distributions.** Distributions have some basic properties that will be important in later results and applications of the theory.

(1) **Translation**.

Note that for distributions from Example 1.4, by a change of variables

$$\int f(t-\tau)\phi(t)dt = \int f(t)\phi(t+\tau)dt$$

which motivates the following definition for translation of a general distribution.

Definition 1.6. Let τ be a linear map such that $\tau\phi(x) = \phi(x-y)$, denoted by $\tau_y\phi(x)$. Then,

$$\langle \tau_y F, \phi \rangle \; := \; \langle F, \tau_{-y}\phi \rangle.$$

(2) **Differentiation**.

Note that for distributions from Example 1.4, in the case of $\mathbb{R}^1$, by integration by parts

$$\begin{aligned}
\left\langle \frac{d}{dx}F, \phi \right\rangle &= \int_{-\infty}^{\infty} \frac{d}{dx} f(x)\phi(x)\,dx \\
&= [f(x)\phi(x)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)\frac{d}{dx}\phi(x)\,dx \\
&= \int_{\infty}^{\infty} f(x)\Big(-\frac{d}{dx}\phi(x)\Big)\,dx \qquad \text{[Since } \phi \text{ is compactly supported]} \\
&= \left\langle F, -\frac{d}{dx}\phi \right\rangle
\end{aligned}$$

which motivates the following definition.

Definition 1.7. For $F \in \mathcal{D}'$, $\langle \partial^\alpha F, \phi \rangle := (-1)^{|\alpha|}\langle F, \partial^\alpha \phi \rangle$.

(3) **Multiplication by Smooth Function**.

For multiplication of a distribution by a smooth function, we observe that for $U \subset \mathbb{R}^n$ and $\psi \in C^\infty(U)$, if $Tf = \psi f$, then $T' = T|C_c^\infty(U)$.

Definition 1.8. (Multiplication by smooth function). If $U \subset \mathbb{R}^n$, $\psi \in C^\infty(U)$, and $F \in \mathcal{D}'(U)$, then

$$\langle \psi F, \phi \rangle \; = \; \langle F, \psi\phi \rangle$$

where $\psi F \in \mathcal{D}'(U)$.

Remark 1.9. Since $\psi \in C^\infty(U)$ and $\phi \in C_c^\infty(U)$, it follows that $\psi\phi \in C_c^\infty(U)$.

Remark 1.10. For multiplication of a distribution by a smooth function, the product rule for a multi-index given by

$$\partial^\alpha(F\psi) := \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \big(\partial^\beta F\big)\big(\partial^{\alpha-\beta}\psi\big)$$

holds for differentiation of the product of the smooth function and the distribution.

(4) **Convolution**.

We observe that for $\phi \in C_c^\infty$ and an open set V, where $V = \{x : x - y \in U \text{ for y} \in \text{supp}(\phi)\}$, if f is locally integrable on U, then

$$f * \phi(x) = \int f(x-y)\phi(y)\,dy = \int f(y)\phi(x-y)\,dy = \int f\big(\tau_x\widetilde{\phi}\big)$$

where $\widetilde{\phi}(x) = \phi(-x)$, is well-defined for every $x \in V$.

Definition 1.11. (Convolution). If $U \subset \mathbb{R}^n$ and V is an open set such that $V = \{x : x - y \in U \text{ for y} \in \text{supp}(\phi)\}$, then the convolution $F * \phi$ is the function defined on V by

$$F * \phi(x) := \langle F, \tau_x\widetilde{\phi} \rangle.$$

We end this section with a discussion of the Dirac Delta functional, also known as the point mass at the origin, which is one of the most important distributions for the application of distribution theory to other areas of mathematics. The Dirac Delta functional is the distribution such that

$$\langle \delta, \phi \rangle = \phi(0)$$

The Dirac Delta functional acts as the multiplicative identity for convolution. If $\psi \in C_c^\infty$, we have

$$\begin{aligned} \delta * \psi(x) &= \psi * \delta(x) \\ &= \int \psi(x-y)\delta(y)dy \\ &= \langle \delta, \tau_x \widetilde{\psi} \rangle \\ &= \tau_x \widetilde{\psi}(0) = \psi(x). \end{aligned}$$

One can think about the Dirac Delta functional intuitively as the derivative of a jump discontinuity, i.e as the derivative of the Heaviside step function $H = \chi_{(0,\infty)}$, which can be seen as follows:

$$\langle H', \phi \rangle \;=\; -\langle H, \phi' \rangle \;= -\int_0^\infty \phi' dx = \phi(0) = \langle \delta, \phi \rangle.$$

This result can be generalized for a function with finitely many arbitrary jump discontinuities.

1.3. **Spaces of Distributions.** Next, we will investigate some of the properties of the space of distributions and subsets of this space and their relations to more general spaces.

Definition 1.12. If $F \in \mathcal{D}'$ and U is open in $\mathbb{R}^n$, we say that F vanishes on U, or $F = 0$, if $\langle F, \phi \rangle = 0$ for all $\phi \in C_c^\infty(U)$.

This definition of the value of a distribution on an open set allows for a definition of equality of distributions.

Definition 1.13. If $F, G \in \mathcal{D}'$, we say that $F = G$ on U if $F - G = 0$ on U.

Before we talk about the spaces themselves, we must first prove a result about where a distribution F vanishes, so that we can then understand the support of a distribution, which will require two lemmas.

Lemma 1.14. *If $f \in L^1$ and $g \in C^k$, then $f * g \in C^k$ and* $\text{supp}(f * g) \subset \text{supp}(f) + \text{supp}(g) = \{x + y : x \in \text{supp}(f),\ y \in \text{supp}(g)\}$

Lemma 1.15. *Let $U \subset \mathbb{R}^n$ be open and let $K \subset U$ be compact, so K has an open cover $\{U_{\alpha_j}\}_1^n$. Then, there exists $\psi_1, \dots, \psi_n \in C_c^\infty$ such that $\sum_1^m \psi_i = 1$ and* $\text{supp}(\psi_i) \subset U_{\alpha_j}$ *and is compact.*

Theorem 1.16. *Let $\{U_\alpha\}$ be a collection of open sets in $\mathbb{R}^n$ and let $U = \bigcup_\alpha U_\alpha$. If $F \in \mathcal{D}'$ and F vanishes on each U_α, then F vanishes on U.*

Proof. Let $\phi \in C_c^\infty(U)$. Then, since $\text{supp}(\phi)$ is compact and U is an open cover, there exist $\alpha_1, \dots, \alpha_m$ such that $\text{supp}(\phi) \subset \bigcup_1^m U_{\alpha_j}$. By Lemma 1.15, we can choose

$\psi_1, \ldots, \psi_m \in C_c^\infty$ such that $\psi_\text{j} \subset U_{\alpha_j}$ and $\sum_1^m \psi_j = 1$ on $\text{supp}(\phi)$. Then, since F vanishes on $\text{supp}(\psi_j)$ for every j and a distribution is linear,

$$\big\langle F, \phi \big\rangle \;=\; \sum_{j=1}^{m} \big\langle F, \psi_j \phi \big\rangle = 0.$$

□

Thus, for $F \in \mathcal{D}'$, we can take the union of all open sets on which F vanishes, which will lead to a largest open set on which it vanishes. So, it now makes sense to define the support of a distribution because we have a way of finding such a set.

Definition 1.17. If $F \in \mathcal{D}'$ and $\{U_\alpha\}$ is the set of open sets on which F vanishes. Then,

$$\text{supp}(F) = \Big(\bigcup U_\alpha\Big)^c.$$

This definition of the support of F then allows us to examine the behavior of a new space of distributions, in particular its relation to C^∞.

Definition 1.18. If U is an open set in $\mathbb{R}^n$, then $\mathscr{E}'(U)$ is the set of all distributions whose support is a compact subset of U.

Similarly to C_c^∞, for an open set U in $\mathbb{R}^n$, $C^\infty(U)$ is a Fréchet space, with norms defined for the derivatives of a function. However, for C^∞, we use a countable family of seminorms.

Definition 1.19. Let $\{V_m\}_1^\infty$ be an increasing sequence of open precompact sets, i.e their closure is compact, such that $V_j \subset U$ for all $j \in \mathbb{N}$ and $U = \bigcup_1^\infty V_j$. Then, for each $m \in \mathbb{N}$ and each multi-index α we define the seminorm

$$\|f\|_{[m,\alpha]} = \sup_{x \in \overline{V_m}} \big|\partial^\alpha f(x)\big|.$$

Then, $\partial^\alpha f_j \to \partial^\alpha f$ uniformly on compact sets for all α if and only if $\|f_j - f\|_{[m,\alpha]} \to 0$ for all m, α. Thus, we can now prove the following surprising result about the relation between C_c^∞ and C^∞.

