REGULARITY OF SOLUTIONS TO THE FRACTIONAL LAPLACE EQUATION

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Abstract. We explore properties of the fractional Laplacian, particularly for negative exponent, which allows us to examine the solutions of the fractional Laplace equation. We prove several regularity results involving Sobolev and Besov spaces for Bessel potentials. These results are then easily extended to solutions of the fractional Laplace equation.

Contents

1. Introduction

In this paper, we wish to explore properties of the fractional Laplacian and, more particularly, the fractional Laplace equation, which are generalizations of the usual Laplacian and Laplace equation. The fractional Laplacian appears in various partial differential equations and has applications in other areas, such as probability and mathematical finance. For instance, just as the usual Laplace equation has connections to Brownian motion, the fractional Laplace equation has connections to the more general Levy processes. However, we will restrict our discussion to regularity of solutions to the fractional Laplace equation.

First we wish to motivate the definition of the fractional Laplacian. Consider the Laplacian $\Delta f = \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j^2}$ of a sufficiently nice function f (it suffices to take f in the Schwartz space, $\mathcal{S}$). For the remainder of this paper, we will use $\hat{f}$ to denote

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the Fourier transform of a function f. Note that

$$\begin{aligned}\widehat{(\Delta f)}(x) &= \int_{\mathbb{R}^n} \Delta f(y) e^{-2\pi i y\cdot x}\,dy \\ &= \sum_{j=1}^{n} \int_{\mathbb{R}^n} \frac{\partial^2 f}{\partial y_j^2}(y) e^{-2\pi i y\cdot x}\,dy \\ &= \sum_{j=1}^{n} \int_{\mathbb{R}^n} f(y)(-2\pi i x_j)^2 e^{-2\pi i y\cdot x}\,dy \\ &= -4\pi^2 \sum_{j=1}^{n} x_j^2 \hat{f}(x) \\ &= -4\pi^2 |x|^2 \hat{f}(x),\end{aligned}$$

where we have integrated by parts twice and used the rapid decay of both f and its first-order partial derivatives at infinity. Thus, we have the relation

$$\widehat{(-\Delta f)}(x) = (2\pi|x|)^2 \hat{f}(x). \tag{1.1}$$

We therefore choose to define the fractional Laplacian from this expression.

Definition 1.2. For $2\alpha \in (-n, n)$, the fractional Laplacian is the operator Δ^α satisfying the relation

$$\widehat{(-\Delta^\alpha f)} = (2\pi|x|)^{2\alpha} \hat{f}$$

for all $f \in \mathcal{S}$.

Note that in light of (1.1), it is clear that when $\alpha = 1$ the above definition agrees with the usual definition of the Laplacian.

Using this operator, we may define the fractional Laplace equation for a domain $\Omega \subset \mathbb{R}^n$:

$$(-\Delta)^\alpha u = f \quad \text{on } \Omega,$$

for some function f. Note that when $\alpha = 1$, this is precisely the usual Laplace equation. We will explore the regularity of solutions to this equation in the full space (i.e. when $\Omega = \mathbb{R}^n$). In order to do this we will explore regularity of solutions to a similar equation and then show that these results hold for the fractional Laplace equation.

This paper is outlined as follows. In Section 2, we will define two operators that give the solutions to our two equations. We will discuss different representations of these operators and will prove some basic properties. We complete this discussion in Section 3, in which we prove that the operator for the fractional Laplace equation is bounded for certain L^p spaces. Our goal is to prove several regularity results about solutions to the fractional Laplace equation, which we achieve in Section 6. However, these results require us to introduce certain function spaces, namely Sobolev and Besov spaces, that give some measure of regularity of functions. We do this in Section 4. Furthermore, we compare the regularity of solutions to the fractional Laplace equation to those of the usual Laplace equation in Section 5.

2. Preliminaries

We have already defined the fractional Laplacian Δ^α. Note that our definition works for negative α, so if

$$(-\Delta)^\alpha u = f$$

for $\alpha > 0$, then

$$u = (-\Delta)^{-\alpha} f.$$

Thus, our main concern will be the operator $(-\Delta)^{-\alpha}$.

Definition 2.1. For $0 < 2\alpha < n$, we define the α Riesz potential to be

$$I_\alpha f = c_\alpha \int_{\mathbb{R}^n} |x-y|^{2\alpha - n} f(y)\, dy, \tag{2.2}$$

where

$$c_\alpha = \frac{\Gamma\left(\frac{n}{2} - 2\alpha\right)}{\pi^{n/2} 2^{2\alpha} \Gamma(\alpha)}.$$

We claim that $I_\alpha f = (-\Delta)^{-\alpha} f$. To see this, note that for $\delta > 0$, the Fourier transform of $e^{-\pi\delta|x|^2}$ is $e^{-\pi|x|^2/\delta}\delta^{-n/2}$. Then by Plancherel's formula, we have that for any $\varphi \in C_c^\infty$ (the space of compactly supported infinitely differentiable functions),

$$\int_{\mathbb{R}^n} e^{-\pi\delta|x|^2} \overline{\varphi(x)}\, dx = \int_{\mathbb{R}^n} e^{-\pi|x|^2/\delta} \delta^{-n/2} \overline{\hat{\varphi}(x)}\, dx.$$

Multiplying both sides by $\delta^{\frac{n-2\alpha}{2}-1}$ and integrating, we see that

$$\int_0^\infty \delta^{\frac{n-2\alpha}{2}} \int_{\mathbb{R}^n} e^{-\pi\delta|x|^2} \overline{\varphi(x)}\, dx\, d\delta = \int_0^\infty \int_{\mathbb{R}^n} e^{-\pi|x|^2/\delta} \delta^{-\alpha-1} \overline{\hat{\varphi}(x)}\, dx\, d\delta.$$

Using Fubini's theorem, we have that the left side of this is simply

$$\Gamma\left(\frac{n-2\alpha}{2}\right) \pi^\alpha \int_{\mathbb{R}^n} |x|^{2\alpha-n} \overline{\varphi(x)}\, dx,$$

and the right side is

$$\Gamma(\alpha)\, \pi^{-\alpha} \int_{\mathbb{R}^n} |x|^{-2\alpha} \overline{\hat{\varphi}(x)}\, dx.$$

But then we have that

$$\widehat{|x|^{2\alpha-n}} = \frac{\Gamma(\alpha)}{\Gamma\left(\frac{n-2\alpha}{2}\right)} \pi^{-2\alpha+n/2} |x|^{-2\alpha} = \frac{(2\pi|x|)^{-2\alpha}}{c_\alpha}.$$

If we let $K_\alpha = c_\alpha |x|^{2\alpha-n}$, we see that $I_\alpha f$ is the convolution of f and K_α, i.e. $I_\alpha f(x) = (K_\alpha * f)(x)$. Thus, we have that

$$\widehat{I_\alpha f}(x) = \widehat{K_\alpha * f}(x) = \hat{K_\alpha} \hat{f} = (2\pi|x|)^{-2\alpha} \hat{f}(x).$$

But then it is clear that $I_\alpha f = (-\Delta)^{-\alpha} f$.

