QUASI-PREFERENCE: CHOICE ON PARTIALLY ORDERED SETS

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Abstract. A preference relation is a total order on a finite set and a quasi-preference relation is a partial order. This paper first introduces the classic axiomatic property of preference relation, and develops a similar axiom for the quasi-preference relation.

Contents

1. Preliminaries

1.1. Introduction.

The standard theory of choice has already been well developed. The basic assum ption of rational preference is that our world is a totally ordered set as the real line. If someone prefers analysis to algebra and algebra to topology, then under rational preference he must prefer analysis to topology. Also, under rational preference, he must have a preference among analysis and any other field in math or other subjects. But, is it really reasonable?

To overcome the shortcoming of rational preference, we could use a concept of quasi-preference to capture some irrational choices that are not allowed in rational preference. For example, if someone set analysis to 4, algebra to 3, topology to 2, and number theory to 1, then he could decide he prefer one subject to the other if their difference value is larger than 1. So, in this case, he would prefer analysis to topology, but not algebra.

This paper is an attempt to theorize the concept of quasi-preference in math languages, just like what has been done to preference relation. The structure of the paper is laid out as two parts: the first part introduces with preference, and the second part develops quasi-preference.

1.2. **Basic Definition.**

Thoughout the paper it is assumed that X is a finite set.

Definition 1.1. A binary relation R on the set X is a subset of $X \times X$.

For simplicity, in this paper if a and b in X has the relation R, the notation aRb or $a \succ b$ is used. The latter is used because the relations discussed in the paper are intuitively similar to the usual "larger" order relation. But formally, $a \succ b$ means only $(a, b) \in R$. Also, we use $a \not\succ b$ to denote $(a, b) \notin R$.

There is a large list of properties that a relation can satisfy. Here are some properties that are used in this paper:

Definition 1.2. A relation on X is

(1) *reflexive* if aRa for all $a \in X$;
(2) *irreflexive* if aRa does not hold for any $a \in X$;
(3) *asymmetric* if aRb implies that bRa does not hold;
(4) *transitive* if aRb and bRc together imply aRc;
(5) *negatively transitive* if neither aRb nor bRc holds, then aRc does not hold as well;
(6) *acyclic* if a_1Ra_2, a_2Ra_3, $\cdots a_{n-1}Ra_n$, then a_nRa_1 does not hold;

Note that if a relation is not reflexive, then it is not always irreflexive. Also the negative transitivity seems intuitively strange, but it is different from simple transitivity. Indeed, neither of them implies the other. Consider the usual order $>$ on $\mathbb{R}$, the order is clearly both transitive and negatively transitive. However, consider a partial order $>$ on $\mathbb{R}^2$ defined as the following: $(a, b) > (c, d)$ if $a > c$ and $b = d$. Then such partial order is transitive but not negatively transitive. On the other hand, consider the set $X = \{a, b, c\}$ and the relation $R = \{(a, b), (b, a), (b, c), (c, b)\}$, the relation is not transitive but it is negatively transitive.

The standard choice theory has the following definition of a preference relation:

Definition 1.3. A relation $\succ$ on X is a preference relation if it is a total order on X.

The definition of total order involves the notion of both $\succ$ and $\succeq$, silimar to $>$ and $\geq$ on $\mathbb{R}$. On this paper we do not care $\succeq$, thus we use the alternative definition as following:

Definition 1.4. A relation $\succ$ on X is a preference relation if it is both asymmetric and negatively transitive.

It is possible to construct $\succeq$ from $\succ$ given the conditions so this definition would match the prior one, but it is not important for the paper, and thus the detail is skipped. Rather, the property of the preference relation is important:

Proposition 1.5. *If a relation $\succ$ is a preference relation, then it is irreflexive, transitive and acyclic.*

Proof.
(1) Asymetry directly implies irreflexivity.
(2) Let $a \succ b$ and $b \succ c$. Then by asymmetry $c \not\succ b$. Suppose $a \not\succ c$, by negatively transitivity we have $a \not\succ b$, a contradiction. Thus $a \succ c$.
(3) since $\succ$ is transitive, if it is not acyclic, then it is not irreflexive, a contradiction.

□

One of the most important consequence of the setup of the preference theory is the following theorem, which allows people's preferences to be calculated as numbers.