Proposition 1.20. *If U is an open set in $\mathbb{R}^n$, then $C_c^\infty(U)$ is dense in $C^\infty(U)$.*

Proof. Let $\{V_m\}_1^\infty$ be an increasing sequence of open precompact sets, such that $V_j \subset U$ for all $j \in \mathbb{N}$ and $U = \bigcup_1^\infty V_j$. Then, for each m, by the C^∞ Urysohn Lemma, we can choose $\psi_m \in C_c^\infty(U)$ such that $\psi_m = 1$ on $\overline{V_m}$. If $\phi \in C^\infty(U)$, then $\psi_m \phi \in C_c^\infty$. Since $\overline{V_m}$ is an increasing sequence of compact sets, for $m \geq m_0$, $V_{m_0} \subset V_m$. So,

$$\|\psi_m\phi - \phi\|_{[m_0,\alpha]} = \sup_{x \in \overline{V_{m_0}}} \big|\partial^\alpha\big(\psi_m(x)\phi(x) - \phi(x)\big)\big| = \sup_{x \in \overline{V_m}} \big|\partial^\alpha\big(\phi(x) - \phi(x)\big)\big| = 0.$$

So, $\psi_m\phi$ converges uniformly to ϕ in $C_c^\infty(U)$. Thus, $\phi \in C^\infty(U)$ is a limit point of $C_c^\infty(U)$ and therefore $C_c^\infty(U)$ is dense in $C^\infty(U)$. □

The following bound for a continuous linear functional will be very useful for proving the continuity of a given linear functional, although we omit the proof for sake of space.

Proposition 1.21. *If a continuous linear functional G is defined on a Fréchet space $\mathscr{F}$ with a set of seminorms $\{q_\beta\}_{\beta\in A}$, then there exist $N \in \mathbb{N}$, and $C > 0$ such that*

$$\left|\langle G, \phi\rangle\right| = C \sum_{|\beta|\leq N} \|\phi\|_{q_\beta}$$

for every $\phi \in \mathscr{F}$.

Lemma 1.22. *If $f \in C_c^\infty$, then for every α, $\partial^\alpha f$ is bounded and uniformly continuous and for $g \in C_c^\infty$, $\partial^\alpha(f * g) = (\partial^\alpha f) * g$.*

Theorem 1.23. *If U is an open set in $\mathbb{R}^n$, $\mathscr{E}'(U)$ is the dual space of $C^\infty(U)$. More precisely: If $F \in \mathscr{E}'(U)$, then F extends uniquely to a continuous linear functional on $C^\infty(U)$; and if G is a continuous linear functional on $C^\infty(U)$, then $G|C_c^\infty \in \mathscr{E}'(U)$.*

Proof. Let $F \in \mathscr{E}'(U)$. Choose $\psi \in C_c^\infty(U)$ such that $\psi = 1$ on $\text{supp}(F)$. Define the linear functional G on $C^\infty(U)$ by $\langle G, \phi\rangle = \langle F, \psi\phi\rangle$. Since F is continuous on $C_c^\infty(\text{supp}(\psi))$, using the C_c^∞ norms, we have from Proposition 1.21 that there exists $N \in \mathbb{N}$ and $C > 0$ such that

$$\left|\langle G, \phi\rangle\right| \leq C \sum_{|\alpha|\leq N} \|\partial^\alpha(\psi\phi)\|_u$$

for every $\phi \in C^\infty(U)$. Let $\{V_m\}_1^\infty$ be an increasing sequence of open precompact sets, such that $V_j \subset U$ for all $j \in \mathbb{N}$ and $U = \bigcup_1^\infty V_j$. Choose m sufficiently large so that $\text{supp}(\phi) \subset V_m$. Since $\psi \in C_c^\infty$, by Lemma 1.22, $|\partial^\beta \psi|$ is bounded for every β. Also, $\text{supp}(\psi)$ is compact and, for every γ, $\partial^\gamma \phi$ is continuous, so $\partial^\gamma\phi(\text{supp}(\psi))$ is compact and thus bounded. So, by the product rule,

$$\begin{aligned}
\left|\langle G, \phi\rangle\right| &\leq C \sum_{|\alpha|\leq N} \left(\sup_{x\in\mathbb{R}^n} \left(\sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!}(\partial^\beta\psi)(\partial^\gamma\phi)\right)\right)\\
&= C \sum_{|\alpha|\leq N} \left(\sup_{x\in\text{supp}(\psi)} \left(\sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!}(\partial^\beta\psi)(\partial^\gamma\phi)\right)\right)\\
&\leq C' \sum_{|\alpha|\leq N} \sup_{x\in\text{supp}(\psi)} |\partial^\alpha\phi|\\
&\leq C' \sum_{|\alpha|\leq N} \|\phi\|_{[m,\alpha]}.
\end{aligned}$$

Therefore, $\left|\langle G, \phi\rangle\right|$ is bounded and since it is a linear functional, this implies that G is continuous. By Proposition 1.20, G is unique because we can construct a sequence that approaches G since $C_c^\infty(U)$ is dense in $C^\infty(U)$ and the uniqueness of sequential limits implies the uniqueness of G.

Next, let G be a continuous linear functional on C^∞. Then, by Proposition 1.21, there exists C, m, N such that

$$|\langle G, \phi\rangle| \leq C \sum_{|\alpha|\leq N} \|\phi\|_{[m,\alpha]}$$

for every $\phi \in C^\infty(U)$. For a compact set $K \subset U$, since $\|\phi\|_{[m,\alpha]} \leq \|\partial^\alpha\phi\|_u$ and $|\partial^\alpha\phi(K)|$ is bounded, it follows that G is continuous on $C_c^\infty(K)$. Therefore, $G|C_c^\infty \in$

$\mathcal{D}'(U)$. Now, if $\text{supp}(\phi) \cap \overline{V_m} = \emptyset$, then $\|\phi\|_{[m,\alpha]} = 0$ for every m and every α, so $\langle G, \phi \rangle = 0$. Thus, $\text{supp}(G) \subset \overline{V_m}$ and so $G|C_c^\infty(U) \in \mathscr{E}'$. □

1.4. **Tempered Distributions and the Fourier Transform.** Next, we discuss tempered distributions which are a special class of distributions that will be important for applications of distribution theory to partial differential equations because, as we shall see, they allow for the Fourier transform of a distribution. Tempered distributions act on functions in the Schwartz Space $\mathscr{S}$, which consists of C^∞ functions such that it and its derivatives go to zero faster than any negative power of $|x|$. Formally, we use the following norm:

Definition 1.24. For any nonnegative integer N and any multi-index α, we define

$$\|f\|_{(N,\alpha)} = \sup_{x \in \mathbb{R}^n} (1+|x|)^N \big|\partial^\alpha f(x)\big|.$$

Then, using this norm we define the Schwartz Space

$$\mathscr{S} = \{f \in C^\infty : \|f\|_{(N,\alpha)} < \infty \;\; \forall N, \alpha\}$$

Remark 1.25. $\mathscr{S}$ is a Fréchet space, i.e it is a complete, metrizable, locally convex topological vector space with the norms $\|\cdot\|_{(N,\alpha)}$.

Before we define a tempered distribution, we must first establish a result about the relation between C_c^∞ and $\mathscr{S}$.