This gives us a more useful expression for the fractional Laplacian. From this, however, it is clear that although K_α behaves nicely locally, its behavior at infinity is less favorable, particularly if 2α is close to n. To correct for this, we define a similar operator $\mathscr{J}_\alpha = (I - \Delta)^{-\alpha}$, where I is the identity operator. We may then consider the analogous equation

$$(I - \Delta)^\alpha u = f,$$

or

$$u = (I - \Delta)^{-\alpha} f = \mathscr{J}_\alpha(f).$$

It will be much easier to prove regularity results for $\mathscr{J}_\alpha$ and then check that these results carry over to I_α. As with I_α, we can represent $\mathscr{J}_\alpha f$ as a convolution of f with a kernel. In this case, we want the kernel

$$G_\alpha(x) = \frac{1}{(4\pi)^\alpha \Gamma(\alpha)} \int_0^\infty e^{-\pi|x|^2/\delta} e^{-\delta/4\pi} \delta^{\alpha - n/2} \frac{d\delta}{\delta}.$$

For a proof that $\mathscr{J}_\alpha f = G_\alpha * f$, see [3, p. 132]. We note that for any α, $\mathscr{J}_\alpha$ is a bounded operator from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$. This follows from the fact that if $f \in L^p$, then

$$\|\mathscr{J}_\alpha f\|_p = \|G_\alpha * f\|_p \leq \|G_\alpha\|_1 \|f\|_p = \|f\|_p,$$

so $\mathscr{J}_\alpha f \in L^p$. It will also be useful later on to note that

$$\hat{G_\alpha}(x) = (1 + 4\pi^2|x|^2)^{-\alpha}.$$

3. Boundedness of I_α

Although we have defined the operator I_α on the Schwartz space, we wish to consider its action on more general spaces, for instance on $L^p(\mathbb{R}^n)$. First we need to discover when the integral (2.2) converges for $f \in L^p(\mathbb{R}^n)$. If we have almost everywhere convergence for this expression, then we may explore the possibility that I_α is a bounded operator from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for some p, q. In the proof of the theorem below, we use the following interpolation result. Suppose $p_0, q_0, p_1, q_1 \in [1, \infty]$ such that $p_0 \leq q_0$, $p_1 \leq q_1$, and $q_0 \neq q_1$ and

$$\frac{1}{p} = \frac{1-t}{p_0} + \frac{t}{p_1} \qquad \frac{1}{q} = \frac{1-t}{q_0} + \frac{t}{q_1}$$

for $t \in (0, 1)$. If T is a sublinear map defined on $L^{p_0}(\mathbb{R}^n) + L^{p_1}(\mathbb{R}^n)$ and T is of weak types (p_0, q_0) and (p_1, q_1), then T is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. For a proof of this result, see [1, 3.8].

Theorem 3.1. *Let $0 < 2\alpha < n$ and $1 \leq p < q < \infty$ such that $\frac{1}{q} = \frac{1}{p} - \frac{2\alpha}{n}$. Then for $f \in L^p(\mathbb{R}^n)$,*

$$I_\alpha f(x) = c_\alpha \int_{\mathbb{R}^n} |x - y|^{2\alpha - n} f(y)\, dy$$

converges for almost every x and, moreover, if $p \neq 1$, there exists a constant C such that

$$\|I_\alpha f\|_q \leq C\|f\|_p.$$

Proof. As noted above, $I_\alpha f$ is really the convolution of f with the kernel $K_\alpha(x) = c_\alpha |x|^{2\alpha - n}$. For convenience, we fix $c > 0$ and write $K_\alpha = K_1 + K_2$, where

$$K_1(x) = \begin{cases} K_\alpha(x) & |x| \leq c \\ 0 & |x| > c \end{cases} \qquad K_2 = \begin{cases} 0 & |x| \leq c \\ K_\alpha(x) & |x| > c. \end{cases}$$

Then we have that $I_\alpha f = K_\alpha * f = K_1 * f + K_2 * f$. Thus, to show almost everywhere convergence of $I_\alpha f$, it suffices to show that both $K_1 * f$ and $K_2 * f$ are finite almost everywhere. Now, $-n < 2\alpha - n$, so

$$c_\alpha \int_{|x| \leq c} |x|^{2\alpha - n}\, dx < \infty,$$

i.e. $K_1 \in L^1(\mathbb{R}^n)$. Thus, if $f \in L^p(\mathbb{R}^n)$, it follows from Fubini's Theorem that $\|K_1 * f\|_p \leq \|K_1\|_1 \|f\|_p < \infty$, so $K_1 * f \in L^p(\mathbb{R}^n)$. But this implies that $K_1 * f$ is finite almost everywhere.

Now let p' be the dual exponent to p. Then since $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{p} - \frac{2\alpha}{n} = \frac{1}{q}$, we have that

$$\frac{1}{q} = \frac{p'(n-2\alpha) - n}{np'}.$$

Thus, since $q \neq \infty$ and q, p', n are positive, we have that $p'(n-2\alpha) > n$, i.e. $p'(2\alpha - n) < -n$. But then

$$c_\alpha \int_{|x| \geq c} |x|^{p'(2\alpha - n)}\, dx$$

converges, so $K_2 \in L^{p'}(\mathbb{R}^n)$. Then by Hölder's inequality,

$$\|K_2 * f\|_1 \leq \|K_2\|_{p'} \|f\|_p.$$

Hence, $K_2 * f \in L^1(\mathbb{R}^n)$ and thus is finite almost everywhere. But then we have that $I_\alpha f$ is finite almost everywhere.

In order to show that I_α is a bounded operator from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, we will show that for all $\lambda > 0$,

$$m(\{x : |I_\alpha f(x)| > \lambda\}) \leq A \|f\|_p^q \lambda^{-q}, \tag{3.2}$$

for some constant A depending only on p and q, i.e. that I_α is weak-type (p, q). Then, in particular, I_α is weak-types $(1, q)$ and (p, q), so we may apply the Marcinkiewicz Interpolation Theorem (stated above) to obtain the result.