Theorem 1.6. *If a relation $\succ$ on X is a preference relation, then (X,$\succ$) is order isomorphic to a subset A of $\mathbb{R}$ with the usual order $>$. That is, there exists a function $f\colon X \to \mathbb{R}$ such that if for any $a, b \in X$ satisfying $a \succ b$, then $f(a) > f(b)$.*

The proof of the theorem is not the focus of the paper and thus it is skipped. A proof and further details of the theorem can be found in [1].

Next we turn to the notion of choice on such sets.

Definition 1.7. A function c: $\mathcal{P}(X) \to \mathcal{P}(X)$ is a choice function if for all non-empty $A \subset X$, c(A) is non-empty and a subset of X.[1]

The intuition behind the choice function is that if someone is given a set of possible selections, then that person always selects some fixed things. Besides this notion of choice, we can also construct another notion of choice by the relation $\succ$:

Definition 1.8. A candidate choice function c: $\mathcal{P}(X) \to \mathcal{P}(X)$ generated by the relation $\succ$ is defined by $c(A, \succ) = \{a \mid \forall b \in A,\ b \not\succ a\}$.

[1]$\mathcal{P}(X)$means a power set of X.

The intuition behind the candidate function is that if someone is given a set of possible selections, he only chooses the selection that no other selection is better than what he has chosen. Note that a candidate choice function may not be a choice function. For example, suppose $X = \{a, b, c\}$ with relation $a \succ b$, $b \succ c$ and $c \succ a$, then $c(X, \succ) = \emptyset$, which contradicts the definition of a choice function. But we have the following relationship between choice function and candidate choice function:

Proposition 1.9. *A relation $\succ$ is acyclic iff $c(\cdot, \succ)$ is a valid choice function.*

Proof.
(1) Let $\succ$ be acyclic. Suppose $c(A, \succ) = \emptyset$. Take any $a_1 \in A$. Since $a_1 \notin c(A, \succ)$, we can find $a_2 \in A$ such that $a_2 \succ a_1$. Continue the process and we can get a "sequence" $\cdots \succ a_2 \succ a_1$. Since A is finite, the "sequence" must ends at some a_n which is exactly a_1. This contradicts $\succ$ be acyclic, Thus $c(A, \succ)$ is always non-empty.
(2) Let $c(\cdot, \succ)$ be a valid choice function. Suppose $\succ$ is not acyclic, then there exists $a_1 \succ a_2 \succ a_3 \cdots \succ a_n \succ a_1$. Let set A contain all these elements, then $c(A, \succ) = \emptyset$, a contradiction.

□

2. Axiom on Preference Relation

Because usually we do not observe the complete relation $\succ$, we can only see a person's choice behavior, and from his choice behavior we can infer his preference relation. Thus, the person's choice function reveals his preference which is not directly observed. Here I briefly introduce the classic axiomatic property of preference relation in the eyes of choice function. Further details could be found in [1].

Definition 2.1. (Houthakker's axiom) If $a, b \in A \cap B$, and if $a \in c(A)$, $b \in c(B)$, then $a \in c(B)$.

Intuitively, this axiom means if a is selected by the choice function in A and b is available, then whenever b is chosen and a is avalable, then a must also be chosen. The significance of the axiom is that a choice function satisfying the axiom is equivalent to having a preference relation.

Proposition 2.2. *If a relation $\succ$ is a preference relation, then $c(\cdot, \succ)$ satisfies Houthakker's axiom.*

Proof. Let $a, b \in A \cap B$, $a \in c(A, \succ)$ and $b \in c(B, \succ)$. $a \in c(A, \succ)$ implies $b \not\succ a$, and $b \in c(B, \succ)$ implies $\forall c \in B, c \not\succ b$. By negative transitivity, $\forall c \in B, c \not\succ a$, and thus $a \in c(B)$. □

The other direction is somewhat more difficult.

Proposition 2.3. *If a choice function c satisfies Houthakker's axiom, then it generates a preference relation $\succ$ such that $c = c(\cdot, \succ)$.*

Proof. We construct the relation $\succ$ as following: for distinct elements a, b in X

$$a \succ b \ \ if \ \ c(\{a, b\}) = \{a\}$$

And $a \succ a$ is never true.

Fix a set A. We first show if $a \in c(A)$, then $a \in c(A, \succ)$. Suppose $a \notin c(A, \succ)$, implying there exists $b \in A$ such that $c(\{a, b\}) = b$. But taking $B = \{a, b\}$, by Houthakker's axiom $a \in \{a, b\}$, a contradiction.