Proposition 1.26. *Suppose $\psi \in C_c^\infty$ and $\psi(0) = 1$, and let $\psi^\epsilon(x) = \psi(\epsilon x)$. Then, for any $\phi \in \mathscr{S}$, $\psi^\epsilon \phi \to \phi \in \mathscr{S}$ as $\epsilon \to 0$. In particular, C_c^∞ is dense in $\mathscr{S}$.*

Proof. Let $N \in \mathbb{N}$ and let $\eta > 0$. Then, since $\phi \in \mathscr{S}$, there exists a compact set K such that $(1+|x|)^N |\phi(x)| < \eta$ for $x \notin K$. Given that K is compact and ψ is continuous, ψ is uniformly continuous on K. Thus, for $x \in K$, $\psi(\epsilon x) \to \psi(0) = 1$ as $\epsilon \to 0$. Then, $\|\psi^\epsilon \phi - \phi\|_{(N,0)} = (1+|x|)^N |\phi||\psi^\epsilon - 1| \to 0$ as $\epsilon \to 0$. Now, for the norms involving derivatives, by the product rule,

$$\begin{aligned}
\|\psi^\epsilon \phi - \phi\|_{(N,\alpha)} &= (1+|x|)^N \partial^\alpha (\psi^\epsilon \phi - \phi) \\
&= (1+|x|)^N (\psi^\epsilon \partial^\alpha \phi - \partial^\alpha \phi) + (1+|x|)^N \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \big(\partial^\beta \psi^\epsilon\big)\big(\partial^{\alpha-\beta}\phi\big).
\end{aligned}$$

Since $\phi \in \mathscr{S}$, $(1+|x|)^N \partial^{\alpha-\beta}\phi$ is bounded and since $\psi \in C_c^\infty$, by Lemma 1.22, ψ is bounded. So,

$$|\partial^\beta \psi^\epsilon(x)| = \epsilon^{|\beta|} |\partial^\beta \psi(x)| \le C_\beta \epsilon^{|\beta|}.$$

Therefore, there exists $C > 0$ such that

$$(1+|x|)^N \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \big(\partial^\beta \psi^\epsilon\big)\big(\partial^{\alpha-\beta}\phi\big) \le C\epsilon \to 0 \qquad \text{as } \epsilon \to 0.$$

So, $\|\psi^\epsilon \phi - \phi\|_{(N,\alpha)}$ approaches 0 as ϵ approaches 0. Since $\phi \in \mathscr{S}$ and $\psi^\epsilon \phi \in C_c^\infty$, this proves that C_c^∞ is dense in $\mathscr{S}$. □

Now, we define a tempered distribution as a continuous linear functional on $\mathscr{S}$. Then, the set of all tempered distributions is denoted by $\mathscr{S}'$. In light of Proposition 1.26, it makes sense to consider $\mathscr{S}'$ as the distributions that extend continuously from C_c^∞ to $\mathscr{S}$. For example, compactly supported distributions are tempered. For a tempered distribution, the definition for differentiation, translation, and convolution still hold, however, multiplication by a smooth function requires an extra restriction to preserve $\mathscr{S}$ and $\mathscr{S}'$ in the map $F \to \psi F$, namely that the function $\psi \in C^\infty$ be slowly increasing.

Definition 1.27. A function f on $\mathbb{R}^n$ is *slowly increasing* if for every α, there exists $N(\alpha)$ such that

$$|\partial^\alpha \psi(x)| \le C_\alpha (1 + |x|)^{N(\alpha)}.$$

In other words, a function is slowly increasing if it has at most polynomial growth at infinity. Hence, every polynomial is slowly increasing. Now, we have our new definition for multiplication by a smooth function because it is clear that $\mathscr{S}$ and $\mathscr{S}'$ are closed under multiplication by such a function.

Definition 1.28. If $\psi \in C^\infty$ and ψ is slowly increasing and $F \in \mathscr{S}'$, then

$$\langle F\psi, \phi \rangle \;=\; \langle F, \psi\phi \rangle.$$

The major reason for using tempered distributions instead of our original definition of distribution is that the Fourier transform is naturally defined for tempered distributions. This is due to following theorem:

Theorem 1.29. *The Fourier transform $\mathscr{F}$ maps $\mathscr{S}$ continuously into itself.*

Also, observe that for $f, g \in \mathscr{S}$, by Fubini's Theorem

$$\int \widehat{f}(y) g(y)\, dy = \int\int f(x) g(y) e^{-2\pi i x y}\, dx\, dy = \int f(x) \widehat{g}(x)\, dx.$$

So, since $\widehat{g} \in \mathscr{S}$, the Fourier transform is a continuous linear map from $\mathscr{S}'$ to itself.

Definition 1.30. For $F \in \mathscr{S}'$ and $\phi \in \mathscr{S}$, the Fourier transform of a distribution is defined by

$$\langle \widehat{F}, \phi \rangle \;=\; \langle F, \widehat{\phi} \rangle.$$

We then also have $\langle \check{F}, \phi \rangle = \langle F, \check{\phi} \rangle$, and thus it is clear that the Fourier inversion theorem formula still holds for distributions. So, the Fourier transform is an isomorphism on $\mathscr{S}'$.

Proposition 1.31. *For $F \in \mathscr{S}'$ or $F \in L^1(\mathbb{R}^n)$, we have the following properties*

(a) $(\tau_y F)^\wedge = e^{-2\pi i \xi y} \widehat{F}$
(b) $\tau_y \widehat{F} = (e^{2\pi i y x} F)^\wedge$
(c) $\partial^\alpha \widehat{F} = [(-2\pi i x)^\alpha]^\wedge$
(d) $(\partial^\alpha F)^\wedge = (2\pi i \xi)^\alpha \widehat{F}$
(e) $(F * \psi)^\wedge = \widehat{\psi}\widehat{F}$

Since compactly supported distributions are tempered distributions, the Fourier transform holds in $\mathscr{E}'$ and so we have the following theorem:

Theorem 1.32. *If $F \in \mathscr{E}'$, then $\widehat{F}$ is a slowly increasing C^∞ function, and it is given by $\widehat{F}(\xi) = \langle F, E_{-\xi} \rangle$, where $E_\xi(x) = e^{2\pi i \xi x}$.*

Proof. Let $g(\xi) = \langle F, E_{-\xi} \rangle$. Then, $\partial^\alpha g(\xi) = \langle F, \partial_\xi^\alpha E_{-\xi} \rangle = (-2\pi i)^{|\alpha|} \langle F, x^\alpha E_{-\xi} \rangle$, which is clearly in C^∞. By Theorem 1.23, g extends uniquely to a continuous linear functional on C^∞. So, by Proposition 1.21, there exist C, m, N such that

$$\left|\partial^\alpha g(\xi)\right| \le C \sum_{|\beta| \le N} \sup_{|x| \le m} \left|\partial^\beta [x^\alpha E_{-\xi}(x)]\right| \le C'(1 + m)^{|\alpha|}(1 + |\xi|)^N.$$

Therefore, g is slowly increasing.

So, it remains to show that $g = \widehat{F}$. Since C_c^∞ is dense in $\mathscr{S}$, it suffices to show that $\int g\phi = \langle F, \widehat{\phi} \rangle$ for $\phi \in C_c^\infty$. Now, $g\phi \in C_c^\infty$, so such an integral exists and we can approximate this integral by its Riemann sums. Thus, for $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\int g\phi \, d\xi - \sum_{j=1}^{N} g(\xi_j)\phi(\xi_j)\Delta\xi_j < \epsilon.$$

Since $\phi \in C_c^\infty$, it follows that, for x in a compact set, the sums $S(x) = \sum \phi(\xi_j)e^{-2\pi i \xi_j x}\Delta\xi_j$ and their derivatives with respect to x converge uniformly to $\widehat{\phi}(x)$ and its derivatives, respectively. Therefore, since F is a continuous linear functional on C^∞,

$$\int g\phi = \lim_{n\to\infty} \sum_{j=1}^{n} \langle F, E_{-\xi_j} \rangle \phi(\xi_j)\Delta\xi_j = \lim_{n\to\infty} \langle F, \sum_{j=1}^{n} \phi(\xi_j)E_{-\xi_j}\Delta\xi_j \rangle = \langle F, \widehat{\phi} \rangle$$

which completes the proof. □

Now, as a result of this theorem, the Fourier transform of the Dirac Delta functional is the constant function 1, shown by

$$\langle \delta, E_{-\xi} \rangle = E_{-\xi}(0) = 1.$$

We end this section by proving an interesting property of the Schwartz space and stating three results that will be useful for the application of distribution theory to partial differential equations.

Definition 1.33. We say a function f *vanishes at infinity* if for every $\epsilon > 0$, the set $\{x : |f(x)| \geq \epsilon\}$ is compact. Then, we associate with such functions the space $C_0(X)$ which we define to be the set of all functions $f \in C(X)$ such that f vanishes at infinity.

Proposition 1.34. $\mathscr{S} \subset C_0(\mathbb{R}^n)$.

Proof. Suppose, to reach a contradiction, that there exists $f \in \mathscr{S}$ such that $f \notin C_0(\mathbb{R}^n)$. Then, there exists $\epsilon > 0$ such that $U = \{x : |f(x)| \geq \epsilon\}$ is not compact. Then, by the Heine-Borel theorem, U is unbounded. For any $N \in \mathbb{N}$, $f \in \mathscr{S}$, so we have that $(1+|x|)^N|f(x)| < \infty$. But, for any $M > 0$, since U is unbounded, we can find $x \in U$ such that $|x| > \sqrt[N]{\frac{M}{\epsilon}} - 1$. Then, $(1+|x|)^N|f(x)| > M$, which is a contradiction. So, $f \in C_0(\mathbb{R}^n)$. □

Proposition 1.35. *The Fourier Transform, $\mathscr{F}$, is an isomorphism of $\mathscr{S}$ onto itself.*

Lemma 1.36. *Riemann-Lebesgue Lemma. $\mathscr{F}(L^1(\mathbb{R}^n)) \subset C_0(\mathbb{R}^n)$.*

We observe that since $\mathscr{F}$ is an isomorphism, this implies that $\mathscr{F}^{-1}(L^1(\mathbb{R}^n)) \subset C_0(\mathbb{R}^n)$.