Note that it suffices to show that (3.2) holds for $f \in L^p(\mathbb{R}^n)$ with $\|f\|_p = 1$. This follows from the fact that $f/\|f\|_p$ has p-norm 1, so given (3.2) for functions of norm 1, we have that

$$m(\{x : |I_\alpha f(x)| > \lambda\}) = m\left(\left\{x : \left| I_\alpha \left(\frac{f}{\|f\|_p}\right)(x)\right| > \frac{\lambda}{\|f\|_p}\right\}\right) \leq A \frac{\|f\|_p^q}{\lambda^q}.$$

Now note that

$$\begin{aligned} m(\{x : |I_\alpha f(x)| > \lambda\}) &= m(\{x : |K_\alpha * f(x)| > \lambda\}) \\ &\leq m(\{x : |K_1 * f(x)| > \lambda/2\}) + m(\{x : |K_2 * f(x)| > \lambda/2\}). \end{aligned}$$

Furthermore, we have that

$$\frac{\lambda^p}{2^p} m(\{x : |K_1 * f(x)| > \lambda/2\}) \leq \int_{\mathbb{R}^n} |K_1 * f(x)|^p\, dx = \|K_1 * f\|_p^p \leq \|K_1\|_1^p \|f\|_p^p = \|K_1\|_1^p.$$

Now,

$$\|K_1\|_1 = c_\alpha \int_{|x| \leq c} |x|^{2\alpha - n}\, dx = A \int_0^c r^{2\alpha - n + n - 1} = A c^{2\alpha}.$$

From the above two expressions it then follows that

$$m(\{x : |K_1 * f(x)| > \lambda/2\}) \leq A c^{2\alpha p} \lambda^{-p}.$$

Since this holds for any $c > 0$, we can choose $c = \lambda^{-q/n}$. Then $c^{2\alpha p} = \lambda^{p-q}$, so for $\|f\|_p = 1$, we have

$$m(\{x : |K_1 * f(x)| > \lambda/2\}) \leq A \lambda^{-q} = A \|f\|_p^q \lambda^{-q}.$$

Now we claim that $\|K_2 * f\|_\infty = 0$. By Hölder's inequality, we already have that

$$\|K_2 * f\|_\infty \leq \|f\|_p \|K_2\|_{p'}.$$

Using $c = \lambda^{-q/n}$ as above, as well as the fact that $p'(n-2\alpha) - n = np'/q$, we have

$$\begin{aligned}\|K_2\|_{p'}^{p'} &= c_\alpha \int_{|x|\geq c} |x|^{p'(2\alpha-n)}\,dx \\ &= B\int_c^\infty r^{p'(2\alpha-n)+n-1}\,dr \\ &= Bc^{p'(2\alpha-n)+n} \\ &= Bc^{-\frac{np'}{q}} \\ &= B\lambda^{p'}.\end{aligned}$$

Thus, $\|K_2 * f\|_\infty \leq B\|f\|_p\lambda$ for all $\lambda > 0$, so we must have that $\|K_2 * f\|_\infty = 0$. But then $K_2 * f = 0$ almost everywhere, so $m(\{x : |K_2 * f(x)| > \lambda/2\}) = 0$. Hence, we have that

$$m(\{x : |I_\alpha f(x)| > \lambda\}) \leq m(\{x : |K_1 * f(x)| > \lambda/2\}) \leq A\|f\|_p^q\lambda^{-q},$$

as desired. □

4. Sobolev and Besov Spaces

Before proceeding to our regularity results, we need to define two function spaces that provide some measure of regularity and in which we will look for solutions to our differential equations.

Suppose $f \in C^k(\mathbb{R}^n)$ for $k \geq 1$ and f is locally integrable. We will denote by $\frac{\partial^\alpha f}{\partial x^\alpha}$ the partial derivative $\frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$ for $\alpha = (\alpha_1, \dots, \alpha_n)$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n \leq k$. Let $\varphi \in C_c^\infty$. Then integration by parts gives us

$$\int_{\mathbb{R}^n} f\frac{\partial^\alpha \varphi}{\partial x^\alpha}\,dx = (-1)^{|\alpha|}\int_{\mathbb{R}^n} \frac{\partial^\alpha f}{\partial x^\alpha}\varphi\,dx, \tag{4.1}$$

where the boundary terms are zero since φ has compact support. We use this expression to define the notion of a weak derivative.

Definition 4.2. A function $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ (i.e. a locally integrable function f) is weakly differentiable in $L^1_{\text{loc}}(\mathbb{R}^n)$ if there exists a function $g \in L^1_{\text{loc}}(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} f\frac{\partial^\alpha \varphi}{\partial x^\alpha}\,dx = (-1)^{|\alpha|}\int_{\mathbb{R}^n} g\varphi\,dx$$

holds for all $\varphi \in C_c^\infty$. We say that g is the α-th weak derivative of f, and use the notation $g = \frac{\partial^\alpha f}{\partial x^\alpha}$.

From (4.1), it is clear that if f is differentiable, then the weak derivative coincides with the ordinary derivative. The notion of weak derivatives is particularly useful in partial differential equations because it allows one to find "weak" solutions (those only weakly differentiable) to problems for which classical solutions may not exist. Accordingly, we will define spaces of weakly differentiable functions with a norm giving some measure of regularity of these functions. We will then be able to measure regularity of solutions to equations when they fall into these spaces.

Definition 4.3. The Sobolev space $W^{k,p}(\mathbb{R}^n)$ is the space consisting of all functions in $L^p(\mathbb{R}^n)$ whose α-th weak derivatives exist and are in $L^p(\mathbb{R}^n)$ for all multi-indices

α with $|\alpha| \leq k$. The norm of a function $f \in W^{k,p}$ is defined as

$$\|f\|_{W^{k,p}} = \|f\|_p + \sum_{0<|\alpha|\leq k} \left\| \frac{\partial^\alpha f}{\partial x^\alpha} \right\|_p .$$

In fact, $W^{k,p}$ is a Banach space under this norm. This follows easily from the fact that if $\{f_k\}$ is a Cauchy sequence in $W^{k,p}$, then $\{\frac{\partial^\alpha f_k}{\partial x^\alpha}\}$ is a Cauchy sequence in the Banach space L^p. In the special case when $p = 2$, $W^{k,p}$ is a Hilbert space, which we will denote by H^k.

Let $y > 0$. We define

$$P_y(x) = \frac{a_n y}{(|x|^2 + y^2)^{\frac{n+1}{2}}},$$

where $a_n = \Gamma(\frac{n+1}{2})/\pi^{\frac{n+1}{2}}$. (Note that P_y is the Poisson kernel, i.e. the solution to the Laplace equation in the upper half plane with $f = 0$ can be obtained by convolution of P_y with the boundary function.)

Definition 4.4. For $\alpha > 0$, $1 \leq p \leq \infty$, $1 \leq q < \infty$, and $k = \lceil \alpha \rceil$, we define the Besov space $B^\alpha_{p,q}$ to be

$$B^\alpha_{p,q} = \left\{ f \in L^p(\mathbb{R}^n) : \left(\int_0^\infty \left(y^{k-\alpha} \left\| \frac{\partial^k (P_y * f)}{\partial y^k} \right\|_p \right)^q \frac{dy}{y} \right)^{1/q} < \infty \right\}.$$

We define the norm of $f \in B^\alpha_{p,q}$ to be

$$\|f\|_{B^\alpha_{p,q}} = \|f\|_p + \left(\int_0^\infty \left(y^{k-\alpha} \left\| \frac{\partial^k (P_y * f)}{\partial y^k} \right\|_p \right)^q \frac{dy}{y} \right)^{1/q}.$$

For $q = \infty$, we define

$$B^\alpha_{p,\infty} = \left\{ f \in L^p(\mathbb{R}^n) : \sup_{y>0} y^{k-\alpha} \left\| \frac{\partial^k (P_y * f)}{\partial y^k} \right\|_p < \infty \right\}$$

with the norm

$$\|f\|_{B^\alpha_{p,\infty}} = \|f\|_p + \sup_{y>0} y^{k-\alpha} \left\| \frac{\partial^k (P_y * f)}{\partial y^k} \right\|_p .$$

Note that Besov spaces are complete with respect to these norms.