Then we show if $a \notin c(A)$, then $a \notin c(A, \succ)$. We choose $b \in c(A)$, and we claim $c(\{a, b\}) = b$, for if $a \in c(\{a, b\})$ then by Houthakker's axiom $a \in c(A)$, a contradiction. Thus $b \succ a$ and $a \notin c(A, \succ)$.

Since asymmetry is implied by the construction, we only need to prove negative transitivity. Let $a \not\succ b$ and $b \not\succ c$, we want to prove $a \not\succ c$. Suppose $a \succ c$, then we have $c(\{a, c\}) = \{a\}$. By Houthakker's axiom, $c \notin c(\{a, b, c\})$. Since $c \in c(\{b, c\})$, by Houthakker's axiom $b \notin c(\{a, b, c\})$. Since $b \in c(\{a, b\})$, by Houthakker's axiom $a \notin c(\{a, b, c\})$. Then $c(\{a, b, c\})$ is empty, a contradiction.

□

Now we have completed prove the equivalence of preference relation and the axiom. Since the axiom is somewhat intuitively not easy to understand, we want to split it into easier statements.

Definition 2.4. (Sen's Property α) If $a \in A \subset B$ and $a \in c(B)$, then $a \in c(A)$.

In Sen's original paper [2], he describes the property as following: If the world champion in some game is a Pakistani, then he must also be the champion of Pakistan. From Houthakker's axiom to Sen's Property α is straightforward:

Proposition 2.5. *If a choice function c satisfies Houthakker's axiom, then it satisfies Sen's Property α.*

Proof. Let $a \in A \subset B$ and $a \in c(B)$. Since $a \in A \cap B$, take both a and b in Houthakker's axiom to be a here, the result is immediate. □

This property has a nice feature as following:

Proposition 2.6. *Let $\succ$ be a relation on X, then $c(\cdot, \succ)$ satisfies Sen's Property α.*

Proof. Let $a \in A \subset B$ and $a \in c(B, \succ)$. Then $\forall b \in B$, $b \not\succ a$. Thus $\forall b \in A$, $b \not\succ a$. Hence $a \in c(A, \succ)$. □

Note that this property holds true even if $c(\cdot, \succ)$ is not a valid choice function.

3. Quasi-preference Relation

The introduction gives a brief description of what a quasi-preference looks like. Here I give two additional examples to show the motivation of the concept of quasi-preference.

Example 3.1. (Threshold on Preference) Suppose someone has to choose between a list of goods valued as $1, 2, 3, 4, \cdots 10$. His preference is given by $a \succ b$ if $a > b+1$. Then this relation is not a preference relation since it is not negatively transitive.

Example 3.2. (Bundle) Suppose someone has to choose amount of two goods. So one choice would be a subset of R^2. He preference is given by $(a, b) \succ (c, d)$ if both $a > b$ and $c > d$ hold. Then the relation is clearly not preference relation, but it still retains some sense of rationality.

Definition 3.3. A quasi-preference relation is a relation that only satisfies irreflexity and transitivity.

The choice function and the candicate choice function are determined in the similar way.

Definition 3.4. (Axiom of revealed quasi-preference) Let A, B and X be finite and non-empty sets satisfying $a \in A \subset B \subset X$. Let c be a choice function on X,

(1) if $a \in c(B)$, then $a \in c(A)$.
(2) if $a \notin c(B)$, then there exists $b \in c(B)$ such that $c(\{a, b\}) = \{b\}$.

Note that the first statement of the axiom is exactly Sen's Property α. Now we prove choice function satisfying this axiom is equivalent to having a quasi-preference relation.