Proposition 1.37. *If, for some $C > 0$, $|f(x)| \leq C|x|^{-a}$ on a ball B in $\mathbb{R}^n$ for some $a < n$, then $f \in L^1(B)$. However, if $|f(x)| \geq C|x|^{-n}$ on B, then $f \notin L^1(B)$.*

2. Application to Partial Differential Equations

The primary application of distribution theory is finding solutions for partial differential equations. Due to the space of smooth and compactly supported functions on which distributions act, distributions allow us to find 'nice' solutions to partial differential differential equations, where 'nice' usually means infinitely continuously differentiable, particularly through convolution of a distribution with a smooth function. In addition to methods of convolution, distributions allow for the construction of Sobolev spaces, which use properties of Hilbert spaces and Fourier transforms to find solutions to partial differential equations.

2.1. The Fundamental Solution.

We begin by examining the fundamental solution, which is important because it acts as a base solution from which all other solutions can be easily found. For this section, we follow Folland's *Lectures on PDE* [3].

For a given linear partial differential equation of order m

$$P(\partial)u = g \qquad \text{where } P(\partial) = \sum_{|\alpha|\leq m} c_\alpha \partial^\alpha \tag{2.1}$$

where c_α are constants, the fundamental solution F is a distribution such that $P(\partial)(F * g) = g$, i.e $u = F * g$, which solves our equation. Now, we observe that it suffices to find a distribution F such that $P(\partial)F = \delta$ for the following reason:

$$\begin{aligned} P(\partial)(F * g) &= P(\partial)\langle F, \tau_x \widetilde{g}\rangle && \text{[By Def. 1.11]} \\ &= \langle P(\partial)F, \tau_x \widetilde{g}\rangle && \text{[Since } F \text{ is linear]} \\ &= \langle \delta, \tau_x \widetilde{g}\rangle && \text{[By assumption]} \\ &= \delta * g \\ &= g && [\delta \text{ is multiplicative identity]} \end{aligned}$$

Thus, the problem of finding a fundamental solution to a linear partial differential equation is now significantly simplified because each linear partial differential operator has a set of fundamental solutions associated to it, which are the distributions F such that $P(\partial)F = \delta$.

For example, we can find a fundamental solution for the Laplacian. In order to do so, we shall use the Fourier Transform on a tempered distribution, which will exhibit how distributions allow us to find such solutions. To find a fundamental solution, we first explore some rather messy results regarding the Fourier Transform that will be used to produce the desired solution.

Lemma 2.2. *If $f(x) = e^{-\pi a|x|^2}$ and $a > 0$, then $\widehat{f}(\xi) = a^{\frac{n}{2}} e^{\frac{-\pi|\xi|^2}{a}}$.*

Proposition 2.3. *For $n > 2$ and $0 < k < n$, let F_k be a tempered distribution which is represented by the locally integrable function $F_k(\xi) = |\xi|^{-k}$. Then,*

$$\widetilde{F_k} = \frac{\Gamma(\frac{n-k}{2})}{\Gamma(\frac{k}{2})} \pi^{k-\frac{n}{2}} |x|^{k-n}$$

where $\Gamma(x)$ is the Gamma function.

Proof. First, we observe that for $r > 0$ and $k > 0$, by a change of variables, where we let $r = st$,

$$\int_0^\infty e^{-rt} t^{k-1} dt = \int_0^\infty e^{-s} s^{k-1} r^{-k} ds$$
$$= r^k \Gamma(k).$$

So,

$$r^{-k} = \frac{1}{\Gamma(k)} \int_0^\infty e^{-rt} t^{k-1} dt.$$

Now, let $r = \pi|\xi|^2$ and replace k by $\frac{k}{2}$. Then,

$$F_k(\xi) = |\xi|^{-k} = \frac{\pi^{\frac{k}{2}}}{\Gamma(\frac{k}{2})} \int_0^\infty e^{-\pi|\xi|^2 t} t^{\frac{k}{2}-1} dt. \tag{2.4}$$

However, we cannot directly compute the Fourier Transform because $F_k \notin L^1(\mathbb{R}^n)$ because it is only locally integrable. This is where we use the properties of the Fourier Transform on a tempered distribution. For every $\phi \in \mathscr{S}$,

$$\left\langle e^{-\pi|\xi|^2 t}, \widehat{\phi} \right\rangle \; = \; \left\langle \widehat{e^{-\pi|\xi|^2 t}}, \phi \right\rangle.$$

Then, using Proposition 2.3,

$$\int_{\mathbb{R}^n} e^{-\pi|\xi|^2 t} \widehat{\phi}\, d\xi = \int_{\mathbb{R}^n} t^{-\frac{n}{2}} e^{-\pi|\xi|^2 t} \phi(x)\, dx.$$

Next, we multiply both sides by $t^{\frac{k}{2}-1}$ and integrate with respect to t over all positive real numbers,

$$\int_0^\infty \int_{\mathbb{R}^n} e^{-\pi|\xi|^2 t} t^{\frac{k}{2}-1} \widehat{\phi}\, d\xi\, dt = \int_0^\infty \int_{\mathbb{R}^n} e^{-\frac{\pi}{t}|x|^2} t^{\frac{k-n}{2}-1} \phi(x)\, dx\, dt.$$

For the left-hand side, we can change the order of integration because each integrals is uniformly convergent with respect to the given variable and both double integrals are absolutely convergent. So,

$$\text{LHS}: \quad \int_0^\infty \int_{\mathbb{R}^n} e^{-\pi|\xi|^2 t} t^{\frac{k}{2}-1} \widehat{\phi}\, d\xi\, dt = \int_{\mathbb{R}^n} \int_0^\infty e^{-\pi|\xi|^2 t} t^{\frac{k}{2}-1} \widehat{\phi}\, dt\, d\xi$$
$$= \int_{\mathbb{R}^n} \widehat{\phi}(\xi) |\xi|^{-k} \frac{\Gamma(\frac{k}{2})}{\pi^{\frac{\alpha}{2}}} d\xi \qquad \text{[By (2.4)]}$$
$$= \left\langle \frac{\Gamma(\frac{k}{2})}{\pi^{\frac{k}{2}}} |\xi|^{-k}, \widehat{\phi} \right\rangle.$$

By the same reasoning, we can change the order of integration for the right-hand side. So,

$$\text{RHS}: \int_0^\infty \int_{\mathbb{R}^n} e^{-\frac{\pi}{t}|x|^2} t^{\frac{k-n}{2}-1} \phi(x)\, dx\, dt = \int_{\mathbb{R}^n} \int_0^\infty e^{-\frac{\pi}{t}|x|^2} t^{\frac{k-n}{2}-1} \phi(x)\, dt\, dx$$
$$= \int_{\mathbb{R}^n} \int_0^\infty e^{\pi|x|^2 s} s^{\frac{n-k}{2}-1} \phi(x)\, ds\, dx \qquad \left[\text{Let } s = \frac{1}{t}\right]$$
$$= \frac{\Gamma(\frac{n-k}{2})}{\pi^{\frac{n-k}{2}}} \int_{\mathbb{R}^n} |x|^{k-n} \phi(x)\, dx \qquad \text{[By (2.4)]}$$
$$= \left\langle \frac{\Gamma(\frac{n-k}{2})}{\pi^{\frac{n-k}{2}}} |x|^{k-n}, \phi \right\rangle.$$

Bringing the left and right hand sides back together, we have

$$\begin{aligned}\left\langle \widehat{F_k}, \phi \right\rangle &= \left\langle F_k, \widehat{\phi} \right\rangle \\ &= \left\langle |\xi|^{-k}, \widehat{\phi} \right\rangle \\ &= \left\langle \frac{\Gamma(\frac{n-k}{2})}{\Gamma(\frac{k}{2})} \pi^{\frac{k-n}{2}} |x|^{k-n}, \phi \right\rangle.\end{aligned}$$

Therefore,

$$\widehat{F_k}(\xi) = \frac{\Gamma(\frac{n-k}{2})}{\Gamma(\frac{k}{2})} \pi^{\frac{k-n}{2}} |x|^{k-n}.$$