While the expressions in this definition may seem overwhelming, it is most important to note the roles that p, q, α play in controlling the regularity of functions in $B^\alpha_{p,q}$. Clearly, p determines integrability of functions, as well as the starting norm, and α controls the smoothness of functions. Less obvious is that q plays a subtler role in determining smoothness.

5. A Regularity Result for the Usual Laplace Equation

We now move to our regularity results. First we wish to find some relationship between the number of weak derivatives of f and the number of weak derivatives of $I_\alpha f$. To motivate this, we first consider the usual Laplace equation.

Suppose u is a solution to $-\Delta u = f$ that is sufficiently smooth and vanishes sufficiently quickly at infinity to justify the following calculation:

$$\begin{aligned}\int_{\mathbb{R}^n} |f|^2\,dx &= \int_{\mathbb{R}^n} |\Delta u|^2\,dx \\ &= \sum_{i,j}\int_{\mathbb{R}^n} u_{x_ix_i}u_{x_jx_j}\,dx \\ &= -\sum_{i,j}\int_{\mathbb{R}^n} u_{x_ix_ix_j}u_{x_j}\,dx \\ &= \sum_{i,j}\int_{\mathbb{R}^n} u_{x_ix_j}u_{x_ix_j}\,dx \\ &= \int_{\mathbb{R}^n} |Du|^2\,dx.\end{aligned}$$

Here we have integrated by parts twice, noting that the boundary terms disappear due to the decay of u. The above calculation shows that if $f \in L^2$, then so are the second-order derivatives of u. Moreover, we can estimate the L^2 norms of these second-order derivatives with the L^2 norm of f. Now suppose $f \in H^1$. Differentiating our differential equation (in the weak sense), we then have

$$-\Delta u_{x_i} = f_{x_i}.$$

Repeating the above calculation, we will find that we can again estimate the L^2 norm of the third-order derivatives of u by the L^2 norm of the first-order derivatives of f. This shows that $u \in H^3$. More generally, if we have that $f \in H^k$, after differentiating the differential equation and repeating the above calculation, we will have that $u \in H^{k+2}$. Thus we find that, at least in the case when u is sufficiently smooth and well-behaved at infinity, we can guarantee that u has at least two more L^2 derivatives than f. We find that a similar statement holds for the fractional Laplace equation.

Theorem 5.1. *Suppose $f \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ for $q = 2$ and p as in Theorem 3.1. Then $u = I_\alpha f \in H^{2\alpha}$.*

Note that 2α may not be an integer. In this case, we may either conclude that $u \in H^{\lfloor 2\alpha \rfloor}$, or we may conclude that u is in the fractional Sobolev space $H^{2\alpha}$ (see [2, p. 297]). In the proof of this theorem, we will use the fact that if $u \in L^2$, then for any $k \geq 0$, $u \in H^k$ if and only if $(1 + |x|^k)\hat{u} \in L^2$. See [2, p. 297] for a proof of this fact.

Proof. Since $f \in L^2(\mathbb{R}^n)$, we know that $\hat{f} \in L^2(\mathbb{R}^n)$. Since $f \in L^p(\mathbb{R}^n)$, we have by Theorem 3.1 that $u \in L^2(\mathbb{R}^n)$, so $\hat{u} \in L^2(\mathbb{R}^n)$ as well. Note that by definition of $I_\alpha = (-\Delta)^{-\alpha}$, we know that

$$\hat{u}(x) = (2\pi|x|)^{-2\alpha}\hat{f}(x).$$

Furthermore, we have that

$$(1 + |x|^{2\alpha})\hat{u} = \hat{u} + (2\pi)^{-2\alpha}\hat{f} \in L^2.$$

Thus, by the fact above, we have that $u \in H^{2\alpha}$. ∎

Theorem 5.1 shows us that, indeed, as with the usual Laplace equation, solutions to the fractional Laplace equation have more weak derivatives in L^2 than the starting function. In fact, we did not even need to assume that f had any weak derivatives to begin with.

6. Regularity of Solutions to the Fractional Laplace Equation

Now we move to some regularity results involving Besov spaces. We will first prove the results for $\mathscr{J}_\alpha$ and then show that they also hold for I_α, as desired.

Theorem 6.1. *Let $\alpha > 0$ and $\beta \geq 0$. Then for $f \in B^\alpha_{p,q}$, $\mathscr{J}_\beta f \in B^{\alpha+\beta}_{p,q}$.*

Remark 6.2. In fact, it turns out that $\mathscr{J}_\beta$ is an isomorphism from $B^\alpha_{p,q}$ to $B^{\alpha+\beta}_{p,q}$ and the norms $\|f\|_{B^\alpha_{p,q}}$ and $\|\mathscr{J}_\beta f\|_{B^{\alpha+\beta}_{p,q}}$ are equivalent. However, we need only the weaker statement for now.

For a proof of the stronger result, see [3, p. 149].

For convenience of notation, we will introduce a new space for the next two theorems.

Definition 6.3. For $1 \leq p \leq \infty$ and $\alpha > 0$, we define the Bessel potential spaces $\mathscr{L}^p_\alpha$ to be

$$\mathscr{L}^p_\alpha = \mathscr{J}_\alpha(L^p),$$

that is, $f \in \mathscr{L}^p_\alpha$ if and only if $f = \mathscr{J}_\alpha g$ for some $g \in L^p$. The norm of $f \in \mathscr{L}^p_\alpha$ is $\|f\|_{\alpha,p} = \|g\|_p$, where $\mathscr{J}_\alpha g = f$.

Note that this norm is well-defined, for if

$$\int_{\mathbb{R}^n} G_\alpha(x-y)g(y)\,dy = \int_{\mathbb{R}^n} G_\alpha(x-y)h(y)\,dy,$$

then

$$\int_{\mathbb{R}^n} G_\alpha(x-y)(g(y)-h(y))\,dy = 0,$$

so $g = h$ almost everywhere and $\|g\|_p = \|h\|_p$.