Proposition 3.5. *If $\succ$ is a quasi-preference relation, then $c(\cdot, \succ)$ satisfies Axiom of revealed quasi-preference.*

Proof. We only need to prove $c(\cdot, \succ)$ satisfies Statement 2 in the axiom. Let $A \subset B \subset X$ and $a \notin c(B, \succ)$. By definition of $c(B, \succ)$, we can find $a_1 \in B$ such that $a_1 \succ a$. If a_1 is in $c(B, \succ)$, then a_1 is the element we want. Otherwise suppose a_1 is not in $c(B, \succ)$, and then we can find $a_2 \in B$ such that $a_2 \succ a_1$. If a_2 is still not in $c(B, \succ)$, then we can continue the process until we get $a_n \in c(B, \succ)$ with $a_n \succ a_{n-1}$, $a_{n-1} \succ a_{n-2}$, $\cdots$, $a_1 \succ a$ because X is finite. By transitivity of $\succ$, this means $a_n \succ a$ and $c(\{a_n, a\}, \succ) = a_n$. □

Before proving the other side, we first prove the following lemma:

Lemma 3.6. *(Transitivity of choice function) Let c be a choice function on X satisfying the Axiom of Revealed Quasi-preference. Let a, b ,$c \in X$, if $c(\{a,b\}) = \{a\}$ and $c(\{b,c\}) = \{b\}$, then $c(\{a,c\}) = \{a\}$.*

Proof. We first claim $c(\{a,b,c\}) = \{a\}$. Suppose $b \in c(\{a,b,c\})$, then by Statement 1 of the Axiom, we have $b \in c(\{a,b\})$, a contradiction. Similarly, if $c \in c(\{a,b,c\})$, we have $c \in c(\{b,c\})$, another contradiction. Since $c(\{a,b,c\})$ is non-empty, we must have $c(\{a,b,c\}) = \{a\}$.

Then we prove $c(\{a,c\}) = \{a\}$. By Statement 2 of the Axiom, since c is not in $c(\{a,b,c\})$, there must be some element in $c(\{a,b,c\})$ that is "larger" than c. As a is the only element in $c(\{a,b,c\})$, we have $c(\{a,c\}) = \{a\}$.

□

The two statements in the Axiom are both necessary in proving the transitivity of choice function. If Statement 1 is missing, a counter-example would be $c(\{a,b,c\}) = \{a,b,c\}$, $c(\{a,b\}) = \{a\}$, $c(\{b,c\}) = \{b\}$ and $c(\{a,c\}) = \{c\}$. If Statement 2 is missing, a counter-example would be $c(\{a,b,c\}) = \{a\}$, $c(\{a,b\}) = \{a\}$, $c(\{b,c\}) = \{b\}$ and $c(\{a,c\}) = \{a,c\}$.

Now we can prove the converse:

Proposition 3.7. *If a choice function c on X satisfies the Axiom of revealed quasi-preference, then there exists a quasi-preference relation $\succ$ on X such that $c = c(\cdot, \succ)$*

Proof. We construct the relation $\succ$ as following: for distinct elements a, b in X

$$a \succ b \ \ if \ \ c(\{a,b\}) = \{a\}$$

And $a \succ a$ is never true.

We first prove $c = c(\cdot, \succ)$. Let $A \in X$ and $a \in c(A)$. Take any element $b \in A$ distinct from a. Since $\{a,b\} \subset A$, by Statement 1 in the Axiom we have $a \in c(\{a,b\})$. Then $b \succ a$ cannot be true because otherwise we would have $c(\{a,b\}) = \{b\}$. Since this is true for all $b \in A$ distinct from a, by definition of $c(A, \succ)$ we have $a \in c(A, \succ)$. Hence we have $c(A) \subset c(A, \succ)$.

On the other hand, let $a \in c(A, \succ)$. Suppose a is not in $c(A)$, then by Statement 2 of the Axiom, there exists $b \in c(A)$ such that $c(\{a,b\}) = \{b\}$, implying $b \succ a$, a contradiction to $a \in c(A, \succ)$. Thus a must be in $c(A)$ and $c(A) \supset c(A, \succ)$. With $c(A) \subset c(A, \succ)$ we have $c = c(\cdot, \succ)$.

$\succ$ is automatically irreflexive by the construction, and thus at last we only need to prove $\succ$ is transitive. Suppose $a \succ b$ and $b \succ c$, which are equivalent to $c(\{a, b\}) = a$ and $c(\{b, c\}) = b$. By the lemma above, we have $c(\{a, c\}) = a$ and thus $a \succ c$. This completes the proof.

$\square$

The real challenge for quasi-preference theory is to find an order isomophism of the choice set with the quasi-preference relation to some mathematical structure, for which currently I have no idea.

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References

[1] David Kreps. Notes On the Theory of Choice. Westview Press, 1988.

[2] Amartya Sen. Choice Functions and Revealed Preference. The Review of Economic Studies, Vol.38,1970.