Now, since $|\xi|^k \in \mathbb{R}$, $\widehat{F_k} = \widecheck{F_k}$ by Parseval's Theorem, which completes the proof. □

Note that Proposition 1.37 implies that F_k can only be locally integrable and thus this restriction in the proposition was required, as a result of which we had to use distributions to prove this equality. Now, for a distribution K to be a fundamental solution to the Laplacian, we must have

$$g = \Delta(K * g) = (\Delta K) * g. \tag{2.5}$$

From Proposition 1.31 (d), we have

$$\widehat{\Delta K}(\xi) = -4\pi^2 |\xi|^2 \widehat{K}(\xi).$$

Therefore, by the Convolution Theorem, taking the Fourier Transform of both sides of (2.5),

$$\widehat{g}(\xi) = (\Delta K * g)^{\wedge} = \widehat{\Delta K}\widehat{g} = -4\pi^2 |\xi|^2 \widehat{K}(\xi)\widehat{g}(\xi). \tag{2.6}$$

Thus, $\widehat{K}(\xi) = -\frac{1}{4\pi^2|\xi|^2}$. Then, by Proposition 2.3,

$$K(x) = -4\pi^2 \widecheck{F_2} = -4\pi^2 \Gamma\left(\frac{n-2}{2}\right) \pi^{2-\frac{n}{2}} |x|^{2-n}.$$

However, for $n = 2$, the Gamma function blows up in K and the fundamental solution does not depend upon x, so we have to look at $\widecheck{F_k}$ as k approaches n. Let $G_k(\xi) = (2\pi\xi)^{-k}$ and let $R_k = \widecheck{G_k}$ for $0 < k < 2$. Then, by Proposition 2.3, we have

$$R_k(x) = \frac{\Gamma(\frac{2-k}{2})}{2^k \pi^{\frac{n}{2}} \Gamma(\frac{k}{2})} |x|^{k-2}.$$

Since $\widehat{f}(\xi) = (2\pi|\xi|)^k \widehat{f}(\xi)\widehat{R_k}(\xi) = (2\pi|\xi|)^k (f * R_k)^{\wedge}(\xi)$, it follows that for $f \in \mathscr{S}$,

$$f = (-\Delta)^{\frac{k}{2}} (f * R_k). \tag{2.7}$$

Now, notice that $(-\Delta)^{\frac{k}{2}} c = 0$ for $c \in \mathbb{R}$, so we can replace R_k by $R_k - c$ in (2.7). We thus choose c in such a way that the limit as k approaches n will exist. Take

$$c_k = \frac{\Gamma(\frac{2-k}{2})}{2^k \pi^{\frac{n}{2}} \Gamma(\frac{k}{2})}$$

and define $R_k' = R_k - c$. Then,

$$R_k' = \frac{\Gamma(\frac{2-k}{2})}{2^k \pi^{\frac{n}{2}} \Gamma(\frac{k}{2})} (|x|^{k-2} - 1) = \frac{2\Gamma(\frac{2-k}{2} + 1)}{2^k \pi^{\frac{n}{2}} \Gamma(\frac{k}{2})} \cdot \frac{|x|^{k-2} - 1}{2 - k}.$$

So, taking the limit as k approaches n, using l'Hôpital's Rule, $R_2'(x) = \frac{-\log|x|}{2\pi}$. From (2.7), we see that for $k = 2$, we have $K(x) = -R_2'(x) = \frac{\log|x|}{2\pi}$. Finally, we must ensure that K is a tempered distribution. Since $\mathscr{F}^{-1}$ maps $\mathscr{S}'$ to itself, it suffices to show that $G_2(\xi)$ can be represented as a tempered distribution F_2. Define the functional F_2 on $\mathscr{S}$ as follows

$$\langle F_2, \phi \rangle = \int_{|\xi|\leq 1} \frac{\phi(\xi)-\phi(0)}{(2\pi|\xi|)^2} d\xi + \int_{|\xi|>1} \frac{\phi(\xi)}{(2\pi|\xi|)^2} d\xi.$$

By the Mean Value Theorem for $\mathbb{R}^n$, we know that $|\phi(\xi) - \phi(0)| \leq c|\xi|$, so by Proposition 1.37, the first integral converges and since $\phi \in S$, the second integral converges and $\langle F_2, \phi \rangle$ is bounded. Since boundedness implies continuity for a functional and it is clearly linear, the functional defined by F_n on $\mathscr{S}$ is a tempered distribution. Now, if $\phi(0) = 0$, $\langle F_2, \phi \rangle = \int \phi(\xi) G_2(\xi) d\xi$, so $F_2 = G_2$ on $\mathbb{R}^2 \backslash \{0\}$. Similar to above where we subtracted an infinite constant c_2 from R_2, we can subtract an infinite multiple of δ from F_2 to obtain G_2 using the consequence of Theorem 1.32, and thus K is a tempered distribution. So, we have found our fundamental solution for $n = 2$. It can easily be shown that $\Delta F = \delta$.

We now introduce a special set of spaces of functions, which will allow us to prove a major result regarding the relationship between the regularity of a PDE and its solution, namely elliptic regularity.

2.2. **Sobolev Spaces.** For our discussion of Sobolev spaces and elliptic regularity, we follow Folland's *Real Analysis* [4]. A Sobolev space for a given $k \in \mathbb{N}$ is a subspace of $\mathscr{S}'$ consisting of those tempered distributions whose derivatives up to k are in L^p. However, we restrict ourselves to L^2 because L^2 is a Hilbert space, which allows for some important properties of Sobolev spaces, which in turn give rise to their applicability. However, we actually don't use this definition, we instead use an extension of this definition that allows for Sobolev spaces to be defined for $k \in \mathbb{R}$. First, we motivate the alternate definition by showing that the two are equivalent for $k \in \mathbb{N}$.

Lemma 2.8. *$\mathscr{S}$ is dense in L^2, and thus $\mathscr{S} \subset L^2 \subset \mathscr{S}'$.*

Proof. (See [1]). Let $U \subset \mathbb{R}^n$. Then, by inner measure, there exists $K \subset U$ such that $\mu(U\backslash K) = \mu(U) - \mu(K) < \epsilon$. By the C^∞ Urysohn Lemma, there exists $f_U \in C_c^\infty$ such that $f_U = 1$ on K and $0 \leq f_U \leq 1$ and $\text{supp}(f) \subset U$. Then, for χ_U, the characteristic function for U,

$$\int_{\mathbb{R}^n} |\chi_U - f_U|^2 \, d\mu = \int_{U\backslash K} |\chi_U - f_U|^2 \, d\mu < \epsilon.$$

Then, for a collection of open set $\{U_j\}$, the simple function $g = \sum_1^n a_j \chi_{U_j}$ is approximated by $F = \sum_1^n f_{U_j}$. From the basic properties of L^p spaces, the set of simple functions in the form of g are dense in L^2, so it follows that C_c^∞ is dense in L^2 and since $C_c^\infty \subset \mathscr{S}$, $\mathscr{S}$ is also dense in L^2. Therefore, since L^2 is a Hilbert space, we have a rigged Hilbert space, also known as a Gelfand triple, where $\mathscr{S} \subset L^2 \subset \mathscr{S}'$. □

Proposition 2.9. *For $k \in \mathbb{N}$, $f \in \{y \in \mathscr{S}' : \partial^\alpha h \in L^2 \text{ for } |\alpha| \leq k\}$ if and only if $g_k \widehat{f} \in L^2$.*

Proof. ($\Longrightarrow$). Suppose $f \in \mathscr{S}'$ and $\partial^\alpha g \in L^2$ for $|\alpha| \leq k$. Since $|\xi|^k$ and $\sum_{j=1}^n |\xi_j|^k$ are homogeneous of degree k, and non-vanishing for $\xi \neq \mathbf{0}$, we have

$$(1+|\xi|^2)^{\frac{k}{2}} \leq C_0(1+|\xi|^k) \leq C_0\bigg(1 + C\sum_{j=1}^n |\xi_j|^k\bigg).$$

Then,

$$\begin{aligned}
\|(1+|\xi|^2)^{\frac{k}{2}}\widehat{f}\|_2 &\leq \bigg\|C_0(|\widehat{f}| + C|\widehat{f}|\sum_{j=1}^n |\xi_j|\bigg\|_2 \\
&\leq C_0\bigg\|\widehat{f}\|_2 + C_0C\|\widehat{f}\sum_{j=1}^n |\xi_j^k\bigg\|_2 && \text{[By Minkowski's Inequality]} \\
&= C_0\|\widehat{f}\|_2 + C'\bigg\|\bigg(\sum_{j=1}^n \partial_j^k f\bigg)^\wedge\bigg\|_2 && \text{[By Prop. 1.36 (d)]} \\
&= C_0\|f\|_2 + C'\bigg\|\sum_{j=1}^n \partial_j^k f\bigg\|_2 && \text{[By Parseval's Theorem].}
\end{aligned}$$

Since L^2 is closed under addition and multiplication by constants, and by assumption both terms in the latter inequality are in L^2, it follows that $(1+|\xi|^2)^{\frac{k}{2}}\widehat{f} \in L^2$.