Theorem 6.4. *Suppose $1 < p < \infty$ and k is a nonnegative integer. Then $\mathscr{L}^p_k = W^{2k,p}$. Moreover, the norms $\|\cdot\|_{W^{2k,p}}$ and $\|\cdot\|_{k,p}$ are equivalent.*

Proof. The bulk of the proof consists in proving the following lemma:

Lemma 6.5. *If $\alpha \geq 1/2$, then $f \in \mathscr{L}^p_\alpha$ if and only if $f \in \mathscr{L}^p_{\alpha-1/2}$ and $\frac{\partial f}{\partial x_j} \in \mathscr{L}^p_{\alpha-1/2}$ for $j = 1, \ldots, n$. Furthermore, the norm $\|f\|_{\alpha,p}$ is equivalent to the norm*

$$\|f\|_{\alpha-1/2,p} + \sum_{j=1}^n \left\| \frac{\partial f}{\partial x_j} \right\|_{\alpha-1/2,p}.$$

Once we have proven this, we note that $\mathscr{J}_0 f = (I-\Delta)^0 f = f$, so $\mathscr{L}^p_0 = \mathscr{J}_0(L^p) = L^p$. But clearly $W^{0,p} = L^p$, so the statement holds for $k = 0$. If we have shown that the statement holds for some $k \geq 0$, we show that it holds for $k+1$ as follows. By the claim, $f \in \mathscr{L}^p_{k+1}$ if and only if $f \in \mathscr{L}^p_{k+1/2}$ and $\frac{\partial f}{\partial x_i} \in \mathscr{L}^p_{k+1/2}$ for all j. By assumption $f, \frac{\partial f}{\partial x_i} \in \mathscr{L}^p_{k+1/2}$ if and only if $f, \frac{\partial f}{\partial x_i} \in W^{2k+1,p}$ (in particular, $f \in L^p$.) But then we know that $f \in L^p$ and $\frac{\partial f}{\partial x_i} \in W^{2k+1,p}$ if and only if $f \in W^{2k+2,p}$. Thus, it suffices to prove the lemma.

Proof of lemma. First suppose that $f \in \mathscr{L}^p_\alpha$, i.e. $f = \mathscr{J}_\alpha g$ for $g \in L^p \cap C_c^\infty$. Using the fact that $\widehat{G_\alpha}(x) = (1+4\pi^2|x|^2)^{-\alpha}$, we see that

$$\begin{aligned}\widehat{\frac{\partial f}{\partial x_j}}(x) &= 2\pi i x_j \hat{f}(x)\\ &= 2\pi i x_j (1+4\pi^2|x|^2)^{-\alpha}\hat{g}(x)\\ &= \frac{ix_j}{|x|}\frac{2\pi|x|}{(1+4\pi^2|x|^2)^{1/2}}(1+4\pi^2|x|^2)^{-\alpha+1/2}\hat{g}(x)\\ &= \widehat{R_j(\mu * g)}(x)\widehat{G_{\alpha-1/2}}(x),\end{aligned}$$

where R_j is the Riesz transform

$$R_j h(x) = \lim_{\epsilon\to 0} c_n \int_{|y|\geq\epsilon} \frac{y_i}{|y|^{n+1}} h(x-y)\,dy$$

and μ is a finite measure on $\mathbb{R}^n$ such that

$$\hat{\mu}(x) = \frac{2\pi|x|}{(1+4\pi^2|x|^2)^{1/2}}$$

(the existence of such a measure is proven in [3, p. 133]). It is easy to see that this final equality holds when we note that $\hat{R_j h}(x) = \frac{ix_j}{|x|}\hat{h}(x)$. Thus, we have that

$$\frac{\partial f}{\partial x_j} = \mathscr{J}_{\alpha-1/2}(R_j(\mu * g)). \tag{6.6}$$

To extend this argument to all $g \in L^p$, we proceed as follows. First note that

$$\begin{aligned}\|\mu * g\|_p &= \left(\int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^n} g(x-y)\,d\mu(y)\right|^p dx\right)^{1/p}\\ &\leq \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|g(x-y)|^p\,dx\right)^{1/p} d\mu(y)\\ &= \int_{\mathbb{R}^n}\|g\|_p\,d\mu(y)\\ &= C\|g\|_p,\end{aligned}$$

since μ is a finite measure. Thus, $\mu * g \in L^p$, so since R_j is a bounded operator from L^p to L^p (see [3, p. 39]), we have that $R_j(\mu * g) \in L^p$. Now we take a sequence $\{g_m\}$ in C_c^∞ that approximates g in the L^p norm. Then we have

$$\begin{aligned}\|\mathscr{J}_{\alpha-1/2}(R_j(\mu * g)) - \mathscr{J}_{\alpha-1/2}(R_j(\mu * g_m))\|_p &\leq C\|R_j(\mu * g) - R_j(\mu * g_m)\|_p\\ &\leq C\|\mu * (g-g_m)\|_p\\ &\leq C\|g-g_m\|_p.\end{aligned}$$

But this last quantity goes to zero, so we have that

$$\mathscr{J}_{\alpha-1/2}(R_j(\mu * g_m)) \to \mathscr{J}_{\alpha-1/2}(R_j(\mu * g))$$

in the L^p norm, which implies that $\frac{\partial f_m}{\partial x_j} \to \frac{\partial f}{\partial x_j}$ in the $\mathscr{L}_{\alpha-1/2}$ norm (where $f_m = \mathscr{J}_\alpha g_m$). Hence, $\frac{\partial f}{\partial x_j} = \mathscr{J}_{\alpha-1/2}(R_j(\mu * g))$ for any $g \in L^p$.

Moreover, we see that

$$\left\| \frac{\partial f}{\partial x_j} \right\|_{\alpha-1/2,p} = \|R_j(\mu * g)\|_p \le \|\mu * g\|_p \le C\|g\|_p = C\|f\|_{\alpha,p}.$$

We also have that $f = \mathscr{J}_\alpha g = \mathscr{J}_{\alpha-1/2}(\mathscr{J}_{1/2} g)$. Thus,

$$\|f\|_{\alpha-1/2,p} = \|\mathscr{J}_{1/2} g\|_p \le \|g\|_p = \|f\|_{\alpha,p}.$$

But then

$$\|f\|_{\alpha-1/2,p} + \sum_{j=1}^{n} \left\| \frac{\partial f}{\partial x_j} \right\|_{\alpha-1/2,p} \le C\|f\|_{\alpha,p}.$$

This proves the forward direction.

Conversely, suppose that $f = \mathscr{J}_{\alpha-1/2} g$ and $\frac{\partial f}{\partial x_j} = \mathscr{J}_{\alpha-1/2} g_j$ for $g, g_j \in L^p$. First we wish to show that $\frac{\partial g}{\partial x_j} = g_j$, so that $g \in W^{1,p}$. Let $\varphi \in C_c^\infty$. Then

$$\begin{aligned}
\int_{\mathbb{R}^n} f(x) \frac{\partial \varphi}{\partial x_j}(x)\, dx &= \int_{\mathbb{R}^n} \mathscr{J}_{\alpha-1/2} g(x) \frac{\partial \varphi}{\partial x_j}(x)\, dx \\
&= \int_{\mathbb{R}^n} g(x) \mathscr{J}_{\alpha-1/2} \left(\frac{\partial \varphi}{\partial x_j} \right)(x)\, dx \\
&= \int_{\mathbb{R}^n} g(x) \int_{\mathbb{R}^n} G_{\alpha-1/2}(x-y) \frac{\partial \varphi}{\partial x_j}(y)\, dy\, dx \\
&= -\int_{\mathbb{R}^n} g(x) \int_{\mathbb{R}^n} \frac{\partial G_{\alpha-1/2}}{\partial x_j}(x-y)\varphi(y)\, dy\, dx \\
&= -\int_{\mathbb{R}^n} g(x) \frac{\partial}{\partial x_j} \mathscr{J}_{\alpha-1/2}\varphi(x)\, dx.
\end{aligned}$$