($\Longleftarrow$). Suppose $(1+|\xi|^2)^{\frac{k}{2}}\widehat{f} \in L^2$. For $|\alpha| \leq k$ and $|\xi| \geq 1$,

$$|\xi^\alpha| \leq |\xi|^k \leq (1+|\xi|^2)^{\frac{k}{2}}.$$

and for $|\xi| < 1$,

$$|\xi^\alpha| \leq 1 \leq (1+|\xi|^2)^{\frac{k}{2}}.$$

So, $|\xi^\alpha|\widehat{f}(\xi) \leq (1+|\xi|^2)^{\frac{k}{2}}\widehat{f}(\xi)$, and thus $\xi^\alpha\widehat{f} \in L^2$. By Propostion 1.36 (d) for tempered distributions, we know that $\xi^\alpha\widehat{f} = \frac{1}{(2\pi)^{|\alpha|}}\big(\partial^\alpha f\big)^\wedge \in L^2 \subset \mathscr{S}'$. By Plancherel's Theorem, $\partial^\alpha f \in L^2$. Since $\mathscr{F}$ is a continuous linear map from $\mathscr{S}'$ to itself, we have $\partial^\alpha f \in \mathscr{S}'$ and setting $\alpha = \mathbf{0}$ completes the proof. □

Now, it is clear that the requirement that $g_s\widehat{f} \in L^2$ allows for $s \in \mathbb{R}$, so instead of defining Sobolev spaces by those tempered distributions whose derivatives up to k are in L^p (for our purposes L^2), which is restricted to $s \in \mathbb{N}$, we will use the more general requirement that $g_s\widehat{f} \in L^2$ for our definition of Sobolev spaces. This expression depends upon the multiplication of the Fourier Transform of a tempered distribution with the function $g_s(\xi) = (1+|\xi|^2)^{\frac{s}{2}}$, and although it is clear that $g_s \in C^\infty$, as discussed with regards to Definition 1.27 and 1.28, to preserve $\mathscr{S}$ and $\mathscr{S}'$, we must ensure that g_s is slowly increasing.

Lemma 2.10. *For every* $s \in \mathbb{R}$, $\partial^\alpha g_s(\xi) \leq C_\alpha(1+|\xi|)^{s-|\alpha|}$.

Sobolev spaces depend upon the following map:

Definition 2.11. The map Λ_s is defined by

$$\Lambda_s f = \big(g_s(\xi)\widehat{f}\big)^\vee.$$

Since $\mathscr{S}'$ is preserved by multiplication by a slowly increasing function and by the inverse Fourier Transform, Λ_s is a continuous linear operator on $\mathscr{S}'$. We are now ready to define a Sobolev space.

Definition 2.12. If $s \in \mathbb{R}$, we define the *Sobolev space* H_s to be

$$H_s = \{F \in \mathscr{S}' : \Lambda_s F \in L^2\}.$$

Thus, from this definition, a Sobolev space is the set of all tempered distributions such that when the linear operator Λ_s is applied to F, $F \in L^2$.

The space H_s is equipped with the inner product

$$\langle f, h \rangle_{(s)} = \int (\Lambda_s f)(\overline{\Lambda_s h}).$$

Also, we equip the space with the following L^2 norm:

$$\|f\|_{(s)} = \left(\int |\widehat{f}(\xi)|^2 (1 + |\xi|^2)^s \, d\xi \right)^{\frac{1}{2}} = \|(1 - \Delta)^{\frac{s}{2}} f\|_2.$$

Remark 2.13. H_s is a Hilbert space, i.e it is complete with respect to the norm defined above.

Also, there are some properties that follow immediately from the definition. Firstly, notice that $H_0 = L^2$. Secondly, since $|\xi|^\alpha \leq (1 + |\xi|^2)^{\frac{|\alpha|}{2}}$, it follows that ∂^α is a bounded linear map from H_s to $H_{s-|\alpha|}$ for all s and α. Thirdly, since

$$\Lambda_t(\Lambda_s f) = \left[\left(((1 + |\xi|^2)^{\frac{s}{2}} \widehat{f})^\vee \right)^\wedge (1 + |\xi|^2)^{\frac{t}{2}} \right]^\vee = \left(\left(1 + |\xi|^2\right)^{\frac{s+t}{2}} \widehat{f} \right)^\vee.$$

we see that Λ is an isomorphism from H_{s-t} to H_s, which preserves the inner product. Finally, it is clear that for $t > s$, $H_t \subset H_s$.

Before using Sobolev spaces for some important theorems in partial differential equations, we explore some more interesting properties of Sobolev spaces.

Proposition 2.14. *$\mathscr{S} \subset H_s$ for every $s \in \mathbb{R}$.*

Proof. Let $h \in \mathscr{S}$ and let $s \in \mathbb{R}$. Then, by Lemma 2.8, $h \in \mathscr{S}'$ and by Definition 1.30, $\widehat{h} \in \mathscr{S}'$. Then, $\widehat{h}(\xi) \leq C_N(1 + |\xi|)^{-N}$ for all N. Take $N = \frac{n+s+1}{2}$. Then, we have

$$(1 + |\xi|^2)^{\frac{s}{2}} \widehat{h} \leq C_{N,s}(1 + |\xi|)^{\frac{s}{2}} (1 + |\xi|)^{-\frac{n+s+1}{2}} = C_{N,s}(1 + |\xi|)^{-\frac{n+1}{2}}.$$

So, by Proposition 1.37, $(1 + |\xi|^2)^{\frac{s}{2}} \widehat{h} \in L^2$ and hence $h \in H_s$. Therefore, $\mathscr{S} \subset H_s$ for every $s \in \mathbb{R}$. □

We now prove an important result for Sobolev Spaces that, under certain conditions, embeds them in the space of functions whose derivatives, up to some k, vanish at infinity. We define this space as follows:

$$C_0^k = \{f \in C^k(\mathbb{R}^n) : \partial^\alpha f \in C_0, \ |\alpha| \leq k\}$$

Theorem 2.15. ***The Sobolev Embedding Theorem.*** *Suppose $s > k + \frac{1}{2}n$.*

(a) If $f \in H_s$, then $\widehat{\partial^\alpha f} \in L^1$ and $\|\widehat{\partial^\alpha f}\|_1 \leq C_{k-s} \|f\|_{(s)}$ for $|\alpha| \leq k$.
(b) $H_s \subset C_0^k$

Proof. Proof of (a). We observe that

$$\begin{aligned}\frac{1}{(2\pi)^{|\alpha|}}\int |\widehat{\partial^\alpha f}(\xi)| d\xi &= \int |\xi^\alpha \widehat{f}(\xi)|\, d\xi \\ &\leq \int (1+|\xi|^2)^{\frac{k}{2}} |\widehat{f}(\xi)|\, d\xi \\ &\leq \left[\int (1+|\xi|^2)^s |\widehat{f}(\xi)|^2\, d\xi\right]^{\frac{1}{2}} \left[\int (1+|\xi|^2)^{k-s} d\xi\right]^{\frac{1}{2}} \\ &= \|f\|_{(s)} \left[\int (1+|\xi|^2)^{k-s}\, d\xi\right]^{\frac{1}{2}}\end{aligned}$$

The second inequality is given by the Schwarz Inequality. Since $2(k-s) < -n$, it follows from Proposition 1.40 that the integral is finite, which proves (a).

Proof of (b). Let $f \in H_s$. Then, $\partial^\alpha f \in L^2$ for all α. So, $\partial^\alpha f \in L^1$ for all α, and by part (a) $\widehat{\partial^\alpha f} \in L^1$, so we can apply the Fourier Inversion Theorem to $\widehat{\partial^\alpha f}$. Then, by the Riemann-Lebesgue Lemma, $\partial^\alpha f \in C_0$ for every such that $|\alpha| \leq k$, which proves (b). □

Corollary 2.16. *If $f \in H_s$ for all s, then $f \in C^\infty$*

Proof. This follow directly from part (b) of the theorem. □

We now prove a nice inequality that will help in the mechanics of our theorems about Sobolev spaces.

Lemma 2.17. *For every $\xi, \nu \in \mathbb{R}^n$ and for every $s \in \mathbb{R}$,*

$$(1+|\xi|^2)^s(1+|\nu|^2)^{-s} \leq 2^{|s|}(1+|\xi-\nu|^2)^{|s|}.$$

Proof. By the Triangle Inequality, $|\xi| \leq |\xi - \nu| + |\nu|$. So, since $a^2 + b^2 > 2ab$ for $a, b \in \mathbb{R}$, $|\xi|^2 \leq 2(|\xi-\nu|^2 + |\nu|^2)$. Thus,

$$1 + |\xi|^2 \leq 2(1+|\xi-\nu|^2)(1+|\nu|^2)$$

If $s \geq 0$, raise both sides to the sth power and we're done. If $s < 0$, then interchange ξ and ν and raise both sides to the $-s$th power to obtain

$$(1+|\nu|^2)^{-s} \leq 2^{-s}(1+|\xi|^2)^{-s}(1+|\xi-\nu|^2)^{-s}$$

which is the desired result. □

This allows us to show that under certain conditions we can multiply functions in H_s by a C_0 function and remain in H_s.

Theorem 2.18. *Suppose that $\phi \in C_0(\mathbb{R}^n)$ and that $\widehat{\phi}$ is a function that satisfies*

$$\int (1+|\xi|^2)^{\frac{a}{2}} |\widehat{\phi}|\, d\xi = C < \infty$$

for some $a > 0$. Then, the map $M_\phi(f) = \phi f$ is a bounded operator on H_s for $|s| \leq a$.

Proof. Since Λ_s is a map from $L^2 = H_0$ to H_s, which preserves the inner product, it suffices to show that $\Lambda_s M_\phi \phi_{-s}$ is a bounded operator on L^2. We find that

$$\begin{aligned}(\Lambda_s M_\phi \Lambda_{-s} f)^\wedge &= \left[\left((1+|\xi|^2)^{\frac{s}{2}}(\phi\Lambda_{-s}f)^\wedge\right)^\vee\right]^\wedge \\ &= (1+|\xi|^2)^{\frac{s}{2}}\left(\widehat{\phi} * (\Lambda_{-s}f)^\wedge\right)(\xi) \\ &= \int (1+|\xi|^2)^{\frac{s}{2}}(1+|\nu|^2)^{-\frac{s}{2}}\widehat{\phi}(\xi-\nu)\widehat{f}(\nu)\, d\nu.\end{aligned}$$

Let $K(\xi,\nu) = \int (1+|\xi|^2)^{\frac{s}{2}}(1+|\nu|^2)^{-\frac{s}{2}}\widehat{\phi}(\xi-\nu)$. Then, by Lemma 2.16,

$$|K(\xi,\nu)| \leq 2^{\frac{|s|}{2}}(1+|\xi-\nu|^2)^{\frac{|s|}{2}}|\widehat{\phi}(\xi-\nu)|.$$

Then, we see that, since $|s| \leq a$, $\int |K(\xi,\nu)| d\xi$ and $\int |K(\xi,\nu)|$ are bounded by $2^{\frac{a}{2}}C$. So, $(\Lambda_s M_\phi \Lambda_{-s} f)^\wedge \in L^2$ and it is a bounded operator. Since $\Lambda_s M_\phi \Lambda_{-s} f \in H_s$, it is in $L^1 \cap L^2$, so by Plancherel's Theorem, $\Lambda_s M_\phi \Lambda_{-s} f$ is bounded operator on L^2. □

Corollary 2.19. *If $\phi \in \mathscr{S}$, then M_ϕ is a bounded operator on H_s for all $s \in \mathbb{R}$.*

Proof. This follows from the fact that $\phi \in \mathscr{S}$ implies $\widehat{\phi}$ satisfies the condition of the theorem for every $a > 0$. □

The following result, which allows us to find a subsequence of distributions that converges in Sobolev spaces, is very important for the application of Sobolev spaces to solving partial differential equations.

For the application of Sobolev spaces, it is important to look at local smoothness properties, so we define localized Sobolev spaces.

Definition 2.20. The *localized Sobolev space* H_s^{loc} is the set of all distributions $f \in \mathcal{D}'(U)$ such that for every precompact open set V with $\overline{V} \subset U$, there exists $g \in H_s$ such that $g = f$ on V.

We can show that a distribution on U being locally Sobolev is equivalent to all smoothings of the distribution being Sobolev.

Proposition 2.21. *A distribution $f \in \mathcal{D}'(U)$ is in $H_s^{loc}(U)$ if and only if $\phi f \in H_s$ for every $\phi \in C_c^\infty$.*

Proof. ($\Longrightarrow$). Let $f \in H_s^{loc}(U)$ and let $\phi \in C_c^\infty(U)$. Then, let $V \subset U$ be a neighborhood of supp(ϕ). Clearly $\overline{V}$ is compact and $\overline{V} \subset U$. So, there exists $g \in H_s$ such that $f = g$ on V. Since $C_c^\infty \subset \mathscr{S}$, we know that $\phi \in \mathscr{S}$, so it follows from Corollary 2.19 that $\phi f = \phi g \in H_s$.

($\Longleftarrow$). Let V be a precompact set such that $\overline{V} \subset U$. By the C^∞ Urysohn Lemma, there exists $\phi \in C_c^\infty(U)$ such that $\phi = 1$ on $\overline{V}$. Now, $\phi f \in H_s$ and $\phi f = f$ on V, so $f \in H_s^{loc}(U)$. □

2.3. **Elliptic Regularity.** As we shall see, for elliptic operators, it is convenient to use $P(D)$ instead of $P(\partial)$, where $D = (2\pi)^{-|\alpha|}$. It is also useful to use a polynomial representation of the operator, known as the *symbol*.

Definition 2.22. We define the *symbol* of a linear partial differential operator $P(D)$ of order m as

$$P(\xi) = \sum_{|\alpha| \leq m} c_\alpha \xi^\alpha.$$

The reason for the change in notation is that for a function or distribution f, $(P(D)f)^\wedge = P(\xi)\widehat{f}$.

Definition 2.23. The *principal symbol* P_m is the sum of the top-order terms in its symbol:

$$P_m(\xi) = \sum_{|\alpha|=m} c_\alpha \xi^\alpha.$$

Now, we are ready to understand elliptic operators and their properties.

Definition 2.24. We call a partial differential operator $P(D)$ of order m *elliptic* if $P_m(\xi) \neq 0$ for all $\xi \in \mathbb{R}^n$ such that $\xi \neq \mathbf{0}$.

As an example, Δ is an elliptic operator in $\mathbb{R}^n$ because $P_2(\xi) = \sum_{i=1}^n c_i\xi_i^2$, which is clearly only zero when $\xi = \mathbf{0}$. However, the heat operator $\partial_t - \Delta$ is not elliptic in $\mathbb{R}^{n+1}$ because $P_2(\xi) = -\sum_{i=2}^{n+1} c_i\xi_i^2$ and therefore for any $t \in \mathbb{R}\backslash\{0\}$, $\xi = (t, 0, \dots, 0)$ is nonzero, but $P_2(\xi) = 0$. Similarly, the wave operator $\partial_t^2 - \Delta$ is not elliptic on $\mathbb{R}^{n+1}$.

Lemma 2.25. *Suppose that P(D) is of order m. Then $P(D)$ is elliptic iff there exist $C, R > 0$ such that $|P(\xi)| \geq C|\xi|^m$ when $|\xi| \geq R$.*

Proof. ($\Longrightarrow$). Let $P(D)$ be elliptic. Then, since $P(D)$ is elliptic, on the unit sphere, $C_1 = \inf_{|\xi|=1} |P_m(\xi)| > 0$. Note that P_m is homogeneous of degree of m, so for $\lambda \in \mathbb{R}$ and $|\xi_0| = 1$,

$$|P_m(\lambda\xi_0)| = \lambda^m|P_m(\xi)| \geq \lambda^m C_1 = C_1|\lambda\xi_0|^m.$$

Since every $\xi \in \mathbb{R}^n$ can be written as $\lambda\xi_0$ for some λ and ξ_0 as above, we see that for every ξ, $|P_m(\xi)| \geq C_1|\xi|^m$. Now, $P - P_m$ is of order $m-1$ and $|\xi|^m \geq |\xi^\alpha|$ for $|\alpha| \leq m-1$, so there exists $C_2 > 0$ such that $|P(\xi) - P_m(\xi)| \leq C_2|\xi|^{m-1}$. Therefore, for $|\xi| \geq 2C_2C_1^{-1}$,

$$|\xi|^m \frac{C_1}{2} \geq 2C_2C_1^{-1}\frac{C_1}{2}|\xi|^{m-1} = C_2|\xi|^{m-1}.$$

So,

$$|P(\xi)| \geq |P_m(\xi)| - |P(\xi) - P_m(\xi)| \geq C_1|\xi|^m - C_2|\xi|^{m-1} \geq \frac{1}{2}C_1|\xi|^m$$

which is the desired result.