On the other hand,

$$\int_{\mathbb{R}^n} f(x) \frac{\partial \varphi}{\partial x_j}(x)\, dx = \int_{\mathbb{R}^n} \frac{\partial f}{\partial x_j}(x)\varphi(x)\, dx = \int_{\mathbb{R}^n} \mathscr{J}_{\alpha-1/2} g_j(x)\varphi(x)\, dx = \int_{\mathbb{R}^n} g_j(x) \mathscr{J}_{\alpha-1/2}\varphi(x)\, dx.$$

Combining these, we have that

$$\int_{\mathbb{R}^n} g_j(x) \mathscr{J}_{\alpha-1/2}\varphi(x)\, dx = -\int_{\mathbb{R}^n} g(x) \frac{\partial}{\partial x_j} \mathscr{J}_{\alpha-1/2}\varphi(x)\, dx.$$

Now let $\psi \in C_c^\infty$ and let $\hat{\varphi}(x) = \hat{\psi}(x)(1 + 4\pi^2|x|^2)^\alpha$. Since $\hat{\psi} \in C_c^\infty$, it follows that $\hat{\varphi} \in C_c^\infty$. But then $\hat{\psi}(x) = \hat{\varphi}(x)(1 + 4\pi^2|x|^2)^{-\alpha}$, so $\psi = \mathscr{J}_\alpha \varphi$. Hence, $\mathscr{J}_\alpha$ maps C_c^∞ onto C_c^∞, so

$$\begin{aligned}
\int_{\mathbb{R}^n} g_j(x)\psi(x)\, dx &= \int_{\mathbb{R}^n} g_j(x) \mathscr{J}_\alpha \varphi(x)\, dx \\
&= -\int_{\mathbb{R}^n} g(x) \frac{\partial}{\partial x_j} \mathscr{J}_\alpha \varphi(x)\, dx \\
&= -\int_{\mathbb{R}^n} g(x) \frac{\partial \psi}{\partial x_j}(x)\, dx,
\end{aligned}$$

and $g_j = \frac{\partial g}{\partial x_j}$. Thus, $g \in W^{1,p}$. Then we can approximate g in the L^p norm by C_c^∞ functions $\{g_m\}$ such that $\{\frac{\partial g_m}{\partial x_j}\}$ converges in the L^p norm to $\frac{\partial g}{\partial x_j}$ (see [3, p. 122] for a discussion of this result). As noted above, $\mathscr{J}_{1/2}$ maps C_c^∞ onto C_c^∞, so there are functions $h_m \in C_c^\infty$ such that $g_m = \mathscr{J}_{1/2} h_m$. We will see in Lemma 6.9, below, that there exist finite measures ν and λ on $\mathbb{R}^n$ such that

$$\frac{1}{\widehat{G_1}(x)} = \hat{\nu}(x) + \frac{\hat{\lambda}(x)}{\widehat{K_1}(x)}.$$

If we multiply this equation by $\hat{g_m}$, we get that

$$\hat{h_m} = \hat{\nu}\hat{g_m} + \frac{\hat{\lambda}\hat{g_m}}{\widehat{K_1}}.$$

Now note that

$$\frac{\hat{g_m}}{\widehat{K_1}} = \frac{\widehat{G_1}}{\widehat{K_1}}\hat{h_m} = \hat{\mu}\hat{h_m} = \sum_{j=1}^n \frac{x_j^2}{|x|^2}\hat{\mu}\hat{h_m} = -\sum_{j=1}^n R_j(\widehat{R_j(\mu * h_m)}) = -\sum_{j=1}^n R_j\left(\widehat{\frac{\partial}{\partial x_j} g_m}\right).$$

Hence, taking inverse Fourier transforms, we have that

$$h_m = \mu * g_m - \nu * \left(\sum_{j=1}^n R_j\left(\frac{\partial}{\partial x_j} g_m\right)\right).$$

Now, the Riesz tranforms are bounded operators from L^p to L^p, so it follows that

$$\begin{aligned}
\|h_m\|_p &\le \|g_m * \mu\|_p + \sum_{j=1}^n \left\| R_j\left(\frac{\partial}{\partial x_j} g_m\right)\right\|_p \\
&\le \int_{\mathbb{R}^n} \|g_m\|_p \, d\mu(y) + C\sum_{j=1}^n \left\|\frac{\partial g_m}{\partial x_j}\right\|_p \\
&= C\left(\|g_m\|_p + \sum_{j=1}^n \left\|\frac{\partial g_m}{\partial x_j}\right\|_p\right) \\
&< \infty.
\end{aligned}$$

Note that by repeating this argument, we may prove the same inequalities if we replace h_m with $h_m - h_k$ and g_m by $g_m - g_k$. Since $\{g_m\}$ is Cauchy, this shows that $\{h_m\}$ is Cauchy as well. But then $h_m \to h$ for some $h \in L^p$. Hence,

$$\|\mathscr{J}_\alpha h_m - \mathscr{J}_\alpha h\|_{\alpha,p} = \|\mathscr{J}_\alpha (h_m - h)\|_{\alpha,p} = \|h_m - h\|_p \to 0.$$

Note that $\mathscr{J}_{\alpha-1/2} g_m = \mathscr{J}_{\alpha-1/2}(\mathscr{J}_{1/2} h_m) = \mathscr{J}_\alpha h_m$. It follows that $\mathscr{J}_{\alpha-1/2} g_m \to \mathscr{J}_{\alpha-1/2} g = f$ in the $\mathscr{L}_\alpha^p$ norm. But then $f \in \mathscr{L}_\alpha^p$. Furthermore,

$$\begin{aligned}
\|f\|_{\alpha,p} = \|h\|_p &\le C\left(\|g\|_p + \sum_{j=1}^n \left\|\frac{\partial g}{\partial x_j}\right\|_p\right) \\
&= C\left(\|g\|_p + \sum_{j=1}^n \left\|\frac{\partial g}{\partial x_j}\right\|_p\right) \\
&= C\left(\|f\|_{\alpha-1/2,p} + \sum_{j=1}^n \left\|\frac{\partial f}{\partial x_j}\right\|_{\alpha-1/2,p}\right),
\end{aligned}$$

so the norms are equivalent and the lemma is proved. □

Corollary 6.7. *If $\alpha > 0$, $1 < p < \infty$, and $f \in L^p$, then $\mathscr{J}_\alpha f \in W^{2k,p}$ for $k = \lfloor \alpha \rfloor$.*

Proof. First we note that if $\alpha < \beta$, then $\mathscr{L}^p_\beta \subset \mathscr{L}^p_\alpha$. This follows from the fact that $G_\alpha * G_\beta = G_{\alpha+\beta}$, so $\mathscr{J}_\alpha \circ \mathscr{J}_\beta = \mathscr{J}_{\alpha+\beta}$. Then for $f \in \mathscr{L}^p_\beta$, we have that $f = \mathscr{J}_\beta g$ for some $g \in L^p$, so $f = \mathscr{J}_\alpha(\mathscr{J}_{\beta-\alpha} g)$. But

$$\|\mathscr{J}_{\beta-\alpha} g\|_p \leq \|G_{\beta-\alpha}\|_1 \|g\|_p = \|g\|_p,$$

(since $\|G_\epsilon\|_1 = 1$ for any ϵ) so $\mathscr{J}_{\beta-\alpha} \in L^p$. Hence, $f \in \mathscr{J}_\alpha$, so $\mathscr{L}^p_\beta \subset \mathscr{L}^p_\alpha$.

Now we see that if $f \in L^p$, then

$$\mathscr{J}_\alpha f \in \mathscr{L}^p_\alpha \subset \mathscr{L}^p_k = W^{2k,p},$$

as desired. □

Thus, $\mathscr{J}_\alpha f$ has up to α weak derivatives in L^p, even if f has none.

Theorem 6.8. *Suppose $\alpha > 0$ and $1 < p < \infty$. If $p \leq 2$, then*

$$\mathscr{L}^p_{\alpha/2} \subset B^\alpha_{p,2} \quad \text{and} \quad B^\alpha_{p,p} \subset \mathscr{L}^p_{\alpha/2}.$$

If $p \geq 2$, then

$$B^\alpha_{p,2} \subset \mathscr{L}^p_{\alpha/2} \quad \text{and} \quad \mathscr{L}^p_{\alpha/2} \subset B^\alpha_{p,p}.$$

Proof. If we prove these inclusions for any particular $\alpha > 0$, then by Remark 6.2, and by a similar statement for Bessel potential spaces (under the same assumptions, $\mathscr{J}_\beta$ is an isomorphism from $\mathscr{L}^p_\alpha$ to $\mathscr{L}^p_{\alpha+\beta}$–see [3, p. 135]), the inclusions will hold for all α. Thus, we prove these inclusions for $\alpha = 1$.

For this proof, we will find it convenient to define the operators

$$T_p f(x) = \begin{cases} \left(\int_0^\infty y^p |\nabla U|^{2p} \frac{dy}{y}\right)^{1/p} & 1 < p < \infty \\ \sup_{y>0} y|\nabla U|^2 & p = \infty \end{cases}$$

where $U = P_y * f$ is the convolution of $f \in L^p$ ($p < \infty$) with the Poisson kernel. We will use without proof the fact that both $\|T_2 f\|_p$ and $\|T_\infty f\|_p$ are equivalent to $\sum_{j=1}^n \|\frac{\partial f}{\partial x_j}\|_p$.

First we show that $\mathscr{L}^p_{1/2} \subset B^1_{p,2}$ for $p \leq 2$. Since $2/p \geq 1$, we can apply Minkowski's inequality:

$$\begin{aligned} \left(\int_0^\infty y^2 \left(\int_{\mathbb{R}^n} |\nabla U|^{2p}\, dx\right)^{2/p} \frac{dy}{y}\right)^{p/2} &\leq \int_{\mathbb{R}^n} \left(\int_0^\infty y^2 |\nabla U|^4 \frac{dy}{y}\right)^{p/2} dx \\ &= \|T_2 f\|_p^2 \\ &\leq C \left(\sum_{j=1}^n \left\|\frac{\partial f}{\partial x_j}\right\|_p\right)^2. \end{aligned}$$

This is finite if $f \in W^{1,p}$, so we then have that $\|f\|_{B^1_{p,2}} < \infty$. Thus, $\mathscr{L}^p_{1/2} = W^{1,p} \subset B^1_{p,2}$.

Next we show that if $p \leq 2$, then $B^1_{p,p} \subset \mathscr{L}^p_{1/2}$. Note that

$$\int_0^\infty (y|\nabla U|^2)^2 \frac{dy}{y} \leq \left(\sup_{y>0} y|\nabla U|^2\right)^{2-p} \int_0^\infty (y|\nabla U|^2)^p \frac{dy}{y}.$$

It follows that $T_2 f \leq (T_\infty f)^{1-p/2} (T_p f)^{p/2}$. Then by Hölder's inequality,

$$\|T_2 f\|_p \leq \|(T_p f)^{p^2/2}\|_r^{1/p} \|(T_\infty f)^{p(1-p/2)}\|_q^{1/p} = \|T_p f\|_{p^2 r/2}^{p/2} \|T_\infty f\|_{pq(1-p/2)}^{1-p/2}$$

for r and q any conjugate exponents. If we choose $r = 2/p$ and $q = 1/(1-p/2)$, then we get that the right side of this equation is equal to $\|T_p f\|_p^{p/2}\|T_\infty f\|_p^{1-p/2}$. By the equivalence of norms mentioned above, we then have that

$$C\sum_{j=1}^{n}\left\|\frac{\partial f}{\partial x_j}\right\|_p \leq \|T_2 f\|_p \leq \|T_p f\|_p^{p/2}\|T_\infty f\|_p^{1-p/2}$$
$$\leq C'\|T_p f\|_p^{p/2}\left(\sum_{j=1}^{n}\left\|\frac{\partial f}{\partial x_j}\right\|_p\right)^{1-p/2},$$

which implies that $C''\left(\sum_{j=1}^{n}\|\frac{\partial f}{\partial x_j}\|_p\right) \leq \|T_p f\|_p$. Hence, we are left with

$$C''\sum_{j=1}^{n}\left\|\frac{\partial f}{\partial x_j}\right\|_p \leq \|T_p f\|_p = \left(\int_0^\infty y^p \||\nabla U|^2\|_p^p \frac{dy}{y}\right)^{1/p}$$
$$\leq C'''\left(\int_0^\infty y^p\left\|\frac{\partial^2 U}{\partial y^2}\right\|_p^p \frac{dy}{y}\right)^{1/p}$$
$$\leq C'''\|f\|_{B^1_{p,p}}.$$

Therefore,

$$\|f\|_{W^{1,p}} = \|f\|_p + \sum_{j=1}^{n}\left\|\frac{\partial f}{\partial x_j}\right\|_p \leq \|f\|_p + \left(\int_0^\infty y^2\left\|\frac{\partial^2 U}{\partial y^2}\right\|_p^p \frac{dy}{y}\right)^{1/p} = \|f\|_{B^1_{p,2}}.$$

Thus, $B^1_{p,2} \subset W^{1,p} = \mathscr{L}^p_{1/2}$.