($\Longleftarrow$). Let $P(D)$ be not elliptic. Then, there exists ξ_0 such that $P_m(\xi_0) = 0$ and since $P(D)$ is homogeneous of order m, the same is true for all scalar multiples of ξ_0. Let $R > 0$ and let $C > 0$ be such that $|P(\xi)| \geq C|\xi|^m$ when $|\xi| \geq R$. But, we can find $\lambda \in \mathbb{R}$ such that $\xi = |\lambda\xi_0| \geq R$ and, since $P(D)$ is of order $m-1$ when $\xi = |\lambda\xi_0|$, such that $|P(\xi)| \leq C|\xi|^{m-1}$, which is a contradiction. So, there do not exist $C, R > 0$ such that $|P(\xi)| \geq C|\xi|^m$ for every ξ with $|\xi| \geq R$. ☐

Lemma 2.26. *If $P(D)$ is elliptic of order m, $u \in H_s$, and $P(D)u \in H_s$, then $u \in H_{s+m}$.*

Proof. By Plancherel's Theorem, $(1+|\xi|^2)^{\frac{s}{2}}\widehat{u} \in L^2$ and $(1+|\xi|^2)^{\frac{s}{2}}P\widehat{u} \in L^2$. By Lemma 2.25, for some $R \geq 1$,

$$(1+|\xi|^2)^{\frac{m}{2}} \leq 2^m|\xi|^m \leq C^{-1}2^m|P(\xi)| \qquad \text{for } |\xi| \geq R.$$

Also, for $|\xi| \leq R$, $(1+|\xi|^2)^{\frac{m}{2}} \leq (1+R^2)^{\frac{m}{2}}$. So, for all $\xi \in \mathbb{R}^n$,

$$0 \leq (1+|\xi|^2)^{\frac{(s+m)}{2}}|\widehat{u}| \leq C'(1+|\xi|^2)^{\frac{s}{2}}(|P\widehat{u}|+|\widehat{u}|) \in L^2.$$

Therefore, $(1+|\xi|^2)^{\frac{(s+m)}{2}}|\widehat{u}| \in L^2$, and hence $u \in H_{s+m}$. □

We now prove one of the major applications of Sobolev spaces, which says that the regularity of an elliptic operator applied to a function implies the regularity of the function itself.

Theorem 2.27. ***The Elliptic Regularity Theorem.*** *Suppose that L is a constant-coefficient elliptic linear partial differential operator of order m, Ω is an open set in $\mathbb{R}^n$, and $u \in \mathcal{D}'(\Omega)$. If $Lu \in H_s^{loc}(\Omega)$ for some $s \in \mathbb{R}$, then $u \in H_{s+m}^{loc}(\Omega)$; and if $Lu \in C^\infty(\Omega)$, then $u \in C^\infty(\Omega)$.*

Proof. For the first part, by Proposition 2.21, it suffices to show that if $Lu \in H_s^{loc}(\Omega)$ and $\phi \in C_c^\infty$, then $\phi u \in H_{s+m}$. Let V be a precompact open set such that $\mathrm{supp}(\phi) \subset V \subset \overline{V} \subset \Omega$ and, by the C^∞ Urysohn Lemma, choose $\psi \in C_c^\infty$ such that $\psi = 1$ on $\overline{V}$. Then, $\psi u \in \mathscr{E}'$. So, by Theorem 1.32, $\widehat{\psi u}$ is slowly increasing and therefore for some $\sigma \in \mathbb{R}$, $(1+|\xi|^2)^{\frac{\sigma}{2}}\widehat{\psi u} \in \mathscr{S} \subset L^2$. So, $\psi u \in H_\sigma$. Now, it is clear that if we decrease σ, it is still true that $(1+|\xi|^2)^{\frac{\sigma}{2}}\widehat{\psi u} \in L^2$, and hence $\psi u \in H_\sigma$, so we decrease σ such that $s+m-\sigma = k \in \mathbb{N}$. Set $\psi_0 = \psi$ and $\psi_k = \phi$, and again by C^∞ Urysohn Lemma, choose recursively $\psi_1, \dots, \psi_{k-1} \in C_c^\infty$ such that $\psi_j = 1$ on a neighborhood of $\mathrm{supp}(\phi)$ and $\mathrm{supp}(\psi_j)$ is contained in the set where $\psi_{j-1} = 1$.

We shall prove by induction that $\psi_j u \in H_{\sigma+j}$ for $j \in \mathbb{N} \cup \{0\}$. Before we do so, we observe that for any $\zeta \in C_c^\infty$, the operator $[L,\zeta]$ defined by

$$[L,\zeta]f = L(\zeta f) - \zeta Lf = \sum_{|\alpha|\leq m}\left[b_\alpha \sum_{\beta \leq \alpha}\left(\binom{\alpha}{\beta}\partial^\beta\zeta\partial^{\alpha-\beta}f\right)\right] - \zeta Lf.$$

For $|\alpha| = m$, the $\beta = 0$ in the inner sum cancels with the $-\zeta Lf$ term, so this operator is of order $m-1$. Then, by grouping the $\partial^{\alpha-\beta}f$ terms, in the operator, we see that the coefficients of the operator are linear combinations of derivatives of ζ, and since $\zeta \in C_c^\infty$, these coefficients are C^∞ functions. Also, these coefficients vanish on any open set where ζ is constant and such sets exist because ζ is compactly supported, so in fact these coefficients are C_0 functions. Therefore, if $f \in H_t$, then $\partial^\alpha f \in H_{t-|\alpha|} \subset H_{t-(m-1)}$ for $|\alpha| < m-1$. Since the coefficients are in C_0, it follows from Theorem 2.18 that $[L,\zeta]f \in H_{t-(m-1)}$, which is the key to the proof. Now, we can proceed with the induction.

For $j = 0$, we have $\psi_0 u = \psi u \in H_\sigma$ by our choice of σ. Assume that $\psi_j u \in H_{\sigma+j}$ for $0 \leq j < k$. Then, since $\psi_{j+1} = \psi_{j+1}\psi_j$, we have

$$\begin{aligned} L(\psi_{j+1}) &= \psi_{j+1}Lu + [L,\psi_{j+1}]u \\ &= \psi_{j+1}Lu + [L,\psi_{j+1}]\psi_j u \\ &\in H_s + H_{\sigma+j-(m-1)} = H_{\sigma+j+1-m} \qquad [\text{Since } H_s \subset H_{\sigma+j+1-m}]. \end{aligned}$$

So, since $\psi_{j+1}u = \psi_{j+1}\psi_j u \in H_{\sigma+j} \subset H_{\sigma+j+1-m}$, by Lemma 2.26, $\psi_{j+1}u \in H_{\sigma+j+1}$. Therefore, for every $n \in \mathbb{N}$, $\psi_n u \in H_{\sigma+n}$ by induction.

Let $n = k$. Then, $\phi u = \psi_k u \in H_{\sigma+k} = H_{s+m}$. Since V was an arbitrary precompact open subset of Ω, $\phi u \in H_{s+m}^{loc}(\Omega)$.

For the second part of the proof, suppose $Lu \in C^\infty(\Omega)$ and let $s \in \mathbb{R}$. Then, for every $\phi \in C_c^\infty$, $\phi Lu \in C_c^\infty \subset S$. Since $S \subset H_s$, $\phi Lu \in H_s$. So, it follows from Proposition 2.21 that $Lu \in H_s^{loc}(\Omega)$. Then, from the first part, $u \in H_{s+m}^{loc}(\Omega)$. Now, let $K \subset \Omega$ be compact and let V be a neighborhood of K such that $\overline{V} \subset \Omega$. Then, there exists $g_{s+m} \in H_{s+m}$ such that $u = g_{s+m}$ on V. Since s was arbitrary, $u \in H_t$ on K for every $t \in \mathbb{R}$, where $t = s + m$. So, by Corollary 2.16, $u \in C^\infty(K)$. Then, since K is an arbitrary compact subset of Ω, it follows that $u \in C^\infty(\Omega)$. □

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