Now we will prove that $B^1_{p,2} \subset \mathscr{L}^p_{1/2}$ if $p \geq 2$. Since $p/2 \geq 1$, we can apply Minkowski's inequality to see that

$$\|T_2 f\|_p^2 = \left(\int_{\mathbb{R}^n}\left(\int_0^\infty y^2|\nabla U|^4 \frac{dy}{y}\right)^{p/2} dx\right)^{2/p}$$
$$\leq \int_0^\infty\left(\int_{\mathbb{R}^n} y^p|\nabla U|^{2p}\,dx\right)^{2/p}\frac{dy}{y}$$
$$= \int_0^\infty y^2\||\nabla U|^2\|_p \frac{dy}{y}$$
$$\leq C\int_0^\infty y^2\left\|\frac{\partial^2 U}{\partial y^2}\right\|_p^2 \frac{dy}{y}.$$

But then we have that

$$C\left(\sum_{j=1}^{n}\left\|\frac{\partial f}{\partial x_j}\right\|_p\right)^2 \leq \|T_2 f\|_p^2 \leq \int_0^\infty y^2\left\|\frac{\partial^2 U}{\partial y^2}\right\|_p^2 \frac{dy}{y}.$$

Hence,

$$\|f\|_{W^{1,p}} = \|f\|_p + \sum_{j=1}^{n}\left\|\frac{\partial f}{\partial x_j}\right\|_p \leq \|f\|_p + \left(\int_0^\infty y^2\left\|\frac{\partial^2 U}{\partial y^2}\right\|_p^2 \frac{dy}{y}\right)^{1/2} = \|f\|_{B^1_{p,2}}.$$

Then by Theorem 6.4, we have that

$$B^1_{p,2} \subset W^{1,p} = \mathscr{L}^p_{1/2}.$$

Finally, we show that $\mathscr{L}^p_{1/2} \subset B^1_{p,p}$ if $p \geq 2$. Since

$$\int_0^\infty (y|\nabla U|^2)^p \,\frac{dy}{y} \leq \left(\sup_{y>0} y|\nabla U|^2\right)^{p-2} \int_0^\infty (y|\nabla U|^2)^2 \,\frac{dy}{y},$$

we have that $T_p f \leq (T_\infty f)^{(p-2)/p}(T_2 f)^{2/p}$. Applying Hölder's inequality for conjugate exponents $p/(p-2)$ and $p/2$, we find that

$$\|T_p f\|_p \leq \|T_\infty\|_p^{1-2/p}\|T_2\|_p^{2/p} \leq C\sum_{j=1}^n \left\|\frac{\partial f}{\partial x_j}\right\|_p.$$

This is finite if $f \in W^{1,p}$. But $\|T_p f\|_p^p \geq \int_0^\infty y^p \|\frac{\partial U}{\partial y^2}\|_p^2 \frac{dy}{y}$. Hence, $\|f\|_{B^1_{p,p}} < \infty$, so $\mathscr{L}^p_{1/2} = W^{1,p} \subset B^1_{p,p}$. □

Theorems 6.4 and 6.8 give us a relationship between certain Sobolev and Besov spaces. Together with Theorem 6.8, Corollary 6.7 implies that if $f \in L^p$, then $\mathscr{J}_\alpha f \in B^{2\alpha}_{p,q}$, where $q = 2$ if $p \leq 2$ and $q = p$ if $p \geq 2$. We wish to show that these results also hold for I_α. First we state a lemma.

Lemma 6.9. *Let $\alpha > 0$. Then there exist finite measures μ_α and ν_α on $\mathbb{R}^n$ such that*

$$\frac{1}{\widehat{G_\alpha}(x)} = \widehat{\mu_\alpha}(x) + \frac{\widehat{\nu_\alpha}(x)}{\widehat{K_\alpha}(x)}, \tag{6.10}$$

where K_α and G_α are the kernels corresponding to I_α and $\mathscr{J}_\alpha$.

For a proof of this lemma, see [3, p. 133].

We will see that (6.10) illustrates a useful relationship between $\mathscr{J}_\alpha$ and I_α. Putting the right side of (6.10) over a common denominator and cross multiplying, we get that

$$\widehat{K_\alpha}(x) = \widehat{K_\alpha}(x)\widehat{G_\alpha}(x)\widehat{\mu_\alpha}(x) + \widehat{G_\alpha}(x)\widehat{\nu_\alpha}(x).$$

Multiplying both sides by $\hat{f}$, we get that

$$\widehat{K_\alpha * f}(x) = \widehat{K_\alpha * f}(x)\widehat{G_\alpha}(x)\widehat{\mu_\alpha}(x) + \widehat{G_\alpha * f}(x)\widehat{\nu_\alpha}(x).$$

Taking inverse Fourier transforms, this yields

$$I_\alpha f = \mathscr{J}_\alpha(I_\alpha f) * \mu_\alpha + \mathscr{J}_\alpha f * \nu_\alpha, \tag{6.11}$$

or, since convolution with K_α and G_α commutes, this also implies that

$$I_\alpha f = I_\alpha(\mathscr{J}_\alpha f) * \mu_\alpha + \mathscr{J}_\alpha f * \nu_\alpha. \tag{6.12}$$

Suppose $f \in L^p$. Then by Corollary 6.7, we have that $\mathscr{J}_\alpha f \in W^{2\alpha,p}$. Now, for $\frac{1}{q} = \frac{1}{p} - \frac{2\alpha}{n}$, with $q < \infty$, we have that $W^{2\alpha,p} \subset L^q(\mathbb{R}^n)$ (this is the so-called Sobolev Embedding Theorem–see [3, p. 124]). Thus, $\mathscr{J}_\alpha f \in L^q(\mathbb{R}^n)$. Then by Theorem 3.1 (which can easily be extended to show that I_α is bounded from L^q to L^p as well), $I_\alpha(\mathscr{J}_\alpha f) \in L^p(\mathbb{R}^n)$. Hence, from (6.12) it is clear that $I_\alpha f \in L^p(\mathbb{R}^n)$. But then $\mathscr{J}_\alpha(I_\alpha f)$ and $\mathscr{J}_\alpha f$ are both in $W^{2\alpha,p}$ by Corollary 6.7. It follows from (6.11) that $I_\alpha f \in W^{2\alpha,p}$ as well. Thus, Corollary 6.7 holds if we replace $\mathscr{J}_\alpha$ with I_α. Using this and Theorem 6.8, we obtain the following result for I_α:

Theorem 6.13. *Let $\alpha > 0$ and $f \in L^p$ for $1 < p < \infty$. If $p \leq 2$, then $I_\alpha f \in B^{2\alpha}_{p,2}$. If $\phi \geq 2$, then $I_\alpha f \in B^{2\alpha}_{p,p}$.*

Proof. This follows from the fact that $I_\alpha f \in W^{2\alpha,p} = \mathscr{L}^p_\alpha \subset B^{2\alpha}_{p,q}$, for $q = 2$ if $p \leq 2$ and $q = p$ if $p \geq 2$. □

Thus, we see that for $f \in L^p$, solutions of the fractional Laplace equation have up to 2α derivatives in L^p, even if f has none. Furthermore, they gain additional smoothness as measured by Besov spaces.

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